M.Sc.(Maths) (Part I) Oct-2023

Mathematics : Analysis & Topology (R-2016)

Duration: 3 hrs **Marks: 80**

N.B. 1) Both the sections are compulsory.
2) Attempt **ANY TWO** questions from each section.

Section A

1. (a) Let $f: (\mathrm{X}, \mathrm{d}_1) \to (\mathrm{X}, \mathrm{d}_2)$ be the function then show that the following statements are equivalent **(10)**
i) f is continuous on X.
ii) Inverse image of open set in Y is open set in X.
iii) Inverse image of closed set in Y is closed set in X.

(b) (i) Prove that if $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x < y$, then there exists a $p \in \mathbb{Q}$ such that **(5)**
$x < p < y$.

(ii) Show that if $A \subseteq B$, then $diam(A) \leq diam(B)$. **(5)**

2. (a) Show that continuous image of compact set is compact. **(10)**

(b) (i) Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Show that f carries bounded sets to bounded sets. **(5)**

(ii) Prove that if $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then it is continuous at a. **(5)**

3. (a) If $f: \mathbb{R}^n \to \mathbb{R}^m$, then $Df(a)$ exists if all $D_j f_i(x)$ exists in an open set containing a and if each function $D_j f_i$ is continuous at a. **(10)**

(b) (i) Give an example of a continuous function which is not differentiable. **(5)**

(ii) Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function such that $|f(x)| \leq |x|^3$. Show that f is differentiable twice at 0. **(5)**

4. (a) State and prove Inverse function theorem. **(10)**

(b) (i) Define Lagrange's method of undetermined multipliers. **(5)**

(ii) Find the Taylor expansion of function, **(5)**
$f(x;\ y) = sin(2x + 3y)$ at $(a; b) = (0; 0)$.

Section B

5. (a) Let $f:(X;\tau_1) \to (Y;\tau')$ be any map. Show that the following conditions are equivalent: **(10)**
i) f is continuous on X.
ii) If $H \in \tau'$, then $f^{-1}(H) \in \tau$.
iii) If C is a closed subset of $(Y;\tau')$, then $f^{-1}(C)$ is a closed subset of $(X;\tau)$.
iv) For any subset A of X, $f(c(A))$ is a subset of $c(f(A))$, where $c(A)$ denotes the closure of A and $c(f(A))$ denotes the closure of $f(A)$.

(b) (i) Define Hausdorff space. Further show that a subspace of a Hausdorff space is Hausdorff. **(5)**

(ii) Define Topology on a set. Let X be a non-empty set and τ be the collection of all subsets U of X such that U^c is either finite or is all of X. Show that τ is topology on X. **(5)**

6. (a) Let $(X;\tau)$ be a topological space. When is X said to be separable? Show that a topological space being separable is a topological property. Is being separable a hereditary property? Justify your answer. **(10)**

(b) (i) Define a path connected topological space. Show that the punctured Euclidean plane $\mathbb{R}^2 - \{(0,0)\}$ and the unit sphere in $\mathbb{R}^2$ are path connected. **(5)**

(ii) Let $A \subsetneq X$ and $B \subsetneq Y$. If X, Y are connected topological spaces then show that so is $(X \times Y) - (A \times B)$. **(5)**

7. (a) Show that closed subsets of a compact space is compact. Also prove that compact subset of a Hausdorff space is closed. **(10)**

(b) (i) Show that in the finite complement topology on $\mathbb{R}$, every subspace is compact. **(5)**

(ii) Define *locally compact* topological space. Further show that the space $\mathbb{R}^n$ is locally compact. **(5)**

8. (a) Define limit point compact and sequentially compact space. Show that if a topological space X is limit point compact then X is sequentially compact. **(10)**

(b) (i) Suppose that the space X has a countable basis. Then show that every open covering of X contains a countable sub collection covering X. **(5)**

(ii) Let X be a topological space such that one-point sets are closed in X. Show that X is regular if and only if given a point x of X and a neighbourhood U of x, there is a neighbourhood V of x such that $\bar{V} \subset U$. **(5)**

****