M.Sc. Maths (Sem-II) March-2023

Mathematics : Topology (Rev 2021)

Time: 3 Hours] **[Max. Marks: 80**

Note: 1) All Questions are compulsory

2) Figures to the right indicate full marks

Q.1 (a) Attempt the following **(10)**

(i) Define Topology, Basis for a Topology.
Show that the collection of all rectangular regions in a plane is a basis for a topology.

(b) Attempt any two **(10)**

(i) Define Closed set, Closure of a set.
If A is a closed set, then show that $A = \bar{A}$, where $\bar{A}$ is closure of A.

(ii) Let X be a set. Let $\mathcal{T}_f$ be the collection of all subsets U of X such that $X - U$ is either finite or all of X. Show that $\mathcal{T}_f$ is a topology of X. Which of the subsets $\mathbb{R} - (0, 1), \mathbb{R} - \{0, 1\}$ is open in $\mathbb{R}_f$?

(iii) Define homeomorphism between two topological spaces.
Let $f: X \to Y$ be bijective and continuous. Is it necessary that f should be a homeomorphism? Justify.

Q.2 (a) Attempt the following **(10)**

(i) Define Connected Space.
Let A be connected subspace of X. If $A \subset B \subset \bar{A}$, then show that B is a connected subspace of X.

(b) Attempt any two **(10)**

(i) Prove that the image of a connected space under a continuous map is connected.

(ii) Define topologist's sine curve. Show that it is connected but not path-connected.

(iii) Define a relation ~ on a non-empty set X, by $x \sim y$ if there is a connected subspace containing both x and y. Show that ~ is an equivalence relation. What are the equivalence classes of this relation called?

Q.3 (a) Attempt the following **(10)**

(i) Define compact space. Prove that compact subspace of a Hausdorff space is closed. Give an example to show that compact subsets of a topological space need not be closed in general.

(b) Attempt any two **(10)**

(i) Prove that compactness is a topological property. Show that [0, 1] and (0, 1) are not homeomorphic.

(ii) Let $\mathbb{R}$ be the set of real numbers with usual topology. Is $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ compact subspace of $\mathbb{R}$? Justify your answer.

(iii) Define locally compact space. Show that image of a locally compact space under a continuous map need not be locally compact.

Q.4 (a) Attempt the following **(10)**

(i) Define normal space. Prove that

- Every compact Hausdorff space is normal.
- Every metrizable space is normal.

(b) Attempt any two **(10)**

(i) State Urysohn Lemma. Can there exist a continuous function $f: R^2 \to [0,1]$ such that $f(x,y) = 0$, for $0 \le x, y \le 1$ and $f(x,y) = 1$, for $x = 5$, $0 \le y \le 1$. Justify your answer.

(ii) Define regular space. Show that regularity is a topological property.

(iii) Let X be a topological space. Let one-point sets be closed in X and X be normal. Prove that, given a closed set A and an open set U containing A, there exists an open set V containing A such that $\bar{V} \subset U$.