M.Sc. MATHEMATICS

SEMESTER - II (CBCS)

ANALYSIS-II

SUBJECT CODE : PSMT/ PAMT203

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CONTENTS

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PSMT 203/PAMT 203: ANALYSIS II

Course Outcome:

1. In this course students are expected to understand the basic concepts of measure on an arbitrary measure space X as well as on $\mathbb{R}^n$.

2. They are also expected to study Lebesgue outer measure of sets and measurable sets, measurable functions.

3. Students will be able to understand the concepts of integrals of measurable functions in an arbitrary measure space $(X, \mathcal{A}, \mu)$. Lebesgue integration of complex valued functions and basic concepts of signed measures.

Unit-I: Measures and Measurable Sets (15 Lectures)
Additive set functions, σ-algebra countable additivity, Outer measure, constructing measures, μ^* measurable sets (Definitions due to Carathéodory), μ^* measurable subsets of X forms a σ algebra, measure space $(X, \sum, \mu)$. Lebesgue outer measure in $\mathbb{R}^d$, properties of exterior measure, monotonicity property and countable sub-additivity property of Lebesgue measure, translation invariance of exterior measure, example of set of measure zero. Measurable sets and Lebesgue measure, properties of measurable sets. Existence of a subset of $\mathbb{R}$ which is not Lebesgue measurable.
[Reference for unit I: 1. Andrew Browder, Mathematical Analysis, An Introduction, Springer Undergraduate Texts in Mathematics.
2. Elias M. Stein and Rami Shakarchi, Real Analysis, Measure Theory, Integration and Hilbert Spaces, New Age International Limited, India]

Unit-II: Measurable functions and their Integration (15 Lectures)
Measurable functions on $(X, \sum, \mu)$, simple functions, properties of measurable functions. If $f \geq 0$ is a measurable function, then there exists a monotone increasing sequence (s_n) of non-negative simple measurable functions converging to point wise to the function f. Egorov's theorem, Lusin's theorem. Integral of nonnegative simple measurable functions defined on the measure space $(X, \sum, \mu)$ and their properties. Integral of a non-negative measurable function.
[Reference for unit II: 1. Andrew Browder, Mathematical Analysis, An Introduction, Springer Undergraduate Texts in Mathematics.
2. Elias M. Stein and Rami Shakarchi, Real Analysis, Measure Theory, Integration and Hilbert Spaces, New Age International Limited, India]

Unit-III: Convergence Theorems on Measure space (15 Lectures)
Monotone convergence theorem. If $f \geq 0$ and $g \geq 0$ are measurable functions, then $\int (f + g) d\mu = \int f d\mu + \int g d\mu$, Fatou's lemma, summable functions, vector space of summable functions, Lebesgue's dominated convergence theorem. Lebesgue integral of bounded functions over a set of finite measure, Bounded convergence theorem. Lebesgue and

Riemann integrals: A bounded real valued function on [a, b] is Riemann integrable if and only if it is continuous at almost every point of [a, b]; in this case, its Riemann integral and Lebesgue integral coincide.
[Reference for unit III: 1. Andrew Browder, Mathematical Analysis, An Introduction, Springer Undergraduate Texts in Mathematics.
2. Royden H. L. Real Analysis, PHI]

Unit-IV: Space of Integrable functions (15 Lectures)
Borel set, Borel algebra of $\mathbb{R}^d$. Any closed subset and any open subset of $\mathbb{R}^d$ is Lebesgue measurable. Every Borel set in $\mathbb{R}^d$ is Lebesgue measurable. For any bounded Lebesgue measurable subset E of $\mathbb{R}^d$, given any $\epsilon > 0$ there exist a compact set K and open set U in $\mathbb{R}^d$ such that $K \subseteq E \subseteq U$ and $m(U \setminus K) < \epsilon$. For any Lebesgue measurable subset E of $\mathbb{R}^d$, there exist Borel sets F, G in $\mathbb{R}^d$ such that $F \subseteq E \subseteq G$ and $m(E \setminus F) = 0 = m(G \setminus E)$. Signed Measures, positive set, negative set and null set. Hahn decomposition theorem. Complex valued Lebesgue measurable functions on $\mathbb{R}^d$. Lebesgue integral of complex valued measurable functions, Approximation of Lebesgue integrable functions by continuous functions. The space $L^1(\mu)$ of integrable functions, properties of L^1 integrable functions, Riesz-Fischer theorem.
[Reference for unit IV: 1. Elias M. Stein and Rami Shakarchi, Real Analysis, Measure Theory, Integration and Hilbert Spaces, New Age International Limited, India
2. Royden H. L. Real Analysis, PHI
3. Andrew Browder, Mathematical Analysis, An Introduction, Springer Undergraduate Texts in Mathematics.]

Recommended Text Books

1. Andrew Browder, Mathematical Analysis, An Introduction, Springer Undergraduate Texts in Mathematics.

2. Elias M. Stein and Rami Shakarchi, Real Analysis, Measure Theory, Integration and Hilbert Spaces, New Age International Limited, India

3. Royden H. L. Real Analysis, PHI.

4. Terence Tao, Analysis II, Hindustan Book Agency (Second Edition).

1

LEBESGUE OUTER MEASURE

Unit Structure :

1.0 OBJECTIVE

After going through this chapter you can able to know that

- Concept of σ – Algebra, Measurable set.
- Extension measure in $\mathbb{R}^n$
- Lebesgue measureable set
- Lebesgue outer measure & its properties.

1.1 INTRODUCTION

In this chapter we shall fist study such a verified theory function d-dimensional value based on the notation of a measure, and then we shall use this theory to build a stronger and more flexible theory.

Now if we want to partition the range of a function, we need same way of measuring how much of the domain is sent to a particular region of the partition, To set a feeling function what we are aiming function let us assume that we want to measure the volume of subsets $A, C\mathbb{R}^3$ and that are denote the volume of A by $\mu(A)$.

Then function we have

i) $\mu(A)$ should be non-negative number as ∞.

ii) $\mu(\varnothing)=0$ it will be convenient to assign a volume to the empty set.

iii) If $A_1, A_2, \ldots.., A_n$ are non overlapping disjoint sets then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right)=\sum_{n=1}^{\infty} \mu(A_n)$$

This means that the volume the whole is equal to the sum of the volume of the parts. This problems leads us to the theory of measures where we try to give a notation of measure to subsets of an Euclidean space.

Defenition :

The Euclidean norm on $\mathbb{R}^n$ is $|x|=\left(x_1^2+\ldots.+x_d^2\right)^{1/2}$.

The distance between $x, y \Rightarrow \mathbb{R}^n$ is $|x-y|$

1.2 $\sigma-$ALGEBRA

Definition :

Let X be a set. A collection A of subsets of X is called a $\sigma-$algebra of the following hold.

i) $\varnothing \in A$

ii) $A \in A \Rightarrow X/A \in A$

iii) $A_1, A_2, \ldots., \in A \Rightarrow \bigcup_{i=1}^{\infty} A_i \in A$

Note :

The pair (X, A) is called measurable space and elements of A are called measurable sets.

Example 1 :

Let $X=\{1,2,3\}$ and $b_1=\{\{1\},\{1,2,3\}, X, \varnothing\}$, $b_2=\{1,2,3,\{3\}, X, \varnothing\}$. Check whether b_1 and b_2 are both algebras or not.

Solution :

I) Let $X=\{1,2,3\}$ and b_1 is not $\sigma-$Algebra.

Since it does not contain $\{1\}^C$.

II) b_2 is $\sigma-$Algebra since it satisfies all condition of $\sigma-$Algebra

i.e. $X=b_1$

$\varnothing=b_2$

$\{1,2\}\in b_2$ & $\{1,2\}^C\in b_2$

$\therefore b_2$ is $\sigma-$Algebra.

Example 2 :

A measure on a topological space X whose domain is the Borel algebra is called a Borel measure.

Example : For every $x\in X$, the Dirac measure is given by $\delta_x(A)=\begin{cases}1\,if\ x\in A\\0\,if\ x\notin A\end{cases}$

Definition :

Let μ be a set function whose domain in a class A of subsets of a set X and whose values are non-negative extended reals, we say that μ is contably additive if $\mu\left(\bigcup_{k=1}^{\infty}A_k\right)=\sum_{k=1}^{\infty}\mu(A_k)$ whenever, (A_k) is a sequence of painoise disjoint set in A whose union is also in A.

Theorem :

Let μ be a finitely additive set function, defined on the $\sigma-$Algebra A. Then μ is countably additive iff it has the following property : if $A_n\in A$ and $A_n\subset A_{n+1}$ Anti for each positive integer n, and if $\bigcup_{n=1}^{\infty}A_n\in A$ then $\mu\left(\bigcup_{n=1}^{\infty}A_n\right)=\lim_{n\to\infty}\mu(A_n)$.

Proof :

Suppose μ is countable additive Let $\{A_n\}$ be a sequence of elements in A s.t. $A_1\subseteq A_2\subseteq,......,A=\bigcup_{i=1}^{\infty}A_i\in A$

s.t. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu(A_n)$

Define $B_1 = A_1$

$B_K = A_K / A_{K-1}$ for $K \geq 2$

Examples 3:

Let $\{A_i; i \in I\}$ be collection of $\sigma-$Algebra. Show tha $\bigcap_{i\in I} A_i$ is a $\sigma-$Algebra, but $\bigcup_{i\in I} A_i$ is not in general.

Solution :

Let $Ai = \bigcap_{i\in I} A_i$

To show that A is a $\sigma-$Algebra

a) If $\varnothing \in A$

$\because A_i$ is $\sigma-$Algebra, $\forall i \in I$

$\therefore \varnothing \in A_i \ \forall i \in I$

$\Rightarrow \varnothing \in \bigcap_{i\in I} A_i \Rightarrow \varnothing \in A$

b) Let $A \in \boldsymbol{A}$

$\Rightarrow A = \bigcap_{i\in I} A_i$

$\because A_i$ is $\sigma-$Algebra $\forall i \in I$

$\therefore$ For $A \in A_i \Rightarrow A^C \in A_i \ \forall i \in I$

$\therefore A^c \in \bigcap_{i\in I} A_i$

$\Rightarrow A^c \in A$

c) Let $A_k \in A, \forall k = 1, 2....$

then $A_k \in \bigcap Ai \quad \forall i \in i$

$$\Rightarrow \bigcup_{k=1}^{\infty} A_k \in A_i \quad \forall i$$

$$\Rightarrow \bigcup_{k=1}^{\infty} A_k \in \bigcap_{i \in I} A_i$$

$$\Rightarrow \bigcup_{k=1}^{\infty} A_k \in A$$

$A = \bigcap_{i \in I} A_i$ is a $\sigma -$Algebra

Now, we have to show that $\bigcup A_i$ is not a $\sigma -$Algebra.

Let $X = \{1,2,3\}$

Let $A_1 = \{\phi, X, \{1\}, \{2,3\}\}$

$A_2 = \{\phi, X, \{3\}, \{1,2\}\}$

then A_1 & A_2 are $\sigma -$Algebra but $A_1 \cup A_2$ is not $\sigma -$Algebra. $\{1\} \in A_1 \cup A_2$ but $\{1,3\} \notin A_1 \cup A_2$.

Clearly $Bi \in A \; \forall i$ and $Bi's$ are pairwise disjoint we first show that $A_k = \bigcup_{i=1}^{k} Bi$

By induction on 'k'

The result is trivial when $k = 1$

Assume the result is true for $k - 1$

i.e. $A_{k-1} = \bigcup_{i=1}^{k-1} Bi$

Now $\bigcup_{i=1}^{k} Bi = \bigcup_{i=1}^{k-1} Bi \bigcup B_k$

$$= A_{k-1} \bigcup (A_k / A_{k-1})$$
$$= A_k$$

$\therefore$ The result is true for k.

