M.Sc.(Maths) (Sem-II) July-2023

Mathematics : Analysis II (Rev 2021)

[Time: 3:00 Hrs.] **[Marks: 80]**

Please check whether you have got the right question paper.

N.B:
1. **All questions are compulsory.**
2. **Figures to the right indicate full marks.**
3. **Scientific calculator can be used.**

Q.1 a) Let C be a collection of closed rectangles of $\mathbb{R}^n$, For $R \in C$, let $\vartheta(R)$ denote the volume of R. If for $A \subset \mathbb{R}^n, A \neq \emptyset, \mu^*$ is defined by **10**

$$\mu^*(A) = \inf\{\sum_{n=1}^{\infty} \vartheta(C_k) : C_k \in C \text{ and } A \subset \cup_{n=1}^{\infty} C_k\}$$

then show that μ^* is an exterior measure on $\mathbb{R}^n$.

b) Attempt **any Two** of the following: **10**

i) Construct a non-measurable subset of $\mathbb{R}$. **5**

ii) Show that Lebesgue outer measure of a Cantor set is zero. **5**

iii) Show that open subsets of $\mathbb{R}^d$ is measurable. **5**

Q.2 a) State and prove Egoroff's theorem. **10**

b) Attempt **any Two** of the following: **10**

i) Show that χ_A is Measurable if and only if the set A is measurable. **5**

ii) Let $f, g: \mathbb{R}^d \to \mathbb{R}$ are Lebesgue integrable function then prove that $f + g$ is Lebesgue integrable and $\int_{\mathbb{R}^d} (f + g)\, dm = \int_{\mathbb{R}^d} f\, dm + \int_{\mathbb{R}^d} g\, dm$. **5**

iii) Show that Lebesgue outer measure is invariant under translation. **5**

Q.3 a) Define norm on the vector space $L^1(\mathbb{R})$ of Lebesgue integrable functions. Show that $L^1(\mathbb{R})$ is complete under the suitable norm. **10**

b) Attempt **any Two** of the following: **10**

i) State and prove Monotone convergence Theorem. **5**

ii) Use the dominated, convergence theorem to prove that $\lim_{n\to\infty} n\int_0^1 \sqrt{n}\, e^{n^2x^2}\, dx = 0$. **5**

iii) If $f: \mathbb{R} \to [-\infty, \infty]$ is Lebesgue measurable, and $\int_{\mathbb{R}} |f| dm = 0$ then prove that $f(x) = 0$ for almost everywhere. **5**

Q.4 a) State and prove the Riesz-Fischer theorem. **10**

b) Attempt **any Two** of the following: **10**

i) State and prove Chebychev's Inequality for non-negative measurable function. **5**

ii) Let E be a measurable set such that $0 < \nu(E) < \infty$. Then prove that there exists a positive set $A \subset E$ such that $\nu(A) > 0$. **5**

iii) If $f \in L^1(\mathbb{R}^n)$, then show that $|\int f| \leq \int |f|$. **5**

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