Paper / Subject Code: 82180 / Mathematics. : Calculus on Manifolds (R-2021)

M.Sc. (maths) 80:20 sem - IV date - 14/01/2025

[Duration: 3 Hours] [Marks: 80]

N.B.
1) All questions are compulsory and carry equal marks.
2) Figures to the right indicates full marks.
3) Use of scientific non programmable calculator is allowed.
4) Standard notations have their usual meaning.

1. (a) If T is an inner product on V, there is a basis $v_1, v_2, \cdots, v_n$ for V such that $T(v_i, v_j) = \delta_{ij}$. Show that there is an isomorphism $f : I\!R^n \to V$ such that $T(f(x), f(y)) = \langle x, y \rangle$ for $x, y \in I\!R^n$. (10)

(b) Attempt any two of the following

(i) If $S \in \Im^k(V)$, $T \in \Im^l(V)$, $U \in \Im^m(V)$ then Show that $(S \otimes T) \otimes U = S \otimes (T \otimes U)$. (5)

(ii) If $\omega \in \Lambda^2(V)$ and $\eta \in \Lambda^3(V)$ then prove or disprove: $\omega \wedge \eta = \eta \wedge \omega$. (5)

(iii) If $\omega_1, \omega_2 \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$ then show that $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$ (5)

2. (a) If $A \subset I\!R^n$ is an open set star-shaped with respect to 0 then show that every closed form on A is exact. (10)

(b) Attempt any two of the following

(i) Calculate exterior derivatives of 2−form $(z^2 - xy)dx \wedge dy + (y^2 - xyz)dy \wedge dz$ in $I\!R^3$. (5)

(ii) In $I\!R^3$, let $\omega = xydx + 2zdy - ydz$ and $\alpha : I\!R^2 \longrightarrow I\!R^3$ be given by $\alpha(u, v) = (uv, u^2, 3u + v)$. Calculate $\alpha^*(d\omega)$. (5)

(iii) If $f : I\!R^n \to I\!R$ is differentiable then show that (5)

$$df = D_1 f \cdot dx^1 + D_2 f \cdot dx^2 + \cdots D_n f \cdot dx^n.$$

3. (a) Let $A \subset I\!R^n$ be open and let $g : A \to I\!R^p$ be a differentiable function such that $g'(x)$ has rank p whenever $g(x) = 0$. Then show that $g^{-1}(0)$ is an $(n-p)$−dimensional manifold in $I\!R^n$. (10)

(b) Attempt any two of the following

(i) Define differomorphism and give an example of differomorphism. (5)

(ii) Show that the function $\alpha : [0, 1] \longrightarrow S^1$ given by $\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$ is not a coordinate patch on S^1. (5)

[TURN OVER

(iii) Let $\beta : H^1 \longrightarrow I\!R^2$ be the map $\beta(t) = (t, t^2)$. Let N be image set of β. Show that N is 1−manifold in $I\!R^2$. (5)

4. (a) State and prove Stoke's theorem for a compact two dimensional manifold with boundary. (10)

(b) Attempt any two of the following

(i) Consider vector field $\vec{F} = (y+z)i + (z+x)j + (x+y)k$. Compute $\int\limits_{\partial M} \langle F, n\rangle dA$, where $M \subset I\!R^3$ be a compact oriented two-dimensional manifold with boundary. (5)

(ii) Evaluate $\iint_S \vec{A} \cdot \hat{n}\, ds$ where $\vec{A} = 18z\hat{i}$ - $12\hat{j}$ + $3y\hat{k}$ and S is the part of the plane $2x + 3y + 6z = 12$ included in the first octant. (5)

(iii) Is the vector field $\vec{F} = \dfrac{\vec{r}}{r^3}$ solenoidal? where $\vec{r}$ is a position vector of (x, y, z). Justify your answer. Also evaluate (5)

$$\int\limits_{\partial S} \langle \vec{F}, \vec{n}\rangle dA$$

where S is 2−manifold in $I\!R^3$ given by $S = \{x \in I\!R^3 / |x| = 1\}$.

****