$\therefore$ by introduction is true for all k

$$A_k = \bigcup_{i=1}^{k} Bi \quad \forall\, k \geq 1$$

Note that $A=\bigcup_{k=1}^{\infty}A_k=\bigcup_{k=1}^{\infty}\left(\bigcup_{i=1}^{k}Bi\right)$

$$=\bigcup_{k=1}^{\infty}B_k$$

$\because \mu$ is countably additive, we have

$$\mu(A)=\mu\left(\bigcup_{k=1}^{\infty}B_k\right)=\sum_{K=1}^{\infty}\mu(B_k)$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\mu(B_k)$$

$$=\lim_{n\to\infty}\left(\mu\left(\bigcup_{k=1}^{n}B_k\right)\right)$$

$$=\lim_{n\to\infty}\mu(A_n)$$

Conversely,

Suppose whenever if $A_1\subset A_2\subset A_3....,Ai\in B_r\boldsymbol{A},\bigcup Ai\in\boldsymbol{A}$

Then $\mu\left(\bigcup_{i=1}^{\infty}Ai\right)=\lim_{n\to\infty}\mu(A_n)$

T.S.T. μ is countably additive

Let (A_n) be a pairwise disjoint sets in A.

Define $B_k=\bigcup_{i=1}^{k}Ai$ then $B_k\in\boldsymbol{A}$ and $B_1\subseteq B_2\subseteq.......$

$\therefore$By hypothesis, we have

$$\mu\left(\bigcup_{i=1}^{\infty}Bi\right)=\lim_{n\to\infty}\mu(B_n)$$

But $\bigcup_{i=1}^{\infty}Bi=\bigcup_{i=1}^{\infty}\left(\bigcup_{K=1}^{i}A_k\right)$

$$=\bigcup_{i=1}^{\infty}Ai$$

$$\therefore \mu\left(\bigcup_{i=1}^{\infty} Ai\right) = \mu\left(\bigcup_{i=1}^{\infty} Bi\right)$$

$$= \lim_{n\to\infty} \mu(B_n)$$

$$= \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} Ai\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(Ai)$$

$$= \sum_{i=1}^{\infty} \mu(Ai)$$

Theorem :

Let A be a $\sigma -$Algebra, If (u, v) are measures on A, $t \in \mathbb{R}, t > 0$ and $A \in \boldsymbol{A}$ hen the following are measures on A.

a) $\mu + \vartheta$ defined by $(\mu+\vartheta)(E) = \mu(E) + \vartheta(E) E \in A$

b) $t\mu$, defined by $(t\mu)(\in) = t\mu(E), E \in A$

Proof :

a) $\mu + \vartheta$ defined by $(\mu+\vartheta)(E) = \mu(E) + \vartheta(E), E \in A$ is a measure on A.

$\therefore \mu$ & ϑ are measure on A.

$\therefore$They are countably additive non-negative set function.

$\therefore (\mu+\vartheta)(E)$ is also countably additive non-negative set function whose domain is A.

$\therefore \mu + \vartheta$ is a measure on A.

b) $(t\mu)(E) = t\mu(E)$

$\therefore \mu$ is a measure on A

$\therefore \mu$ is countable additive non negative set function whose domain in A.

$\therefore$for $E \in A$

$(t\mu)(E) = t\mu(E)$ and $t\mu$ is also countably additive non-negative set of function whose domain is A

$\therefore t\mu$ is measure on A.

1.3 EXTENSION MEASURE

Definition :

Let X be a set, A_n Exterior measure or outer measure on X is a non-negative, extended real valued function μ^* whose domain consist of all subsets of X and which satisfies :

a) $\mu^*(\phi)=0$

b) (Monotonicity) if $A\subset B$ then $\mu^*(A)\subseteq\mu^*(B)$

c) (Countable sub-additivity)

For any sequence (A_n) of subsets of X, we have

$$\mu^*\left(\bigcup_{n=1}^{\infty}A_n\right)\leq\sum_{n=1}^{\infty}\mu^*(A_n)$$

Theorem :

Let C be a collection of closed rectangle of $\mathbb{R}^n$, For $R\in C$, let $\vartheta(R)$ denote the volume of R. If μ^* is defined by

$$\mu^*(A)=\inf\left\{\sum_{k=1}^{\infty}\vartheta(C_k);C_k\in C,\bigcup_{k=1}^{\infty}(k)\supset A\right\}$$

For $A\subset\mathbb{R}^n, A\neq\phi$ then μ^* is exterior measure on $\mathbb{R}^n$.

Proof :

T.S.T. μ^* defined by $\mu^*(A)=\inf\left\{\sum_{k=1}^{\infty}\vartheta(C_k);C_k\right\}$ is closed rectangle where $A\subset\mathbb{R}^n$ is on exterior Measure on $\mathbb{R}^n$.

We first shows that

$$\left\{\sum V(C_k);C_k \textit{ is closed set } A\subseteq C_k\right\}\neq\phi$$

Where $A\subset\mathbb{R}^n$

Let $R_k=$ rectangle with side length 'k' and centre origin.

Then $\bigcup_{k=1}^{\infty}R_k=\mathbb{R}^n$

$\therefore$ for any $A \subset \mathbb{R}^n = \bigcup_{k=1}^{\infty} R_k$

$\Rightarrow \{R_k\}$ covers A

$\therefore \left\{ \sum_{k=1}^{\infty} \vartheta(C_k); C_k \text{ closed rectangle } A \subseteq \bigcup_{k=1}^{\infty} C_k \right\} \neq \phi$

We now show $\mu^*(\phi) = 0$

Let $\in > 0$

Let $R = [0, \in^{y_n}] \times \ldots\ldots \times [0, \in^{y_n}]$ be a rectangle in $\mathbb{R}^n$ with $\vartheta(R) = \in \,\&\, \phi \subseteq R$

$\therefore \{R\}$ covers ϕ

$\therefore$ By definition of μ^*, $\mu^*(\phi) < \in$

This is true for any $\in > 0$

$$\mu^*(\phi) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let $A \subseteq B \subseteq \mathbb{R}^n$

T.S.T. $\mu^*(A) \leq \mu^*(B)$

If $\{C_k\}$ Covers B, then $\{C_k\}$ covers A

$\therefore \left\{ \sum_{k=1}^{\infty} \vartheta(C_k) : B \subseteq \bigcup_{k=1}^{\infty} C_k \right\} \subseteq \left\{ \sum \vartheta(C_k) : A \subseteq \bigcup_{k=1}^{\infty} C_k \right\}$

$\Rightarrow \inf \left\{ \sum \vartheta(C_k) : B \subseteq \bigcup_{k=1}^{\infty} C_k \right\} \geq \inf \left\{ \sum \vartheta(C_k) : A \subseteq \bigcup_{k=1}^{\infty} C_k \right\}$

$\Rightarrow \mu^*(A) \leq \mu^*(B) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Let $\{A_n\}$ be a sequence of subsets of $\mathbb{R}^n$ we show that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$$

Let $\in > 0$ by the definition of μ^*

$\exists$ a cover $\{R_{n_i}\}_{i=1}^{\infty}$ of A_n such that

$$\sum_{i=1}^{\infty} \vartheta\left(R_{n_i}\right) < \mu^*\left(A_n\right) + \in/2^n$$

Then $\bigcup_{n=1}^{\infty}\left(\bigcup_{j=1}^{\infty} R_{n_i}\right)$ covers $\bigcup_{n=1}^{\infty} A_n$

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} \vartheta\left(R_n\right)\right)$$

$$\leq \sum_{n=1}^{\infty}\left(\mu^*\left(A_n\right) + \in\!\!\big/_{2^n}\right)$$

$$\leq \sum_{n=1}^{\infty} \mu^*\left(A_n\right) + \sum_{n=1}^{\infty} \in\!\!\big/_{2^n}$$

$$\leq \sum_{n=1}^{\infty} \mu^*\left(A_n\right) + \in$$

$$\leq \sum_{n=1}^{\infty} \mu^*\left(A_n\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

From (1) (2) & (3)

μ^* is an exterior measure on $\mathbb{R}^n$

Note :

By above lemma, the exterior measure lemma attempts to describe the volume of a set $E \subseteq \mathbb{R}^n$ by approximating it from outside. The set E covered by rectangle and if the covering gets finer, with fewer rectangles overlapping the volume of E should be close to the sum of the volumes of the rectangles.

1.4 LEBESGUE OUTER MEASURE

Definition :

μ^* is called the Lebesgue exterior (or outer) measure on $\mathbb{R}^n$ and is denoted by m^*.

Now the consequences of the definition of exterior measure on $\mathbb{R}^n$.

1) If $\{R_k\}$ are countably many rectangles and $E \subset \bigcup R_k$ then $m^*(E) \leq \sum V(R_k)$

2) For a given $\in > 0$ there exist countable many rectangle $\{R_k\}$ with $E \subseteq \bigcup R_k$ such that $m^*(E) \leq \sum_k \vartheta(R_k) \leq m^*(E) + E$.

Example 4:

Show that exterior (or outer) measure of a closed rectangle is its volume i.e. $m^*(R) = V(R)$ where R is a rectangle or a $b_0 \times in\mathbb{R}^n$.

Solution :

Let R be a closed rectangle in $\mathbb{R}^n$

$$tst\ m^*(R) = V(R)$$

Note that $\{R\}$ covers R

$\therefore$ by definition of $m^*(R)$, we get

$$m^*(R) \leq V(R) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Let $\in > 0$

By definition $m^*(R), \exists a$ countable cover $\{R_i\}$ of closed rectangles of R.

$$\boxed{\sum_{i=1}^{\infty} \vartheta(R_i) < m^*(R_i) + \frac{\in}{2}}$$

For each *i* choose an open rectangle S_i such that $R_i \subseteq S_i$ and

$$V(S_i) \leq V(R_i) + \frac{\in}{2^{i+1}}$$

Then $R \subseteq \bigcup_{i=1}^{\infty} R_i \subseteq \bigcup_{i=1}^{\infty} S_i$

$\therefore \{S_i\}_{i=1}^{\infty}$ is an open cover of R

$\because R$ is compact this open cover has a finite sub cover say

$$R \subseteq \bigcup_{i=1}^{m} S_i \text{ (after renaming)}$$

We have

$$V(R) \leq \sum_{i=1}^{m} V(S_i) \leq \sum_{i=1}^{\infty} v(S_i)$$

$$\leq \sum_{i=1}^{\infty}\left(V(R_i)+\frac{\in}{2^{i+1}}\right)$$

$$\leq \sum_{i=1}^{\infty}V(R_i)+\in/2$$

$$< m^*(R)+\in/2+\in/2$$

$$< m^*(R)+\in$$

This is true for any $\in > 0$

$V(R) \leq m^*(R)$

From (1) & (2)

$V(R) = m^*(R)$

Example 5:

Show that exterior (or outer) measure of an open rectangle in $\mathbb{R}^n$ is volume.

Solution :

Let S_i be an open rectangle them $R_i \subseteq S_i$ where S_i is closed rectangle $\Rightarrow \{S_i\}$ is a cover of R.

$\because$ by definition $m^*(R) \leq V(S_i) = V(R)$ .... (1)

Let $\in > 0$ be $\{R_i\}$ be a countable cover of closed rectangle of R such that $\sum_{i=1}^{\infty}V(R_i) < m^*(R)+\in/2$ for each i choose an open rectangle S_i such that $R_i \subseteq S_i$ & $V(R_i)+\in/2^{i+1}$

Then $R \subseteq \bigcup_{i=1}^{\infty}R_i \subset \bigcup_{i=1}^{\infty}S_i$

$\therefore \{S_i\}_{i=1}^{\infty}$ is an open cover of R

$\because R$ is compact. This open cover has a sub cover say

$R \subseteq \bigcup_{i=1}^{m}S_i$ (after renaming)

We have

$$V(R) \leq \sum_{i=1}^{m} V(S_i) \leq \sum_{i=1}^{\infty} V(S_i)$$

$$\leq \sum_{i=1}^{\infty} \left(V(R_i) + \in /_{2^{i+1}} \right)$$

$$\leq \sum_{i=1}^{\infty} V(R_i) + \in/2$$

$$< m^*(R) + \in/2 + \in/2$$

$$< m^*(R) + \in$$

This is true for any $\in > 0$

$\therefore V(R) \leq m^*(R)$ ……………………………….. (2)

From (1) & (2)

$V(R) = m^*(R)$

Example 6:

Show that exterior measure of a point in $\mathbb{R}^n$ is zero.

