[Time: 3:00 Hrs.] **[Marks: 80]**

Please check whether you have got the right question paper.

N.B:
1. All questions are compulsory.
2. Figures to the right indicate full marks.
3. Scientific calculator can be used.

Q.1 A) Let V be a finite-dimensional vector space over the field F. Define for each $\alpha \in V, L_{\alpha(f)} = f(\alpha)$ for $f \in V^*$. Prove that the mapping $\alpha \to L_\alpha$ is and isomorphism of V onto V^{**}. **10**

B) Attempt **any Two** of the following: **10**

i) Does solutions set of a non-homogeneous system of linear equations closed under addition and scalar multiplication? Justify your answer. **5**

ii) Let V be a vector space over the field F. Show that the intersection of any collection of subspaces of V is a subspace of V. **5**

iii) Show that the vectors $x_1 = (1,1,0)$ and $x_2 = (1, i, 1+i)$ are in the subspace W of $\mathbb{C}^3$ spanned by $(1,0,i)$ and $(1+i, 1, -1)$, and that x_1 and x_2 form a basis of W. **5**

Q.2 A) Let $A = \begin{pmatrix} 4 & 4 & 4 \\ -2 & -3 & -6 \\ 1 & 3 & 6 \end{pmatrix}$ compute, **10**

a) Characteristic Polynomial b) Eigenvalues c) Eigenvectors

d) Algebraic and geometric multiplicity.

B) Attempt **any Two** of the following: **10**

i) If α be an eigenvalue of a linear transformation T and $f(x)$ is a polynomial in indeterminate x then show that $f(T)v = f(\alpha)v$. **5**

ii) Let A be a square matrix of real numbers whose columns are non-zero orthogonal vectors. Then show that $A^t A$ is a diagonal matrix. **5**

iii) State and prove the Gram-Schmidt orthogonalization process. **5**

Q.3 A) Let T be an invariant operator on $\mathbb{R}^2$, the matrix of which in standard ordered basis is $A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$. Prove that the only subspaces of $\mathbb{R}^2$ invariant under T are $\mathbb{R}^1$ and the zero subspace. **10**

B) Attempt **any one** of the following: **10**

i) State and prove the Cayley-Hamilton theorem. **10**

ii) If $x_1, x_2, \dots\dots, x_n$ are eigenvector of $A = (a_{ij})_{n\times n}$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \dots\dots, \lambda_n$ then prove that $x_1, x_2, \dots\dots, x_n$ are linearly independent. **10**

Q.4 **A)** Define a bilinear form. Prove that a bilinear form is reflexive if and only if it is either symmetric or alternating. **10**

B) Attempt **any Two** of the following: **10**

i) Let A be an $n \times n$ matrix in F and define the bilinear form $< v, w >= X^t AY$ where X and Y are co-ordinates of v and w in some basis of V. Show that the bilinear form is symmetric if and only if matrix A is symmetric. **5**

ii) For the matrix $A = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 7 & -5 \\ 2 & -5 & 8 \end{pmatrix}$ find non-singular matrix P such that $P^t AP$ is diagonal and also find its signature. **5**

iii) If f is a bilinear form on the n-dimensional vector space V, then prove the following are equivalent **5**

a) $\text{rank}(f) = n$

b) For each non-zero α in V, there is a β in V such that $f(\alpha, \beta) \neq 0$

c) For each non-zero β in V, there is a α in V such that $f(\alpha, \beta) \neq 0$

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