M.Sc.(Maths) (Sem-IV) July-2023

Mathematics : Calculus on Manifolds (R-2021)

[Duration: 3 Hours] [Marks: 80]

N.B. 1) All questions are compulsory and carry equal marks.
2) Figures to the right indicates full marks.
3) Use of scientific non programmable calculator is allowed.
4) Standard notations have their usual meaning.

1. (a) If $\omega \in \Lambda^k(V),\ \eta \in \Lambda^l(V)$ and $\theta \in \Lambda^m(V)$ then show that $\mathrm{Alt}(\mathrm{Alt}(\omega\otimes\eta)\otimes\theta) = \mathrm{Alt}(\omega\otimes\eta\otimes\theta)$. (10)

(b) Attempt any two of the following
(i) Let $S \in \Lambda^k(V)$ and $T \in \Lambda^l(V)$ and $\mathrm{Alt}(T) = 0$ then compute $T \wedge S$. (5)
(ii) If $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$ then show that $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$. (5)
(iii) Let $\omega \in \Lambda^1(V),\ \eta \in \Lambda^2(V)$ and $\theta \in \Lambda^3(V)$. Find the wedge product $(\omega \wedge \eta) \wedge \theta$ in terms of alternating tensor of tensor product of $\omega,\ \eta$ and θ. (5)

2. (a) Define closed and exact forms. Show that every exact form open set A is closed. State and prove the condition on open set A so that every closed form is exact. (10)

(b) Attempt any two of the following
(i) In $\mathbb{R}^2$, let $\omega = uv^3 du \wedge dv$ and $\alpha : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given by $\alpha(x, y, z) = (x^2 + yz, e^{xyz})$. Calculate $\alpha^*\omega$. (5)
(ii) Calculate exterior derivatives of the $2-$ forms $z^2 dx \wedge dy + (z^2 + 2y)dx \wedge dz$ in $\mathbb{R}^3$. (5)
(iii) If ω is a $k-$form on $\mathbb{R}^m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable show that $f^*(d\omega) = d(f^*\omega)$. (5)

3. (a) State coordinate conditions and show that a subset M of $\mathbb{R}^n$ is a $k-$dimensional manifold if and only if for each point $x \in M$ satisfies coordinate condition. (10)

(b) Attempt any two of the following
(i) Is the $n-$Sphere S^n defined by $\{x \in \mathbb{R}^{n+1} : |x| = 1\}$ a $n-$dimensional manifold? Justify your answer. (5)
(ii) Let $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ be given by $\gamma(t) = (\sin 2t)(|\cos t|, \sin t)$ for $0 < t < \pi$. Let M be image set of γ. Is M $1-$manifold without boundary in $\mathbb{R}^3$? Justify your answer. (5)
(iii) The parametric equation of Möbius band is given by (5)

$$\sigma(t,\theta) = ((1 - t\sin\frac{\theta}{2})\cos\theta, (1 - t\sin\frac{\theta}{2})\sin\theta, t\cos\frac{\theta}{2}),\ \ \frac{-1}{2} < t < \frac{1}{2},\ \ 0 < \theta < 2\pi.$$

Prove or disprove: The Möbius strip is a orientable manifold.

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4. (a) If M is a compact oriented $k-$dimensional manifold with boundary and ω is a $(k-1)-$form on M then show that (10)

$$\int_M d\omega = \int_{\partial M} \omega.$$

(b) Attempt any two of the following

(i) Let M be an oriented two-dimensional manifold with boundary in R^3 and let n be the unit outward normal then show that $n^1 dA = dy \wedge dz$. (5)

(ii) Consider vector field $\vec{F} = (y+z)i + (z+x)j + (x+y)k$. Is vector field $\vec{F}$ solenoidal and irrotational? Justify your answer. (5)

(iii) Evaluate $\iint_S \vec{A} \cdot \hat{n}\ ds$ where $\vec{A} = 18z\hat{i}$ - $12\hat{j}$ + $3y\hat{k}$ and S is the part of the plane $2x + 3y + 6z = 12$ included in the first octant. (5)

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