Solution :

Let $a = (a_1, a_2, ..., a_n) \in \mathbb{R}^n$

To show that $m^*\{0\} = 0$

Let $\in > 0$ then the closed rectangle.

$$R = \left[a_1 - \frac{\in^{1/n}}{2}, a_1 + \frac{\in^{1/n}}{2} \right] \times$$

$$\left[a_2 - \frac{\in^{1/n}}{2}, a_2 + \frac{\in^{1/n}}{2} \right] \times ....$$

Covers $\{a\}$

$\therefore$ By definition of $m^*(\{a\})$, we have $m^*(\{a\}) \leq V(R) = \in$

This is true for any $\in > 0$

$\therefore m^*(\{0\}) = 0$

1.5 PROPERTIES OF OUTER MEASURE

Exterior measure has the following properties.

i) (Empty set) The empty set ϕ has exterior measure $m^*(\phi)=0$.

ii) (Positivity) we have $0 \le m^*(A) \le +\infty$ for every subset A of $\mathbb{R}^n$.

iii) (Monotonicity) If $A \subset B \le \mathbb{R}^n$, then $m^*(A) \le m^*(B)$.

iv) (Finite sub-additivity) If $\{A_j\}_{j\in J}$ are a finite collection of subset of $\mathbb{R}^n$ then $m^*\left(\bigcup_{j\in J} A_j\right) \le \sum_{j\in J} m^*(A_j)$

v) (Countable sub-additivity) if $\{A_j\}_{j\in J}$ are a countable collection of subsets of $\mathbb{R}^n$ then $m^*\left(\bigcup_{j\in J} A_j\right) \le \sum_{j\in J} m^*(A_j)$

vi) (Translation invariance) If E is a subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ then $m^*(x+\in) = m^*(\in)$.

Let $x \in \mathbb{R}^n, E \subseteq \mathbb{R}^n$

$tst \quad m^*(x+\in) = m^*(\in)$

Let $\in > 0$, by definition of $m^*(\in)$

$\exists$ a countable cover (R_i) of closed rectangles in $\mathbb{R}^n$ for s.t.

$$\therefore \sum_{i=1}^{\infty} V(R_i) < m^*(E) + \in \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

We now show that $x + E \subseteq \bigcup_{i=1}^{\infty} (x + R_i)$

Let $a \in x + E \Rightarrow a = x + y$

$\Rightarrow a - x = y \in E \subseteq \bigcup_{i=1}^{\infty} R_i$

$\Rightarrow a - x \in R_i$ for some i

$\Rightarrow a \in -x + R_i$ for some i

$\Rightarrow a \in \bigcup_{i=1}^{\infty}(x+R_i)$

$\therefore x+E \subseteq \bigcup_{i=1}^{\infty}(x+R_i)$

$\therefore$By definition of m^*, we have

$$m^*(x+E) \leq \sum_{i=1}^{\infty} V(x+R_i) \quad \ldots\ldots (2)$$

We now show that $V(x+R_i) = V(R_i) \forall_i$

Let $R_i = [a_{iu}, b_{iu}] \times \ldots.. \times [a_{iu}, b_{iu}]$ then

$x+R_i = [x_1 + a_{i1}, x_1 + b_{i1}] \times \ldots\ldots \times [a_{in} \times x_n, b_{in} + x_n]$

$\therefore V(x+R_i) = \prod_{j=1}^{n} (b_{ij} + x_i) - (a_{ij} + x_i)$

$$= \prod_{j=1}^{n} (b_{ij} - a_{ij}) = V(R_i) \quad \ldots\ldots (3)$$

$\therefore$ By 1,2,3 we get

$m^*(x+E) \leq \sum_{i=1}^{\infty} V(x+R_i) = \sum_{i=1}^{\infty} V(R_i) < m^*(E) + \in$

$m^*(x+E) < m^*(E) + \in$

This is true for any $\in > 0$

$$m^*(x+E) \leq m^*(E) + \in \quad \ldots\ldots (4)$$

Let $E' = x+E$ & $y = -x$

Then by (4)

$m^*(y+E') \leq m^*(E')$

$\Rightarrow m^*(-x+x+E) \leq m^*(x+E)$

$$\Rightarrow m^*(E) \leq m^*(x+E) \quad \ldots\ldots (5)$$

By (4) & (5)

$\therefore m^*(x+E) = m^*(E)$

Theorem :

Show that there are uncountable subset of $\mathbb{R}$ whose exterior measure is zero.

Proof :

Define canter set as follows

Let $C_0 = [0,1]$

trisect C_0 and remove the middle open interval to get C_1.

i.e. $C_1 = \left[0, \frac{1}{3}\right] \cup [2/3, 1]$

$$= [0,1] \setminus [1/3, 2/3]$$

repeat this procedure for each interval in C_1 we get C_2

$$C_2 = [0,1] \setminus (1/3, 2/3) \setminus (1/9, 2/9) \setminus (7/9, 8/9)$$
$$= \left[0, 1/9\right] \cup \left[2/9, \frac{1}{3}\right] \cup \left[2/3, 7/9\right]$$

repeating this procedure at each stage we get a sequence of subsets C_i of $[0,1]$ for $i = 0,1,2$

Note that each C_k is a compact subset of $\mathbb{R}$ and $C_0 \supseteq C_, \supseteq C_2$

The Cantor set 'C' is defined as $C = \bigcap_{i=0}^{\infty} C_i$

$C \neq \phi$ because all end points of each C_r is inc and also C is uncountable

We now compute

$$m^*(C_0) = 1, m^*(C_1) = \frac{2}{3} = 1 - \frac{1}{3}$$
$$m^*(C_2) = m^*(C_1) - \frac{2}{9}$$
$$= 1 - \frac{1}{3} - \frac{2}{3^2}$$
$$m^*(C_3) = m^*(C_2) - \frac{2^2}{3^3} = 1 - \frac{1}{3}$$
$$= 1 - \frac{1}{3} - \frac{2}{3^2} - \frac{2^2}{3^3}$$

in general,

$$m^*(C_k) = 1 - \frac{1}{3} - \frac{2}{3^2} - \frac{2^2}{3^3} - \ldots\ldots - \frac{2^{k-2}}{3^{k-1}}$$

$$= \frac{2}{3} - \frac{2}{3}\left[\frac{3}{3} + \frac{2}{3^2} + \ldots\ldots + \frac{2^{k-3}}{3^{k-2}}\right]$$

$$= \frac{2}{3} - \frac{2}{3}\left[\frac{\frac{1}{3}\left(1 - \left(\frac{2}{3}\right)^{k-2}\right)}{\left(1 - \frac{2}{3}\right)}\right]$$

$$= \frac{2}{3}\left[1 - 1 + \left(\frac{2}{3}\right)^{k-1}\right]$$

$$= \left(\frac{2}{3}\right)^k$$

$\because C \subseteq C_k \forall k$

$\Rightarrow m^*(C) \le m^*(C_k) \forall k$

$\Rightarrow m^*(C_1) \le \left(\frac{2}{3}\right) \forall k$

letting $k \to \infty$, we get

$$0 \subseteq m^*(C) \le 0$$
$$= m^*(C) = 0$$

Theorem :

Show that exterior measure of $\mathbb{R}^n$ is infinite.

Proof :

Let $M > 0$ and R be a rectangle s.t. $V(R) = M$

note that $\mathbb{R} \subseteq \mathbb{R}^n$

$\therefore$By monotonicity of m^*

$m^*(R) \le m^*(\mathbb{R}^n)$

But $m^*(R) = V(R) = M$

$\therefore m^*(\mathbb{R}^n) \ge M$

This is true for any $M > 0$

$\therefore m^*(\mathbb{R}^n) = \infty$

Theorem :

If E and $F \subseteq \mathbb{R}^n$ such that $d(E,F) > 0$ then show that $m^*(E \cup F) = m^*(E) + m^*(F)$.

Proof :

Let $E, F \subseteq \mathbb{R}^n$ be s.t. $d(E,F) > 0$ tst $m^*(E \cup F) = m^*(E) + m^*(F)$. By countable subodditivity property $m^*(E \cup F) \leq m^*(E) + m^*(F)$.. (1)

Let $\in > 0$

By the definition of $m^*, \exists$ countable $\{Ri\}$ of closed rectangles in $\mathbb{R}^n$ for $E \cup F$ such that $\sum_i V(Ri) < m^*(E \cup F) + \in$. .... (2)

We categorize the collection $\{Ri\}$ into 3 types :

1) Those intersecting only E
2) Those interescting only F
3) Those intersecting both E & F

Note that if a rectangle $\mathbb{R}$ intersect both E & F, then $d(R) > d(E,F) > 0$ subdivide such the rectangles into rectangles whose diameter is less than $d(E,F)$.

This subrectanlges intersect either E or F not both.

$\therefore$ We can have a contable collection $\{R_2\}$ of rectangles which intersects either E or F but not both.

Let $I_1 = \{i; R_i \cap E \neq \phi\}$

$I_2 = \{i; R_i \cap F \neq \phi\}$

$\Rightarrow I_1 \cap I_2 = \phi$

$\therefore \{R_i\}_{i \in I}$, covers E, we have

$m^*(E) \leq \sum_{i \in I_1} V(R_i)$

Similarly, $m^*(F) \le \sum_{i \in I_2} V(R_i)$

$$\therefore m^*(E) + m^*(F) \le \sum_{i \in I_1} V(R_i) + \sum_{i \in I_2} V(R_i)$$

$$\le \sum_{i=1}^{\infty} V(R_i)$$

$$< m^*(E \cup F) + \in \text{ (by (2))}$$

This is true for any $\in > 0$

$\Rightarrow m^*(E) + m^*(F) \le m^*(E \cup F)$ ……….. (3)

From (1) & (3)

$m^*(E) + m^*(F) = m^*(E \cup F)$

Theorem :

If a subset $E \subseteq \mathbb{R}^n$ is a countable unit of almost disjoint closed rectangle .

i.e. $E = \bigcup_{i=1}^{\infty} R_i$ then show that $m^*(E) = \sum_{i=1}^{\infty} \vee(R_i)$.

