M.sc (maths) (80:20) Analysis
date - 13/01/2024

[Time: 3:00 Hrs.] [Marks: 80]

Please check whether you have got the right question paper.

N.B:
1. **All questions are compulsory.**
2. **Figures to the right indicate full marks.**
3. **Scientific calculator can be used.**

Q.1 a) Let C be a collection of closed rectangles of $\mathbb{R}^n$, For $R \in C$, let $\vartheta(R)$ denote the volume of R. If for $A \subset \mathbb{R}^n, A \neq \emptyset, \mu^*$ is defined by **10**

$\mu^*(A) = \inf\{\sum_{n=1}^{\infty} \vartheta(C_k) : C_k \in C \text{ and } A \subset \bigcup_{n=1}^{\infty} C_k\}$
then show that μ^* is an exterior measure on $\mathbb{R}^n$.

b) Attempt **any Two** of the following: **10**

i) Construct a non-measurable subset of $\mathbb{R}$. **5**

ii) Show that exterior (or outer) measure of an open rectangle in $\mathbb{R}^n$ is its volume. **5**

iii) Show that closed subsets of $\mathbb{R}^d$ is measurable. **5**

Q.2 a) State and prove Egoroff's theorem. **10**

b) Attempt **any Two** of the following: **10**

i) Let f, g be non-negative, simple, measurable functions on $X \subset \mathbb{R}^n$. Define $\int f\, d\mu$. Show that $\int (f+g)d\mu = \int f d\mu + \int g\, d\mu$. **5**

ii) Prove that the outer measure of countable set is zero. **5**

iii) State and prove Borel Cantelli lemma. **5**

Q.3 a) If f is Riemann integrable on closed interval $[a, b]$ then show that f is Lebesgue measurable and $\int_{[a,b]}^{\mathcal{R}} f(x)dx = \int_{[a,b]}^{\mathcal{L}} f(x)dm$. **10**

b) Attempt **any Two** of the following: **10**

i) Prove that a function $f: \mathbb{R}^d \to \mathbb{R} \cup \{\pm \infty\}$ is a Lebesgue measurable function if for every $a \in \mathbb{R}$, $f^{-1}((a, \infty])$ is Lebesgue measurable subset of $\mathbb{R}^d$. **5**

ii) State and prove Monotone convergence Theorem. **5**

iii) Use the dominated, convergence theorem to prove that $\lim_{n\to\infty} n \int_0^1 \sqrt{n}\, e^{n^2 x^2}\, dx = 0$. **5**

Q.4 **a)** Let B be a bounded measurable set in $\mathbb{R}^n$. Prove that for any small quantity $\epsilon > 0$, there exists a compact set K and an open set U with $K \subset B \subset U$ such that measure $m(U \setminus K) < \epsilon$. **10**

b) Attempt **any Two** of the following: **10**

i) Show that a real valued function that is continuous on its measurable domain is measurable. **5**

ii) If $f \in L^1(\mathbb{R}^n)$, then show that $|\int f| \leq \int |f|$. **5**

iii) State and prove Chebychev's Inequality for non-negative measurable function. **5**

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