Time: 3 Hrs **Marks: 80**

Note:

1. All Questions are compulsory
2. All questions carry equal marks

Q.1 **a)** Define a simple group. Show that there is no simple group of order 144. **(10)**

b) Attempt **any two** of the following: **(10)**

i) Define a solvable group. Show that every abelian group is solvable. **(5)**

ii) Show that the direct product of a finite number of nilpotent groups is nilpotent. **(5)**

iii) Find the upper central series of A_4 **(5)**

Q.2 **a)** Prove that the nilradical of a ring is the intersection of all of the prime ideals of the ring. **(10)**

b) Attempt **any two** of the following: **(10)**

i) Show that the nilradical N(R) of a ring R is an ideal. **(5)**

ii) Let $x \in R$. Prove that $x \in$ J(R) iff $1 - xy$ is a unit for all $y \in R$. **(5)**

iii) Define radical ideal I of a ring R. Show that $I \subset Rad(I)$ **(5)**

Q.3 **a)** Let M be an R-module and N be R-submodule of M. Prove that the submodules of the quotient module M/N are of the type U/N, where U is a submodule of M containing N. **(10)**

b) Attempt **any two** of the following: **(10)**

i) Let M be any R-module. Let $x_1, x_2, \ldots, x_n \in M$ for some $n \in \mathbb{N}$. Then show that the set $N = \{\sum_{i=1}^{n} r_i x_i \mid r_i \in R\}$ is a submodule of M. **(5)**

ii) Let M be a free R-module with a basis $\{e_1, e_2, \ldots, e_n\}$. The show that $M \cong R^n$. **(5)**

iii) Show that every finitely generated module is a homomorphic image of a finitely generated free module. **(5)**

Q.4 **a)** Let R be a PID and M a free R-module of rank n. Then prove that every submodule of M is free with a basis having atmost n elements. **(10)**

b) Attempt **any two** of the following: **(10)**

i) If R is a PID, then show that every submodule of the R-module R^n is free of rank atmost n. **(5)**

ii) Let R be a ring and let $M = \langle m \rangle$ be a cyclic R-module. Then prove that $M \cong R/Ann(m)$, where **(5)**

iii) A finitely generated module M over a PID is free if and only if it is torsion free. Prove or disprove. **(5)**