Proof :

Let $E = \bigcup_{i=1}^{\infty} R_i$ where R_i's are almost disjoint closed rectangles.

tpt $m^*(E) = \sum_{i=1}^{\infty} \vartheta(R_i)$

By countably subadditivity proposition of

$$m^*(E) = m^*\left(\bigcup_{i=1}^{\infty} R_i\right) = \sum_{i=1}^{\infty} m^*(R_i) = \sum_{i=1}^{\infty} V(R_i)$$

($\therefore R$ is rectangle $\Rightarrow m^*(R) = V(R)$)

Let $\in > 0$, by definition of m^*, $\exists$ a countable cover $\{R_i\}$ of closed rectangle $\mathbb{R}^n$ for E s.t.

$$\sum_{i=1}^{\infty} V(R_i) < m^*(E) + \in$$

For each i, choose open rectangle S_i s.t. $S_i \subseteq R_i$ & $V(R_i) \leq V(S_i) + \frac{\in}{2^i}$

Note that $d(S_i, S_j) > 0$ for $i \neq j$

$\therefore m^*(S_i \cup S_j) = m^*(S_i) + m^*(S_j)$ for $i \neq j$ …………………….. (1)

Using (1) finite no. of times, we get $m^*\left(\bigcup_{1}^{k} S_i\right) = \sum_{i=1}^{k} m^*(S_i)$

$\because S_i \subseteq R_i \subseteq E \ \forall i$

$\Rightarrow \bigcup_{i=1}^{k} S_i \subseteq E$

$\therefore$By monotonicity

$\therefore m^*(E) \geq \sum_{i=1}^{k} m^*(S_i) = \sum_{i=1}^{k} V(S_i) \ \forall k$

Let $k \to \infty$

$$m^*(E) \geq \sum_{i=1}^{\infty} V(S_i) = \sum_{i=1}^{\infty}\left(V(R_i) - \in/2^i\right)$$

$$\geq \sum_{i=1}^{\infty} V(R_i) - \in$$

This is true for any $\in > 0$

$\Rightarrow m^*(\in) \geq \sum_{i=1}^{\infty} V(R_i)$ ………………………………………….. (2)

From (1) & (2)

$$m^*(\in) = \sum_{i=1}^{\infty} V(R_i)$$

Theorem :

Show that

1) If $m^*(A) = 0$ then $m^*(A \cup B) = m^*(B)$

2) If $m^*(A \Delta B) = 0$ then show that $m^*(A) = m^*(B)$

3) $m^*(A \setminus B) \geq= m^*(A) - m^*(B)$

Proof :

1) As $B \subseteq A \cup B$

By monotonicity

$m^*(B) \leq m^*(A \cup B)$ .... (1)

Also by countable subadditive of m^*

$m^*(A \cup B) \leq m^*(A) + m^*(B)$

$\leq m^*(B)$ .... (2)

From (1) & (2)

$m^*(A \cup B) = m^*(B)$

2) If $m^*(A \Delta B) = 0$ tst $m^*(A) = m^*(B)$

$wk > A\Delta B = (A \setminus B) \cup (B \setminus A)$
$\Rightarrow m^*(A\Delta B) \leq m^*(A \setminus B) + m^*(B \setminus A)$

given that $m^*(A\Delta B) = 0$

$\Rightarrow m^*(A/B) + m^*(B/A) = 0$

but $0 \leq m^*(A/B) \leq m^*(A\Delta B) = 0$

$\Rightarrow m^*(A/B) = 0$

$\therefore m^*(A\Delta B) \leq 0 + m^*(B/A)$

WKT $m^*(A) \geq m^*(A \cap B)$

$m^*(A) = m^*(A \cap B)$

similarly we show that

$m^*(B) = m^*(A \cap B)$

$\therefore m^*(A) = m^*(B)$

3) $m^*(A \setminus B) = m^*(B) - m^*(A)$

Proof :

Since A and B are measurable sets

$\therefore A^C$ is also measurable and we have

$B = A \cup (B / A) \quad \because A \subseteq B$

$B / A = B \cap A^C$ is measurable.

$\therefore B \,\&\, A^C$ is measurable

$\therefore B = A \cup (B \setminus A)$ union of disjoint measurable sets

$\therefore m^*(A \cup B \setminus A) = m^*(A) + m^*(B \setminus A) = m^*(B)$

$\therefore m^*(B \setminus A) = m^*(B) - m^*(A)$

Theorem :

Let $E \subseteq \mathbb{R}^n$ show that $m^*(E) = \inf\{m^*(\Omega); \Omega \supseteq E \,\&\, \Omega\, open\}$

Proof :

Let $E \subseteq \mathbb{R}^n$

tst $m^*(E) = \inf\{m^*(\pi); \pi \supset E \; and \; \pi \; open \; in \; \mathbb{R}^n\}$

Let Ω be open in $\mathbb{R}^n$ s.t. $E \subseteq \Omega$

Then by monotonicity of m^*, $m^*(E) \leq m^*(\Omega)$

$\therefore m^*(E)$ is lower bound of $\{m^*(\Omega); \Omega \supset \in, \Omega\, open\}$

$\therefore m^*(E) \leq \inf\{m^*(\Omega); \Omega \supset E, \pi\, open\}$ .... (1)

Let $\in > 0$, then by definition of m^*

$\exists$ an countable cover $\{R_i\}$ of closed rectangle of E s.t.

$\sum_i V(R_i) \leq m^*(E) + \in/2$

For each *i*m let S_i be open rectangles containing R_i s.t.

$V(R_i) < V(S_i) + \in/2 \, i + 1$

Let $\pi = \bigcup_1^\infty S_i$ then Ω is open & $E \subseteq \bigcup_1^\infty R_i \subseteq \bigcup_1^\infty S_i = \Omega$

$$\therefore m^*(\Omega) = m^*\left(\bigcup_1^\infty S_i\right) \le \sum_{i=1}^\infty m^*(S_i)$$

$$\le \sum_{i=1}^\infty V(S_i)$$

$$< \sum_{i=1}^\infty \left(V(R_i) + \in/2^{i+1}\right)$$

$$< \sum_1^\infty V(R_i) + \in/2$$

$$< m^*(E) + \in/2 + \in/2$$

$$< m^*(E) + \in$$

This is true for any $\in > 0$.

$\therefore m^*(\Omega) \le m^*(\in)$

$\therefore \inf\{m^*(\Omega); \Omega \supset \in, \Omega \; is \; open\}$

$\le m^*(\Omega) \le m^*(E)$

Theorem :

For every subset E of $\mathbb{R}^n, \exists$ a G_z

Subset G of $\mathbb{R}^n$ s.t. $G \ge E$ & $m^*(G) = m^*(E)$

Proof :

Let $E \subseteq \mathbb{R}^n$

we first show that

$m^*(E) = \inf\{m^*(\Omega) \, i \, \Omega \supset E$ and Ω is open subset of $\mathbb{R}^n\}$

Let $\in > 0$,

Then for each $k \in \mathbb{N}, \exists \Omega_k$ open in $\mathbb{R}^n$ & $\pi_k \ge E$ s.t.

$m^*(\pi_k) < m^*(E) + \in/2^k$

let $G=\bigcap_{k=1}^{\infty}\Omega_k$

$\Rightarrow G$ is G_δ-set and $G \geq E$

$\therefore$By monotonicity

$m^*(E) \leq m^*(G)$ .... (1)

Note that $G \leq \Omega_k \quad \forall_k$

$\Rightarrow m^*(G) \leq m^*(\Omega_k) < m^*(E) + \in/2^k$

This is true for any $\in > 0$

$\Rightarrow m^*(G) \leq m^*(E)$ .... (2)

By (1) & (2)

$m^*(G) = m^*(E)$

Thoerem :

There exist a countable collection $\{A_j\}_{j\in J}$ of disjoint subset of $\mathbb{R}$ such that $m^*\left(\bigcup_{j\in J} A_j\right) \neq \sum_{j\in J} m^*(A_j)$

Solution :

Consider rational θ and realy $\mathbb{R}$

$\mathbb{R}/\theta = \{x \neq \theta; x \in \mathbb{R}\}$

We known that any two cosets are either identified or disjoint.

We now show that if $A \in \mathbb{R}/\theta$ then $A \bigcap [0,1] \neq \phi$

Let $A = x + \theta$

Let q be rational number in $[-x, -x+1]$

then $x + q \in [0,1]$

Also, $x + q \in x + \theta = A$

$\therefore x + q \in A \bigcap [0,1] \Rightarrow A \wedge [0,1] \neq \phi$

For each $A \in \mathbb{R} \setminus \theta$ choose

$x_A \in A \bigcap [0,1]$

Let $E = \{x_A ; A \in \mathbb{R}/\theta\}$

By construction $E \subseteq [0,1]$

Let $X = \bigcup_{q \in \theta \bigcap [-1,1]} q + E$

We now show that

$[0,1] \subseteq X \subseteq [-1,2]$

Let $q \in [-1,1] \bigcap \theta$ Note that $E \subseteq [0,1]$

$\therefore$ for any $x \in E, \quad q + x \in [-1,2]$

This is true for any $q \in [-1,1] \bigcap \theta$

Theorem :

There exist a finite collection $\{A_j\}_{j \in J}$ of disjoint subset of $\mathbb{R}$ such that $m^*\left(\bigcup_{j \in J} A_j\right) \neq \sum_{j \in J} m^*\left(A_j\right)$

Proof :

Consider θ & $\mathbb{R}$

$\mathbb{R}/\theta = \{x + \theta / x \in \mathbb{R}\}$

We known that any two cosets are either identical or disjoint.

We now show that if $A \in \mathbb{R}/\theta$ then $A \bigcap [0,1] \neq \phi$

Let $A = x + \theta$

Let q be a rational number in $[-x, -x+1]$ then $x + q \in [0,1]$

Also, $x + q \in x + \theta = A$

$\therefore x + q \in A \bigcap [0,1] \Rightarrow A \bigcap [0,1] \neq \phi$

For each $A \in \mathbb{R} \setminus \theta$ choose $x_A \in A \bigcap [0,1]$

Let $E = \{x_A / A \in \mathbb{R}/\theta\}$

By construction $E \subseteq [0,1]$

Let $X = \bigcup_{q \in \theta \cap [-1,1]} q + E$

We now show that $[0,1] \subseteq X \subseteq [-1,2]$

Let $q \in [-1,1] \cap \theta$

Note that $E \subseteq [0,1]$

$\therefore$ for any $x \in E, \quad q + x \in [-1,2]$

This is true for any $q \in [-1,1] \cap \theta$

There exist a finite collection $\{A_j\}_{j \in J}$ of disjoint subset of ¡ such that $m^*\left(\bigcup_{j \in J} A_j\right) \neq \sum_{j \in J} m^*(A_j)$

Consider $Q\{$¡

ⁱ $|_Q = \{x + Q \mid x \in$ ¡ $\}$

We know that any two cosets one either identical or disjoint.

We know show that if $A \in$ ⁱ $|_Q$ then A I $[0,1] \neq Q$

Let $A = x + Q$

Let q be a rational number in $[-x, -x+1]$ then $x + q \in [0,1]$.

Also $x + q \in x + Q = A$

$\therefore x + q \in A$ I $[0,1] \Rightarrow A$ I $[0,1] \neq Q$

For each $A \in$ ⁱ $|_Q$ choose $x_A \in A$ I $[0,1]$.

Let $E = \{x_A \mid A \in$ ⁱ $|_Q\}$

By construction $E \subseteq [0,1]$

Let $X = \bigcup_{q \in Q \text{I} [-1,1]} q + \in$

We show that $[0,1] \subseteq \times \subseteq [-1,2]$

Let $q \in [-1,1] \text{I } Q$

Note that $E \subseteq [0,1]$

$\therefore x \in E, q + x \in [-1,2]$

$\Rightarrow q + E \subseteq [-1,2]$

This is true for any $q \in [-1,1] \text{I } Q$

Let $y \in [0,1]$

Then $y \in y + 0 \in y + \theta = A$ (say) but $x_A \in A$

$\therefore y - x_A = y \in \theta$

$\therefore y, x_A \in [0,1] \Rightarrow y - x_A \in [-1,1]$

$\Rightarrow q \in [-1,1] \wedge \theta$

$\therefore y \in q + x_A \in q + E$

$\therefore y \in x$

$\therefore [0,1] \subseteq X \Rightarrow [0,1] \subseteq X \subseteq [-1,2]$

$\therefore$ By monotonicity of m^*

$m^*[0,1] \leq M^*(X) \leq m^*[-1,2]$

$1 \subseteq m^*(x) \leq 3$ ……………………………………… (1)

$\because x = \bigcup_{q \in [-1,1] \wedge \theta} q + E$ by countable subadditive and translation invariance of m^*, we get.

$m^*(X) \leq \sum_{q \in [-1,1] \cap \theta} m^*(q + E) = \sum_{q \in [-1,1] \cap \theta} m^*(E)$

By $(1) \Rightarrow m^*(X) \neq 0$

$\Rightarrow m^*(E) \neq 0$

$\therefore$ By Aritimedian property

$\exists n \in \mathbb{N}$ s.t. $m^*(E) > \frac{1}{n}$

Let I be a finite subset of $[-1,1]\cap\theta$ with cardinality $3n$.

