M.Sc (Maths)(Sem-IV) July-2023

Mathematics : Algebra IV (R-2021)

[Time:3:00 Hrs.] **[Marks: 80]**

Please check whether you have got the right question paper.

N.B:
1. **All questions are compulsory.**
2. **Figures to the right indicate full marks.**
3. **Scientific calculator can be used.**

Q.1 a) Let L/F and F/K are field extensions. Prove that $[L:K]$ is finite if and only if $[L:F]$ and $[F:K]$ are finite **10**

b) Attempt **any Two** of the following: **10**

i) Prove that if F is a field of characteristic 0 and a, b are algebraic over F then there is an element c in $F(a,b)$ such that $F(a,b) = F(c)$ **5**

ii) Show that the characteristic of a field is either zero or a prime integer. **5**

iii) Let E be an extension field of the field F. Show that the set of all elements of E that are algebraic over F is a subfield of E. **5**

Q.2 a) Define a splitting field of a polynomial $f(x)$ over a field K. If $f(x)$ is a monic polynomial over a field K, prove that there exists a splitting field of $f(x)$ over K. **10**

b) Attempt **any Two** of the following: **10**

i) Determine the splitting field and its degree over $\mathbb{Q}$ for the polynomial $x^{11} - 1$. **5**

ii) Let F be a field of characteristic $p > 0$. If K is a finite extension of F such that $[K:F]$ is relatively prime to p, then show that K is separable over F. **5**

iii) Find the minimal polynomial of $\sqrt{3} + \sqrt{5}$ over Q. Also find $[Q(\sqrt{3} + \sqrt{5}):Q]$ **5**

Q.3	**a)**	Let K/F be a Galois extension and let $G = G(K/F)$. Show that there is a bijection between the set of subfields E of K containing F and set of subgroups H of G.	**10**
	b)	Attempt **any Two** of the following:	**10**
	i)	Find the fixed fields of $\mathrm{Aut}(\mathbb{Q}(\sqrt{2}))$ and $\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2}))$	**5**
	ii)	Let E be the splitting field of the polynomial $f(x) \in F[x]$ over F. Then show that $\lvert\mathrm{Aut}(E/F)\rvert \leq [E:F]$.Equality hold if $f(x)$ is separable over F.	**5**
	iii)	$Prove\ that$ $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not a Galois extension, but $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ and $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ are Galois extensions.	**5**
Q.4	**a)**	Let F be a field of characteristic 0 , and let $a \in F$. If K is the splitting field of $x^n - a$ over F, then K/F is a Galois extension and $G(K/F)$ is a solvable group.	**10**
	b)	Attempt **any Two** of the following:	**10**
	i)	Find $\mathrm{Aut}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$	**5**
	ii)	Let F be a field of characteristic zero. Let $f(x) \in F[x]$ be an irreducible cubic degree polynomial and K be its splitting field. Then $G(K/F)$ is either isomorphic to A_3 or S_3.	**5**
	iii)	Let $\zeta \in \mathbb{C}$ and let $n \geq 1$. Then prove that ζ is an n^{th} root of unity if and only if $o(\zeta) \mid n$, where $o(\zeta)$ is the order of ζ in $\mathbb{C}^*$.	**5**

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