Paper / Subject Code: 82173 / Mathematics. : Algebra IV (R-2021)

M-SC.(Maths) Part-II (Sem-IV) dt 08/01/25

Time: 3:00 Hours **Marks: 80**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate full marks.
3. Scientific calculator can be used.

Q.1 (a) Define a minimal polynomial for α over F. Prove that if α is algebraic over a field F and $m_\alpha(x)$ is the minimal polynomial for α over F, then $F(\alpha)$ is isomorphic to $F[x]/(m_\alpha(x))$. **(10)**

(b) Attempt **any two** from the following: **(10)**

(i) Prove that the field generated over F by α and β is the field generated by β over the field $F(\alpha)$, generated by α. **(05)**

(ii) Express cos 30° in terms of square roots. **(05)**

(iii) Prove that a 20° angle is not constructible. **(05)**

Q.2 (a) Define a perfect field. Prove that a field F is perfect iff every irreducible polynomial in $F[x]$ is separable. **(10)**

(b) Attempt **any two** from the following: **(10)**

(i) $\mathbb{R}$ is an infinite extension of $\mathbb{Q}$, i.e. $[\mathbb{R} : \mathbb{Q}] = \infty$. Prove or disprove. **(05)**

(ii) Find the splitting field of $x^2 - 9$ over $\mathbb{Q}$. **(05)**

(iii) Let $f(x) \in F[x]$ be irreducible polynomial over F, then show that $f(x)$ has a multiple root in some extension of F iff $f'(x) = 0$ identically. **(05)**

Q.3 (a) Let E/F be finite extension. Then show that E/F is Galois if and only if F is the fixed field of the group of all F-autmorphisms of E. **(10)**

(b) Attempt **any two** from the following: **(10)**

(i) Let char k = 0 Then show that k is contained in some Galois extension of k. **(05)**

(ii) Show that every finite extension of a finite field is Galois. **(05)**

(iii) Let E/K be Galois and F be any extension of K. Then show that EF/F is Galois and G(EF/F) is isomorphic to a subgroup of G(E/K). **(05)**

Q.4 (a) Prove that for a positive integer n, $x^n - 1 = \prod_{d|n} \Phi_d(x)$. **(10)**

(b) Attempt **any two** from the following: **(10)**

(i) Prove that Gal(Q(ζ)/Q) ≅ U(n). **(05)**

(ii) Prove that for any finite extension E of $\mathbb{Q}$, E contains only finite number of roots of unity. **(05)**

(iii) Determine the minimal polynomial for $\zeta = e^{\pi/3}$ over Q. **(05)**