Then $\sum_{q\in I} m^*(E) > 3n\frac{1}{n} = 3$

$\therefore$ by (1) $m^*(x) \neq \sum_{q\in I} m^*(q+E)$

Theorem :

Let $E\subseteq\mathbb{R}^n$ & $\lambda\in\mathbb{R}(\lambda>0)$ show that $m^*(\lambda E)=\lambda^n m^*(E)$

Proof :

To show that $m^*(\lambda E)=\lambda^n m^*(E), \lambda>0$

Let $\in>0$,

$\therefore$ by definition of $m^*(E), \exists$ a countable cover of $\{R_i\}$ of closed rectangle in $\mathbb{R}^n$, for E s.t. $\sum V(R_i)<m^*(E)+\in$

$\because E\subseteq\bigcup_{i=1}^{\infty}R_i \Rightarrow \lambda E\subseteq\bigcup_{i=1}^{\infty}\lambda R_i$

Let $R_i=[a_{i1},b_{i1}]\times....\times[a_{in},b_{in}]$

$$\begin{aligned}\lambda R_i &= \left\{\lambda(x_1,.....,x_n);x_j\in\left[a_{ij},b_{ij}\right]\right\}\\ &= \left\{(\lambda x_1,.....,\lambda x_n);x_j\in\left[a_{ij},b_{ij}\right]\right\}\\ &= \left\{(\lambda x_1,.....,\lambda x_n);\lambda x_j\in\left[\lambda a_{ij},\lambda b_{ij}\right]\right\}\\ &= [\lambda a_{i1},\lambda b_{i1}]\times.....\times[\lambda a_{in},\lambda b_{in}]\end{aligned}$$

$\Rightarrow \lambda R_i$ is a closed rectangle

$\therefore V(\lambda R_i)=\lambda^n V(R_i)$

$\therefore \lambda E\subseteq\bigcup_{i=1}^{\infty}\lambda R_i$ by monotoricity & countable additive property we get

$$\begin{aligned}m^*(\lambda E) &\le \sum_1^{\infty} m^*(\lambda R_i)=\sum_1^{\infty}V(\lambda R_i)=\sum_1^{\infty}\lambda^n V(R_i)\\ &\le \lambda^n\sum V(\lambda R_i)<\lambda^n m^*(E)+\in\end{aligned}$$

This is true for any $\in > 0$

$\therefore m^*(\lambda E) \le \lambda^n m^*(E)$ …………………………………….. (1)

let $E^1 = \lambda E$ & $\mu = \frac{1}{\lambda}$

$\therefore$ by (1)

$m^*(\mu E^1) \le \mu^* m^*[E^1)$

$\Rightarrow m^*\left(\frac{1}{\lambda}\lambda E\right) \le \frac{1}{\lambda^n} m^*(\lambda E)$

$\Rightarrow \lambda^n m^*(E) \le m^*(\lambda E)$ ………………………………… (2)

From (1) & (2)

$m^*(\lambda E) = \lambda^n m^*(E)$

1.6 SUMMARY

In this chapter we have learned about.

- definition of σ -Algebra, bored algebra
- measure on a set.
- The extension Measure
- Lebesgue outer Measure (μ^*) on $\mathbb{R}^n$
- Properties of lebesgue outer measgure.

1.7 UNIT END EXERCISE

1) Let $X = \{a, b, c, d\}$ and $A_1 = \{X, \phi, \{d\}\}$ and $A_2 = \{X, \phi, \{d\}\}$, $\{a, b, c\}$ check whether A_1 & A_2 are both algebra or not. Also check wheter $A_1 \cup A_2$ is an algebra or not.

2) Show that exterior measure at any countable subset of $\mathbb{R}^n$ is zero. Justify the converse?

3) Show that the outer mesuration interval is its length.

4) Show that if $(F_\alpha)\alpha \in I$ is a collection of σ -Algebra on X then $\eta_\alpha F_\alpha$ is also a σ -Algebra on X.

5) If a subset $E \subseteq \mathbb{R}^n$ is a countable union of almost disjoint closed rectangle then show that $m^*(E) = \sum_{i=1}^{\infty} \cup (R_i)$.

6) If A_1 and A_2 are measurable subsets of the closed interval $[a,b]$ then $A_1 - A_2$ is measurable and if $A_1 \subseteq A_2$ then $m(A_1 - A_2) = mA_1 - mA_2$.

7) Show that for any set A, $m^* A = m^*(A+x)$ where $A + x = \{y + x; y \in A\}$

8) Show that for any set A and any $\in > 0$, there exist an open set O such that $A \subseteq 0$ and $m^* 0 \leq m^* A + \in$.

9) Compute the Lebesgue outer measure of $B = [1-2] \cup \{3\}$

10) Prove that if the boundary of $\pi \subset \mathbb{R}^k$ has outer measure zero than π is measureable.

11) Let Ω be an arbitary collection of subsets of a set. Show that for a given $A \in \sigma(C)$ there exists a countable sub-collection C_A of C depdending on A such that $A \subset \sigma(C_A)$.

12) Check that μ^* is an outer measure on R. Not

i) Let X be any seet and $\mu^* : P(X) \to [0, \infty]$ be given by

i) $\mu^*(A) = 0$ if A is countable

=1 otherwise

ii) $\left.\begin{array}{r} \mu^*(A) = 0 \, if \;\; A \; finite \\ 1 \, if \;\; otherwise \end{array}\right\}$ then X be on infinite set

iii) $\mu^*(A) = 0$ if $A = \phi$

= 1 otherwise

❖❖❖❖

2

LEBESGUE MEASURE

Unit Structure :

2.1 OBJECTIVE

After going through this chapter you can able to know that

- Construction of Lebesgue measure in $\mathbb{R}^n$.
- Lebesgue Measurable set in $\mathbb{R}^n$.
- Properties of measurable sets.
- Existance of non-measurable sets.

2.2 INTRODUCTION

In the previous chapter we have studied about Lebesgue outer measure m^* is not countability additive and it cannot be measure. So that we have to cover with subset of $\mathbb{R}^n$ for which m^* is countably additive this subclass a collection at Measurable sets. Now we shall define lebesgue measure of a set using the lebsgue outer measure and discuss properties of lebesgue measure set.

2.3 LEBESGUE MEASURE

Definition - (Lebesgue measurability)

Let E be a subset of $\mathbb{R}^n$ we say that E is Lebesgue measurable, or measurable if we have the identity

$$m^*(A) = m^*(A \cap E) + m^*(A/E)$$

2.3.1 Properties of measurable sets :

Following are the properties of measurable sets :

a) If E is measurable, then $E^C = \mathbb{R}^n / E$ is also measurable.

b) Any set E of exterior (or outer) measure zero is measurable. In particular, any countable set is measurable.

c) If $E_1 \& E_2$ are measurable, then $E_1 \cap E_2$ and $E_1 \cup E_2$ are measurable.

d) (Boolean algebra property) If $E_1, E_2, ...E_n$ are measurable then $\bigcup_1^n E_j$ & $\bigcap_1^n E_j$ are measurable.

e) (Translation in variance) If E is measurable & $x \in \mathbb{R}^n$ then $x + E$ is also measurable, and $m(x+E) = m(E)$.

Lemma : (Finite additivity)

If $(Ei)_{i=1}^k = (E_j)_{j \in J}$ are a finite collection of disjoint measurable sets and any set A, we have

$$m^*\left(A \cap \bigcup_{j \in J} E_j\right) = \sum_{j \in J} m^*(A \cap E_j)$$

Further more we have

$$m\left(\bigcup_{j \in J} E_j\right) = \sum_{j \in J} m(E_j)$$

Proof :

We prove by induction on K

The result is trivial when K=1

Assume result is true for k-1

We prove result for K

Let $E = \bigcup_{i=1}^{k} E_i$

tpt $m^*(A\cap E)=\sum_{i=1}^{k} m^*(A\cap E_i)$

Now E_k is measurable we have for $A\cap E\subseteq\mathbb{R}^n$.

$m^*(A\cap E)=m^*((A\cap E)\cap E_k)+m^*((A\cap E)\cap E_k^C)$

But $(A\cap E)\cap E_k=A\cap E_k$

$(\because E_k\subseteq E)$

$(A\cap E)\cap E_k^C=A\cap(E\cap E_k^C)$

$=A\cap\left(\bigcup_{i=1}^{k-1}E_i\right)$

$\therefore m^*(A\cap E)=m^*(A\cap E_k)+m^*\left(A\cap\left(\bigcup E_i\right)\right)$

$=m^*(A\cap E_k)+\sum_{i=1}^{k} m^*(A\cap E_i)$

$=\sum_{i=1}^{k} m^*(A\cap E_i)$

$\therefore$ The result is true for K

By introduction, it is true for 'n'.

ii) Put $A=\mathbb{R}^n$

Theorem :

If $A\subseteq B$ are two measurable sets then B/A is also measurable & $m(B/A)=m(B)-m(A)$

Proof :

tst B/A is measurable.

Suppose A & B are measurable

$\because$ intersection of two measurable set is measurable & complement of a measurable set is measurable.

$\Rightarrow B/A=B\cap A^C$ is measurable

Note that $B=A\cup(B/A)$

which is a disjoint union.

$\because m$ is finitely additive

$$m(B) = m(A) + m(B - A)$$
$$\Rightarrow m(B/A) = m(B) - m(A)$$

Example 1 :

Let A be a measurable set of finite outer measure that is contained in B show that $m^*(B/A) = m^*(B) - m^*(A)$

$\Rightarrow$ $\because$ A is measurable

By definition for this B

$$m^*(B) = m^*(B \cap A) + m^*(B/A)$$
$$m^*(B) = m^*(A) + m^*(B/A)$$

$\because m^*(A) < \infty$ we get

$$m^*(B/A) = m^*(B) - m^*(A)$$

Example 2 :

Suppose $A \subseteq E \subseteq B$ where A & B are measurable sets of finite measure show that if $m(A) = m(B)$ then E is measurable.

$\Rightarrow \because$ A & B are measurable $\Rightarrow B/A \Rightarrow B \cap A^C$ is measurable.

Note that $B = A \cup (B/A)$ $\quad (\because A \subseteq B)$.

which is a disjoint union.

$\because m$ is finitely additive, we get

$$m(B) = m(A) + m(B/A)$$
$$m(B/A) = 0 \quad (\because m(B) = m(A))$$
$$\because A \subseteq E \subseteq B \Rightarrow E/A \subseteq B \mid A$$
$$m^*(E/A) \subseteq m^*(B/A) = m(B/A) = 0$$
$$\Rightarrow m^*(E/A) = 0$$

$\Rightarrow E/A$ is measurable

$\Rightarrow E = A \cup (E/A)$ is measurable

Example 3 :

Show that if E_1 & E_2 are measurable then

$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$

Solution :

Suppose E_1 & E_2 are measurable not that

$E_1 \cup E_2 = E_1 \cup (E_2/E_1)$ which is a disjoint union.

By finite additie property of 'm'

$m(E_1 \cup E_2) = m(E_1) + m(E_2/E_1)$ ……………………….. (1)

also $E_2 = (E_1 \cap E_2) \cup (E_2/E_1)$

which is a disjoint union.

By finite additivity of 'm'

$m(E_2) = m(E_1 \cap E_2) + m(E_2/E_1)$ ……………………….. (1)

$m(E_2/E_1) = m(E_2) - m(E_1 \cap E_2)$

subs in 1

$m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$

$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$

Theorem :

Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets prove that for any set A, $m^*\left(A \cap \bigcup_1^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k)$.

Proof :

Let $\{E_k\}_{k=1}^{\infty}$ be countable collection of disjoint measurable sets.

Let $A \subseteq \mathbb{R}^n$

tpt $m^*\left(A \cap \bigcup_1^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k)$.

