

Please check whether you have got the right question paper.

- N.B:**
1. All questions are compulsory.
  2. Figures to the right indicate full marks.
  3. Scientific calculator can be used.

- Q.1** a) Let  $C$  be a collection of closed rectangles of  $\mathbb{R}^n$ , For  $R \in C$ , let  $\mathcal{V}(R)$  denote the volume of  $R$ . If for  $A \subset \mathbb{R}^n, A \neq \emptyset, \mu^*$  is defined by
- $$\mu^*(A) = \inf\{\sum_{k=1}^{\infty} \mathcal{V}(C_k) : C_k \in C \text{ and } A \subset \bigcup_{k=1}^{\infty} C_k\}$$
- then show that  $\mu^*$  is an exterior measure on  $\mathbb{R}^n$ . **10**
- b) Attempt **any Two** of the following: **10**
- i) Construct a non-measurable subset of  $\mathbb{R}$ . **5**
  - ii) Show that Lebesgue outer measure of a Cantor set is zero. **5**
  - iii) Show that open subsets of  $\mathbb{R}^d$  is measurable. **5**
- Q.2** a) State and prove Egoroff's theorem. **10**
- b) Attempt **any Two** of the following: **10**
- i) Show that  $\chi_A$  is Measurable if and only if the set  $A$  is measurable. **5**
  - ii) Let  $f, g: \mathbb{R}^d \rightarrow \mathbb{R}$  are Lebesgue integrable function then prove that  $f + g$  is Lebesgue integrable and  $\int_{\mathbb{R}^d} (f + g) \, dm = \int_{\mathbb{R}^d} f \, dm + \int_{\mathbb{R}^d} g \, dm$ . **5**
  - iii) Show that Lebesgue outer measure is invariant under translation. **5**
- Q.3** a) Define norm on the vector space  $L^1(\mathbb{R})$  of Lebesgue integrable functions. **10**
- Show that  $L^1(\mathbb{R})$  is complete under the suitable norm.

b) Attempt **any Two** of the following: 10

i) State and prove Monotone convergence Theorem. 5

ii) Use the dominated, convergence theorem to prove that  $\lim_{n \rightarrow \infty} n \int_0^1 \sqrt{n} e^{n^2 x^2} dx = 0$ . 5

iii) If  $f: \mathbb{R} \rightarrow [-\infty, \infty]$  is Lebesgue measurable, and  $\int_{\mathbb{R}} |f| dm = 0$  then prove that  $f(x) = 0$  for almost everywhere. 5

**Q.4** a) State and prove the Riesz-Fischer theorem. 10

b) Attempt **any Two** of the following: 10

i) State and prove Chebychev's Inequality for non-negative measurable function. 5

ii) Let  $E$  be a measurable set such that  $0 < \nu(E) < \infty$ . Then prove that there exists a positive set  $A \subset E$  such that  $\nu(A) > 0$ . 5

iii) If  $f \in L^1(\mathbb{R}^n)$ , then show that  $|\int f| \leq \int |f|$ . 5

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