[Time: 3:00 Hrs.] **[Marks: 80]**

Please check whether you have got the right question paper.

N.B:
1. All questions are compulsory.
2. Figures to the right indicate full marks.
3. Scientific calculator can be used.

Q.1 **a)** Show that continuous image of compact set is compact. **10**

b) Attempt **any Two** of the following: **10**

i) Prove that if $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$. **5**

ii) Prove that if $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then it is continuous at a. **5**

iii) State and prove Heine-Borel theorem. **5**

Q.2 **a)** Define differentiability of a function $f: E \subset \mathbb{R}^n \to \mathbb{R}^m$ at $a \in E$, where E is open set in $\mathbb{R}^n$. Let $f = (f_1, f_2, \dots\dots\dots, f_m): \mathbb{R}^n \to \mathbb{R}^m$, and $p \in \mathbb{R}^n$. Then show that f is differentiable at p if and only if each $f_i, 1 \leq i \leq m$ is differentiable at p. **10**

b) Attempt **any Two** of the following: **10**

i) Give an example of a continuous function which is not differentiable. **5**

ii) Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function such that $|f(x)| \leq |x|^2$. Show that f is differentiable at 0. **5**

iii) Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t) = g(f(t))$ $(a \leq t \leq b)$, then h is differentiable at x, and $h'(x) = g'(f(x))f'(x)$. **5**

Q.3 **a)** State and prove Inverse function theorem. **10**

b) Attempt **any Two** of the following: **10**

i) State and prove contraction mapping theorem. **5**

ii) Find the extreme value of, $f(x; y) = x^2 + y^3 + 3xy^2 - 2x$. **5**

iii) Does $f(x, y) = x^3 + y^3 - 2xy$ can be expressed by an explicit function $y = g(x)$ in a neighborhood of the point $(1,1)$? **5**

Q.4 **a)** $f: A \to \mathbb{R}$ is non-negative and $\int_A f = 0$, where A is rectangle, then show that $\{x \in A, f(x) \neq 0\}$ has measure zero. **10**

b) Attempt **any Two** of the following: **10**

i) A bounded function $f: A \to \mathbb{R}$ is integrable if and only if for every $\epsilon > 0$ there is a partition P of A such that $U(f, P) - L(f, P) < \epsilon$. **5**

ii) Let $f, g: A \to \mathbb{R}$ be integrable and suppose $f \leq g$. Show that $\int_A f \leq \int_A g$. **5**

iii) Let $f: [0, 1] \times [0, 1] \to \mathbb{R}$ be defined by **5**

$$f(x, y) = 0 \qquad if\ 0 \leq x < \frac{1}{2},$$
$$= 1 \qquad if\ \frac{1}{2} \leq x \leq 1.$$

Show that f is integrable and $\int_{[0,1]\times[0,1]} f = \frac{1}{2}$.

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