By countable subadditivity property of m^* we get,

$$m^*\left(A\cap\left(\bigcup_1^\infty E_k\right)\right)=m^*\left(\bigcup_1^\infty(A\cap E_k)\right)$$

$$\leq\sum_{k=1}^\infty m^*(A\cap E_k) \text{ .... (1)}$$

Also by finite additive property of m, we get

$$m^*\left(A\cap\left(\bigcup_{k=1}^\infty E_k\right)\right)\geq m^*\left(A\cap\bigcup_{k=1}^\infty E_k\right)$$

$$\geq m^*\left(\bigcup_{k=1}^m(A\cap E_k)\right)$$

$$\geq\sum_{k=1}^m m^*(A\cap E_k)$$

This is true for all 'm'

$$m^*\left(A\cap\left(\bigcup_{k=1}^\infty E_k\right)\right)\geq\sum_{k=1}^\infty m^*(A\cap E_k) \text{ .... (2)}$$

from (1) & (2)

$$m^*\left(A\cap\left(\bigcup_1^\infty E_k\right)\right)=\sum_{k=1}^\infty m^*(A\cap E_k)$$

Theorem :

Show that the union of a countable collection of measurable set is measurable.

Proof :

Let $\{A_k\}_{k=1}^\infty$ be a countable collection of measurable sets and $E=\bigcup_{k=1}^\infty A_k$.

tst E is measurable.

Define $B_1=A,$ & for $k\geq 2$

$$B_k=A_k\Big|\bigcup_1^{k-1}A_i$$

Since finite union of complement m-set are measurable

$\Rightarrow B_k$ is measurable.

Clearly B_k's are pairwise disjoint

$$\bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty}\left(A_k \middle| \bigcup_{1}^{k-1} A_i\right)$$

$$= \bigcup_{k=1}^{\infty}\left(A_k \cap \left(\bigcup_{i=1}^{k-1} A_i\right)^C\right)$$

$$= \bigcup_{k=1}^{\infty}\left(A_k \cap \left(\bigcup_{1}^{k-1} A_i\right)^C\right)$$

$$= A_1 \cup \left(A_2 \cap \left(\cap A_1^C\right)\right) \cup \left[A_3 \cap A_1^C \cap A_2^C\right] \cup ....$$

$$= \bigcup_{k=1}^{\infty} A_k = E$$

Example 4 :

Show that the intersections of a countable collection of measurable set is measurable.

$\Rightarrow$ Let A be a subset of $\mathbb{R}^n$ and for $n \in \mathbb{N}$.

Define $F_n = \bigcup_{k=1}^{\infty} B_k \subseteq E$

$\therefore B_k{}'S$ are measurable

$\Rightarrow F_n$ is measurable

$\therefore$ By definition

$$m^*(A) = m^*(A \cap F_n) + m^*\left(A \cap F_n^C\right)$$

$$\because F_n \subseteq E \Rightarrow F_n^C \supseteq E^C \Rightarrow A \cap F_n^C \supseteq A \cap E^C$$

$$\Rightarrow m^*\left(A \cap E^C\right) \subseteq m^*\left(A \cap F_n^C\right)$$

$$\therefore m^*(A) \geq m^*(A \cap F_n) + m^*\left(A \cap E^C\right) \text{ .... (1)}$$

Now

$$m^*\left(A\cap F_n\right)=m^*\left(A\cup\left(\bigcup_1^n B_k\right)\right)$$

$$=m^*\left(\bigcup_{K=1}^n\left(A\cap B_k\right)\right)$$

$$=m^*\left(\bigcup_{K=1}^n\left(A\cap B_k\right)\right)$$

$$=\sum_{K=1}^n m^*\left(A\cap B_k\right)$$

$$=\sum_{K=1}^n m^*\left(A\cap B_k\right)$$

$\therefore$ By (1)

$$m^*\left(A\right)\geq\sum_{k=1}^n m^*\left(A\cap B_k\right)+m^*\left(A\cap E^C\right)$$

$\because$ LHS is independent of n, we have

$$m^*\left(A\right)\geq\sum_1^n m^*\left(A\cap B_k\right)+m^*\left(A\cap E^C\right)$$

But

$$m^*\left(A\cap E\right)=m^*\left(A\cap\left(\bigcup_1^\infty B_k\right)\right)$$

$$=m^*\left(\bigcup_1^\infty\left(A\cap B_k\right)\right)$$

$$\leq\sum_1^\infty m^*\left(A\cap B_k\right)$$

$$m^*\left(A\right)\geq m^*\left(A\cap E\right)+m^*\left(A\cap E^C\right)$$

As $A=\left(A\cap E\right)\cup\left(A\cap E^C\right)$ by countable subadditivity proposition of m^*.

$m^*\left(A\right)\leq m^*\left(A\cap E\right)+m^*\left(A\cap E^C\right)$ ………………………. (3)

By (2) & (3)

$$m^*\left(A\right)=m^*\left(A\cap E\right)+m^*\left(A\cap E^C\right)$$

$\therefore$By definition E is measurable.

Example 5 : Countable additive

If $\{E_j\}_{j\in J}$ are a countable collection of disjoint measurable sets then $\bigcup_{j\in J} E_j$ is measurable and $m\left(\bigcup_{j\in J} E_j\right)=\sum_{j\in J} m(E_j)$

$\Rightarrow$ Without loss of generality we may assume $J=\mathbb{N}$ suppose $\{E_k\}_{k=1}^{\infty}$ be a countable collection of disjoint measurable set we first show that $E=\cup E_k$ measurable let $F_n=\cup E_k$.

then by previous exercise we get E is measurable.

We now show that

$$m(E)=\sum_1^{\infty} m(E_k)$$

By subadditivity proposition of m^*

$$m(E)=m^*(E)=m^*\left(\bigcup_1^{\infty} E_k\right)$$

$$\leq \sum_1^{\infty} m^*(E_k)$$

$$=\sum_{k=1}^{\infty} m(E_k) \text{ .... (*)}$$

By finite additivity property and monotonicity of m^*

we have as $F_n \supseteq E$

$$m(E)\geq m(F_n)=m\left(\bigcup_{k=1}^{n} E_k\right)$$

$$=\sum_{k=1}^{n} m(E_k)$$

$\therefore$LHS is independent of n, we get

$$m(E)\geq \sum_{k=1}^{\infty} m(E_k) \text{ .... (**)}$$

$\therefore$By countable additivity

$$m(E)\geq \sum_{k=1}^{\infty} m(E_k)$$

Example 6 :

Show that every closed and open rectangles in $\mathbb{R}^n$ are measurable.

$\Rightarrow$ Let R be a closed rectangle

tst R is measurable

Let $\in > 0$, Let $A \subseteq \mathbb{R}^n$

by definition of $m^*(A)$

$\exists$ a countable collection of closed rectangle $\{R_i\}_{i=1}^{\infty}$ such that

$A \subseteq \bigcup_{i=1}^{\infty} R_i$ and $\sum_{i=1}^{\infty} V(R_i) < m^*(A) + \in$ .... (1)

we decompose each R_i into finite union of almost disjoint rectangle $\left\{R_i, S_{i_1}, ..., S_{ik}\right\}$ such that $R_i = R_i^1 \cup \left(\bigcup_{j=1}^{k} S_{ij}\right)$.

$R_i^1 = R_i \cap R \subseteq R$ and $S_{i_j} \subseteq R^C$

$\therefore$By finite additive property of M.

$$m(R_i) = m(R_i) + \sum_{j=1}^{k} m\left(S_{i_j}\right)$$

$$\Rightarrow V(R_i) = V(R_i^1) + \sum_{j=1}^{k} V\left(S_{i_j}\right)$$

$$\therefore \sum_{i=1}^{\infty} V(R_i) = \sum_{i=1}^{\infty} V(R_i) + \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} V\left(S_{i_j}\right)\right)$$

Note That $\{R_i\}_{i=1}^{\infty}$ lover $A \wedge R$

$$\left[\because A \cap R \subseteq \left(\bigcup_{i=1}^{\infty} R_i\right) \cap R = \bigcup_{i=1}^{\infty} (R_i \cap R) = \bigcup_{i=1}^{\infty} R_i^1\right]$$

$\left\{S_{i_j}\right\} i, j$ covers $A \cap R^C$

$$\sum_{1}^{\infty} V(R_i) = m^*\left(\bigcup_{1}^{\infty} R_i^1\right) \geq m^*(A \cap R)^C \text{ and } m^*\left(\bigcup_{i,j} S_{ij}\right) \leq m^*\left(A \cap R^C\right)$$

$$m^*(A \cap R) \leq m^*\left(\bigcup_{i,j} S_{ij}\right)$$
$$\leq \sum_{i,j} m^*(S_{ij}) = \sum_{i,j} V(S_{ij})$$

$\therefore$By (1)

$$m^*(A) + \in > \sum_{i=1}^{\infty} V(R_i)$$
$$= \sum_{i=1}^{\infty} V(R_i) + \sum_{i=1}^{\infty}\sum_{j=1}^{k} V(S_{ij})$$
$$\geq m^*(A \cap R) + m^*(A \cap R^C)$$

This is true for any $\in > 0$

$$m^*(A) \geq m^*(A \cap R) + m^*(A \cap R^C)$$

$\therefore$By definition R is measurable.

Example 7 :

Show that every open and closed subsets of $\mathbb{R}^n$ are measurable.

$\Rightarrow$ Let $K = \max\{K_i\}$

Let G be an open subset of $\mathbb{R}^n$ consider the grid of rectangle in $\mathbb{R}^n$ of side length one and whose vertices have integer co-ordinates.

TST G is measurable.

$\therefore$ Number of rectangle in grid is countable and one almost disjoint we ignore all these rectangle contained in G^C.

Now we have two types of rectangle (1) Those rectangle contained in G (2) Those rectangle intersect with G & G^C.

Let C = set of all rectangle contained in G.

We bisect type (2) rectangle into two rectangle each of its side length is ½.

Repeat the process iterating this process for arbitrarily many times we get a constable collections c of almost disjoint rectangle contained in G.

By construction $\bigcup_{R \in C} R \subseteq G$

Let $x \in G$

$\therefore G$ is open

We can choose sufficiently small rectangle in the bisection procedure that contains x is entirely contained in G.

$\therefore x \in \bigcup_{R \in C} R$

$\therefore G \subseteq \bigcup_{R \in C} R$

$\therefore G = \bigcup_{R \in C} R$

$\therefore G$ is countable union of closed rectangle and hence G is measurable.

2.4 OUTER APPROXIMATION BY OPEN SETS

Let $E \subseteq \mathbb{R}^n$ such that E is measurable iff for $\in > 0$, there is an open set Ω containing E for which $m^*(\Omega/E) < \in$.

$\Rightarrow$ Suppose E is measurable

Let $\in > 0$

Suppose $m^*(E) < \infty$

$\therefore$By the definition of $m^*(E)$

$\exists$ a countable collection of open rectangles $\{R_i\}$ such that $E \subseteq \bigcup_{i=1} R_i$

and $\sum_{i=1}^{\infty} V(R_i) < m^*(E) + \in$.

Let $\Omega = \bigcup_{i=1}^{\infty} R_i$ which is countable union of opensets.

$\therefore \Omega$ is open in $\mathbb{R}^n$ and $E \subseteq \Omega$

$\therefore \Omega$ is open, it is measurable

$\therefore \Omega/E$ is measurable

$\Omega = E \cup (\Omega/E)$ which is a countably disjoint union

$m^*(\Omega) = m^*(E) + m^*(\Omega/E)$

$\therefore m^*(\Omega/E) = m^*(\Omega) - m^*(E)$

But

$$\Omega = \bigcup_{i=1}^{\infty} R_i \Rightarrow m^*(\Omega) \leq \sum_{i=1}^{\infty} m^*(R_i) \leq \sum_{i=1}^{\infty} V(R_i)$$

$$\therefore m^*(\Omega/E) \leq \sum_{i=1}^{\infty} V(R_i) - m^*(E) < \in$$

Suppose $m^*(E) = \infty$

For each k

$E_k = E \bigcap R_K$ where

$R_k =$ rectangle with centre origin and side length K

For each k

Then $m^*(E_k) \leq m^*(R_i) = V(R_i) = K^{\infty} < \infty$

$\therefore$ by first case for each K, $\exists$ Ω_k open in $\mathbb{R}^n$ such that

$$E_k \subseteq \Omega_k m^*(\Omega_k / E_k) < \frac{E}{2^k}.$$

Let $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ which is countable union of open set.

$\therefore \Omega$ is open and $E \subseteq \Omega$

$$\begin{aligned} m^*(\Omega/E) &= m^*(\Omega \bigcap E^C) \\ &= m^*\left(\bigcup_{k=1}^{\infty} \Omega_k \bigcap E^C\right) \\ &= m^*\left(\bigcup_{k=1}^{\infty} (\Omega_k / E)\right) \\ &\leq \sum_{k=1}^{\infty} m^*(\Omega_k / E) \\ &\leq \sum_{k=1}^{\infty} m^*(\Omega_k / E_k) \\ &\leq \sum_{k=1}^{\infty} \frac{\in}{2^k} = \in \end{aligned}$$

Conversely suppose for a given $\in > 0$ $\exists$ open set $\Omega \geq E$ such that $m^*(\Omega/E) < \in$.

Tst E is measurable

Let $A \subseteq \mathbb{R}^n$

$\therefore \Omega$ is open

$\Rightarrow \Omega$ is measurable

$m^{\&}(A) = m^*(A \cap \Omega) + m^*(A/\Omega)$

Note that $A/E = (A/\Omega) \cup ((A \cap \Omega/E))$ which is a disjoint union.

$$\therefore m^*(A/E) = m^*(A/\Omega) + m^*((A \cap \Omega)/E)$$
$$\therefore m^*(A \cap E) + m^*(A/E) = m^*(A \cap E) + m^*(A/\Omega) + m^*((A \cap \Omega)/E)$$
$$\leq m^*(A \cap E) + m^*(A/\Omega) + m^*(A \cap \Omega)$$
$$< m^*(A) + \in$$

This is true for any $\in > 0$

$\therefore m^*(A \cap E) + m^*(A/E) \leq m^*(A)$

$\therefore E$ is measurable.

Exercise 8 :

Let $E \subseteq \mathbb{R}^n$ S.T., E is measurable iff for each $\in > 0$ there is G_s set G conlaining E for which $m^*(G/E) = 0$.

Proof : suppose E is measurable

$\therefore$By outer approximation by an open set.

For each $n \in \mathbb{N}$, $\exists$ an open set $\Omega_k \supseteq E$ s.t.

$$m^*(\Omega_k/E) < 1/k$$

Let $G = \bigcap_{k=1}^{\infty} \Omega_k$, then G is a G_δ set ant $E \subseteq G$

$$m^*\left(G/E\right) = m^*\left(\bigcap_{K=1}^{\infty} \Omega_k\, K/E\right)$$
$$= m^*\left(\left(\bigcap_{K=1}^{\infty} \Omega_K\right) \cap E^C\right)$$
$$= m^*\left(\bigcap_{K=1}^{\infty} \left(\Omega_k \cap E^C\right)\right)$$
$$\leq m^*\left(\Omega_K \cap E^C\right)$$
$$\leq m^*\left(\Omega_K / E\right)$$
$$< 1/k$$

This is true for all k

$m^*\left(G/E\right) = 0$

Conversely, suppose $\exists$ G_δ set $G \supseteq E$

s.t. $m^*\left(G/E\right) = 0$

tst E is measurable

Let $A \subseteq \mathbb{R}^n$

$\therefore G$ is countable int of measurable

Set $\Rightarrow$ G is measurable.

$\therefore$By definition

$m^*\left(A\right) = m^*\left(A \cap G\right) + m^*\left(A \cap G^C\right)$

Note that

$A/E = \left(A/G\right) \cup \left(\left(A \cup G\right)/E\right)$

Which is a disjoint union

$\therefore m^*\left(A/E\right) = m^*\left(A/G\right) + m^*\left(\left(A \cap G\right)/E\right)$

$$\therefore m^*\left(A \cap E\right) + m^*\left(A/E\right) = +m^*\left(A \cap E\right) + \left(A/G\right) + m^*\left(A \cap G/E\right)$$
$$\leq m^*\left(A \cap G\right) + m^*\left(A/G\right) + m^*\left(G/E\right)$$
$$\leq m^*\left(A\right) + 0$$
$$\leq m^*\left(A\right)$$

2.5 INNER APPROXIMATION BY CLOSED SETS

Theorem :

Let $E \subseteq \mathbb{R}^n$ S.T. E is measurable iff for each $\in > 0$, there is a closed set $F \subset E$ for which $m^*(E/F) < E$.

Proof :

Suppose E is measurable

$\Rightarrow E^C$ is measurable

Let $\in > 0$

$\therefore$By outer approximative by open seet $\exists$ an open set $\Omega \supset E^C$ s.t.

$m^*(\Omega/E^C) < \in$

Let $E = \Omega^C \Rightarrow F$ is closed & $F \subseteq E$.

Now $m^*(E/F) = m^*(E \cap F^C) = m^*(E \cap \Omega)$

$= m^*(\Omega \cap E) = m^*\left(\Omega \cap \left(E^C\right)^C\right)$

$= m^*(\Omega/E^C) < \in$

Conversely suppose for $\in > 0, \exists$ closed set $F \subseteq E$ such that $m^*(E/F) < E$

Tst E is measurable

Let $A \subseteq \mathbb{R}^n$

$\because$ F is measurable

By definition

$m^*(A) = m^*(A \cap F) + m^*(A/F)$

Note that

$A \cap E = ((A \cap F)/F) + \cup (A \cap F)$ which is disjoint union.

$\therefore m^*(A \cap E) = m^*(A \cap F) + m^*((A \cap E)/F)$

$\therefore m^*(A \cap E) + m^*(A/E)$

$= m^*(A \cap F) + m^*((A \cap E)/F) + m^*(A/E)$

$\leq m^*(A \cap F) + m^*(E/F) + m^*(A/F)$

$< m^*(A) + \in$

Example 9 :

Let E be a set of finite outer measure show that there is an $F\sigma$ set F & a G_δ set G s.t. $F \subseteq E \subseteq G$ & $m^*(F) = m^*(E) = m^*(G)$.

[Ans] $\therefore E$ is measurable for given each k $\exists$ open set G_k and closed set F_k such that $F_k \subseteq E \subseteq G_k$ and $m^*(G_k/F_k) < 1/k$.

Let $G = \bigcap_{k=1}^{\infty} G_k$ & $F = \bigcup_{k=1}^{\infty} F_k$.

Then G is G_S set and F is $F\sigma$ set and $F \subseteq E \subseteq G$.

We now show that $m^*(G) = m^*(E) = m^*(F)$ $G = E \cup (G/E)$ which is disjoint union.

$m^*(G) = m^*(E) + m^*(G/E)$

Now $G/E = G \cap E^C$

$= \left(\bigcap_{k=1}^{\infty} G_k \cap E^C \right)$

$= \bigcap_{k=1}^{\infty} (G_k \cap E^C)$

$= \bigcap (G_k/E) \subseteq G_k/E$

$\subseteq G_k/F_k$

$\therefore m^*(G/E) \leq m^*(G_k/F_k) < 1/k$

This is true for all k

$\therefore m^*(G/E) = 0$

$\therefore m^*(G) = m^*(E)$ .... (1)

$E = F \cup (E/F)$

$m^*(E) = m^*(F) + m^*(E/F)$

$E/F = E \cap F^C = E \cap \left(\left(\bigcup_{k=1}^{\infty} F_k \right)^C \right)$

$= E \cap \left(\bigcup_{k=1}^{\infty} F_k^C \right) = \cap \left(E \cap F_k^C \right)$

$= \bigcap_{k=1}^{\infty} (E/F_k)$

$\subseteq E/F_k$

$\subseteq G_k/F_k$

$m^*(E/F) \le m^*(G_k/F_k) < 1/k$

This is true for all k

$\therefore m^*(E/F) = 0$

Example 10 :

Let E be a set of finite outer measure show that if E is not measure, then there is an open set Ω containing E that has finite outer measure and for which $m^*(\Omega/E) > m^*(\Omega) - m^*(E)$.

Solution :

$\Rightarrow$ Since E is not measurable

$\Rightarrow \exists \in_0 > 0$ for any open set Ω containing E.

$m^*(\Omega/E) \ge \in_0$ .... (1)

$\therefore E$ has finite outer measure.

By definition $\exists$a countable collection of open rectangles $\{R_i\}_{i=1}^{\infty}$ such that $E \subseteq \bigcup_{i=1}^{\infty} R_i$ and $\sum_{i=1}^{\infty} V(R_i) < m^*(E) + \in_0$.

Let $\Omega_0 = \bigcup_{i=1}^{\infty} R_i$

$\Rightarrow E \subseteq \Omega_0$ & Ω_0 open.

$\therefore$By (1) $m^*(\Omega/E) > \in_0$ .... (2)

By countable subadditivity of m^*

$$m^*(\Omega_0) \le \sum_{i=1}^{\infty} m^*(R_i) = \sum_{i=1}^{\infty} V(R_i) < m^*(E) + \epsilon_0$$

$$\therefore m^*(\Omega_0) - m^*(E) < \epsilon_0 \le m^*(\Omega_0 / E)$$

$$\therefore m^*(\Omega_0 / E) > m^*(\Omega_0) - m^*(E)$$

2.6 CONTINUITY FROM ABOVE

Theorem :

If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable set and $m(B_1) < \infty$ then $m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k)$

Proof :

$\Rightarrow B_1 \ge B_2 \ge \ldots$. Be collection of measurable sets and $m(B_1) < \infty$

tst $m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k)$

Let $A_k = B_1 / B_k \ \forall k \ge 1$ then $A_1 \subseteq A_2 \subseteq \ldots\ldots$ and A_k's are measurable ($\therefore B_k$'s are measurable)

$$\therefore \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} (B_1 / B_k) = \bigcup_{k=1}^{\infty} \left(B_1 \cap B_k^C\right)$$

$$= B_1 \cap \left(\bigcup_{k=1}^{\infty} B_k^C\right)$$

$$= B_1 \cap \left(\bigcup_{k=1}^{\infty} B_k\right)^C$$

Let $B = \bigcup_{k=1}^{\infty} B_k$

$$\therefore \bigcup_{k=1}^{\infty} A_k = B_1 \cap B^C = B_1 / B$$

$\therefore$ By continuity from below

$m(B_1 / B) = \lim_{k\to\infty} m(A_k)$ .... (*)

$\therefore B$ and B_1 are measurable

$m(B_1/B) = m(B_1) - m(B)$ and

$m(A_k) = m(B_1/B_k)$

$= m(B_1) - m(B_k)$

$\therefore$By (*)

$$m(B_1) - m(B) \lim_{k\to\infty}\left(m(B_1) - m(B_k)\right)$$
$$= m(B_1) - \lim_{k\to\infty} m(B_k)$$
$$\therefore m(B) = \lim_{k\to\infty} m(B_k)\, i.e. \left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k)$$

Example 11 :

Show by an example that for continuity from aboe the assumption $m(E_1) < \infty$ is necessary.

$\Rightarrow$ Let $B_k = (k, \infty)$ then $B_1 \supseteq B_2 \supseteq ...$ and $m(B_k) = \infty \forall_k$ we now show that $\bigcap_{k=1}^{\infty} B_k = \phi$.

Let $x \in \bigcap_{k=1}^{\infty} B_k \Rightarrow x \in B_k = (k, \infty) \forall k$

$\Rightarrow x > k, \forall\, k$

$\Rightarrow \mathbb{N}$ is bounded by x, which is not possible.

$$\therefore \bigcap_{k=1}^{\infty} B_k = \phi$$
$$\therefore 0 = m(\phi) = m(\cap B_k) \neq \infty = \lim_{k\to\infty} m(B_k)$$

Example 12 :

Show that the continuity of measure together with finite additivity of measure implies countable additivity of measure.

$\Rightarrow$ Let $\{E_k\}$ be a countable collection of disjoint measure sets.

Let $A_k = \bigcup_{i=1}^{k} E_i$

Then A_k's are measurable and $A_1 \subseteq A_2 \subseteq \ldots\ldots$

Also $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty}\left(\bigcup_{i=1}^{k} E_i\right) = \bigcup_{k=1}^{\infty} E_k$

$\therefore$By continvity from below, $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$.

But by the finite additive property

$$m(A_k) = m\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} m(E_i)$$

$$\therefore m\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} m(A_k) = \lim_{k\to\infty} \sum_{i=1}^{k} m(E_i)$$

$$= \sum_{i=1}^{k} m(E_i)$$

$$\therefore m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$$

Definition :

For a measurable set E, we say that a property holds atmost everywhere on E, or it holds for almost all $x \in E$, provided there is a subset E_0 of E for which $m(E_0) = 0$ and the property holds for all $x \in E/E_0$.

2.7 BOREL CANTELLI LEMMA

Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}^n$ belong to Atmost finitely many of the E_k's.

Proof :

Let E_0 be the subset of $\mathbb{R}^n$ such that $E_0 = \{x \in \mathbb{R}^n : x \in E_k$ for infinitely many}

$$E_0 = \bigcap_{k=1}^{\infty}\left(\bigcup_{i=k}^{\infty} E_k\right)$$

We sow that $m(E_0)=0$

Let $F_k=\bigcup_{k=i}^{\infty}E_k$

Then $F_1 \supseteq F_2 \geq \ldots\ldots$ and $\bigcap_{k=1}^{\infty}F_k=E_0$

Note that $\sum_{i=1}^{\infty}m(E_i)<\infty$

Let $L=\sum_{i=1}^{\infty}m(E_i)$

$$\Rightarrow m(F_1)=m\left(\bigcup_{i=k}^{\infty}E_i\right)\leq m\left(\bigcup_{i=k}^{\infty}E_k\right)$$
$$\leq \lim_{k\to\infty} m\left(\bigcup_{i=k}^{\infty}E_i\right)$$
$$\leq \lim_{k\to\infty}\sum_{i=k}^{\infty}m(E_i)$$
$$\leq \lim_{k\to\infty}\left(\sum_{i=k}^{\infty}m(E_i)-\sum_{i=1}^{k-1}m(E_i)\right)$$
$$\leq \lim_{k\to\infty}\left(L-\sum_{i=1}^{k-1}m(E_i)\right)$$
$$\leq L-\sum_{i=1}^{\infty}m(E_i)$$
$$\leq L-L$$
$$=0$$

$\therefore m(E_0)=0$

Example 13 :

Show that there is a non-measurable subset in $\mathbb{R}$.

Solution : $\mathbb{R}\,|\,Q=\{x+Q\,|\,x\in\mathbb{R}\}$

WKT any two cosets are either identical or disjoint.

We now show that

If $A\in\mathbb{R}\,|\,Q$ then $A\bigcap[0,1]=\phi$

Let $A=x+Q$

Let q be a rational number in $[-x,-x+1]$ then $x+q\in[0,1]$

Also $x+q\in x\in Q=A$

$\therefore x+q\in A\bigcap[0,1]$
$\Rightarrow A\bigcap[0,1]\neq\phi$

For each $A\in\mathbb{R}/Q$ choose $x_A\in A\bigcap[0,1]$

Let $E=\{x_A/A\in\mathbb{R}/Q\}$

By construction $E\subseteq[0,1]$

Let $X=\bigcup_{q\in[-1,1]\cap\theta} q+E$

$\therefore$ For any $x\in E, q+x\in[-1,2]$

$\Rightarrow q+E\subseteq[-1,2]$

This is true for any $q\in[-1,1]\bigcap Q$

Let $y\in[-1,1]$ then $y\in y+0\in y+Q=A$ (say)

but $x_A\in A$

$\therefore y-x_A=q\in Q(\because x_A\in A\Rightarrow x_A\in y+Q$ for some $q\in Q\}$

$\because y,x_A\in[0,1]$
$\Rightarrow y-x_A\in[-1,1]$
$\Rightarrow q\in[-1,1]\bigcap Q$
$\therefore y\in=q+x_A\in q+E$
$\therefore y\in X\Rightarrow[0,1]\subseteq X\Rightarrow[0,1]\subseteq X\subseteq[-1,2]$

$\therefore$By monotonicity of m^*

$m^*([0,1])\leq m^*(x)\subseteq m^*([-1,2])$

$1\leq m^*(x)\subseteq 3$

If E is measurable then $q+E$ is measurable and $m(E)=m(q+E)$

$$m\left(\bigcup_{q\in[-1,1]\cap Q} E\right)=\sum_{q\in[-1,1]\cap Q} m(q+E)$$

$$m(X) = \sum_{q\in[-1,1]\cap Q} m(E)$$

$$\therefore 1 \le m(X) \le 3$$

$$\Rightarrow 1 \sum_{q\in[-1,1]\cap Q} m(E) \le 3$$

If $m(E) = 0$ then $\sum_{q\in[-1,1]\cap Q} m(E) = 0$

$\therefore 1 \le 0 \le 3$ and if $m(E) \neq 0$ then $\sum_{q\in[-1,1]\cap Q} m(E) = \infty$

Which is contradictin to (1)

$\therefore E$ is not measurable.

2.8 SUMMARY

In this chapter we have learned about.

- Lebesgue measureable sets.
- Construction of Lebesgue measurable sets in $\mathbb{R}^n$
- Properties of Lebesgue measurable sets
- Non-measurable sets

2.9 UNIT END EXERISES

1. Show that the intersection of a countable collection of measurable sets is measurable.
2. Show tht every open and closed subset of $\mathbb{R}^n$ are measurable.
3. Show that a set E is measurable if and only if for each $\in > 0$, there is a closed set F and open set Ω for which $F \subseteq E \subseteq \Omega$ and $m^*(\Omega/F) < \in$
4. Let E be a measurable set in $\mathbb{R}^n$ and $m(E) < \infty$ show that for any $\in > 0$ there exist a compact set $k \subseteq E$ such that $m^*(E/K) < E$.
5. If $\{A_k\}_{k=1}$ is an ascending collection of measurable sets then $M\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$
6. The outer measure of α, the set of all rational number is '0'.
7. Prove that the outer measure of countable set is zero.
8. Show that the outer Measure of an interval is its length.

❖❖❖❖

3

MEASURABLE FUNCTION

Unit Structure :

3.0 OBJECTIVE

After going through this chaper youcan able to know that

- Measurable function
- Properties of measurable function.
- Concept of simple function

3.1 INTRODUCTION

In the previous chapter we have studied about Lebesgue measure of sets of finite and infinite measures. Now we can discuss Lebesgue Measurability of functions. The definition of measurability of function applies to both bounded and unbounded functions. We also discuss simple function and its Approximation.

3.2 MEASURABLE FUNCTIONS

Definition : We say a function *'f'* on $\mathbb{R}^n$ is extended real valued if it take value on $\overline{\mathbb{R}}$.

Definition : A property is said to hold almost everywhere on a measurable set E provided it holds on E/E_0, where E_0 is a subset of E for which $m(E_0)=0$

Example 1 : Let *f* be a function defined on a measurable subset *E* of $\mathbb{R}^n$. Then the following are equivalent.

1. For each real number C, the set $\{x \in E : f(x) > C\}$ is measurable.
2. For each real number C, the set $\{x \in E; f(x) \geq C\}$ is measurable.
3. For each real number C, the set $\{x \in E; f(x) < C\}$ is measurable.
4. For each real number C, the set $\{x \in E; f(x) \leq C\}$ is measurable.

Solution :

$\Rightarrow (1) \Rightarrow (2)$

Suppose for any $C \in \mathbb{R}$

$\{x \in \mathbb{R}, f(x) > C\}$ is measurable ………. (*)

Let $C \in \mathbb{R}$

tst $\{x \in \mathbb{R}; f(x) \geq C\}$ is measurable

Note that $\{x \in E : f(x) \geq C\} = \bigcap_{n=1}^{\infty} \left\{x \in E; f(x) > C - \frac{1}{n}\right\}$ which is a measurable as countable intersection of measurable set is measurable (by (*))

$\therefore \{x \in E : f(x) \geq C\}$ is measurable

$(2) \Rightarrow (3)$

Suppose $\{x \in E : f(x) \geq C\}$ is measurable

$\{x \in E; f(n) < C\} = \{n \in E; f(x) \geq C\}^C$ which is measurable as complement of measurable set is measurable.

$\therefore \{x \in E; f(x) < C\}$ is measurable.

$(3) \Rightarrow (4)$

Suppose $\{x \in E; f(x) < C\}$ is measurable.

Let $C \in \mathbb{R}$

tst $\{x \in E; f(x) \leq C\}$ is measurable.

Note that

$\{x \in E; f(x) \le C\} = \bigcap_{n=1}^{\infty} \left\{x \in E; f(x) < C + \frac{1}{n}\right\}$ which is measurable as countable intersection of measurable set is measurable set.

$\Rightarrow \{x \in E; f(x) \le C\}$ is measurable.

$(4) \Rightarrow (5)$

Suppose $\{x \in E; f(x) \le C\}$ is measurable.

tst $\{x \in E; f(x) > C\}$ is measurable.

Note that

$\{x \in E; f(x) > C\} = \{x \in E; f(x) \le C\}^C$ which is measurable as complement of measurable set is measurable.

$\Rightarrow \{x \in E; f(x) > C\}$ is measurable.

Definition : An extended real-valued function *'f'* defined $E \subset \mathbb{R}^n$ is said to be Lebesgue measurable or measurable, if its domain E is measurable and it satisfies one of the above four statement i.e. For each real number C, the set $\{x \in E; f(x) \le C\}$ is measurable.

Example 2 : Show that a real vaued function that is continuous on its measurable domain is measurable.

Solution :

Let *'f'* be a continuous function

tst *'f'* is measurable

Let $C \in \mathbb{R}$

Note that, $\{x \in E; f(x) > C\} = f^{-1}(C, \infty)$ but (C, ∞) is open subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ is continuous.

$\therefore f^{-1}(C, \infty)$ is open in E

$\therefore f^{-1}(C, \infty) = G \cap E$ for some G is open subset of $\mathbb{R}^n$ but any open-subset of $\mathbb{R}^n$ is measurable and E is given as measurable.

$\therefore f^{-1}(C, \infty) = G \cap E$ is measurable

$\therefore \{x \in E; f(x) > C\} = f^{-1}(C,\infty)$ is measurable

$\therefore$ By definition

f is measurable.

Example 3 : Let *f* be an extended real valued function on *E*. Sho that

1) F is measurable on E and $f = g$ a.e. on E then g is measurable on E.

2) For a measurable subset D of E, f is measurable on E iff the restriction of F to D and E/D are measurable.

Solution : Suppose *f* is measurable and $f = g$ a.e.

Let $A = \{x \in E : f(x) \neq g(x)\}$

Then as $f = g$ a.e. we have $m(A) = 0$

tst g is measurable.

Let $C \in \mathbb{R}, \{x \in E; g(x) > C\}$

$= \{x \in A; g(x) > C\} \cup \{x \in E/A; g(x) > C\}$

$= \{x \in A; g(x) > C\} \cup \{x \in E/A; f(x) > C\} (\because f = g)$

$(\therefore f = g)$

$= \{x \in A; g(x) > C\} \cup \{x \in E; f(x) > C\} \cap (E/A)\}$

But $\{x \in A; g(x) > C\} \subseteq A$ and $m(A) = 0$

$\therefore$ any subset of measure zero set is measurable

$\Rightarrow \{x \in A; g(x) > C\}$ is measurable

$\because f$ is measurable $\Rightarrow \{x \in E; f(x) > C\}$ is measurable

$\because E$ & A are measurable $(\because m(A) = 0)$

$\Rightarrow E/A$ is measurable

$\therefore \{x \in A; g(x) > C\} \cup [\{x \in E; f(x) > C\} \cap (E/A)]$ is measurable

$\Rightarrow \{x \in E; g(x) > C\}$ is measurable

$\Rightarrow g$ is measurable.

2) $\left\{x \in E; f|_D(x) > C\right\} = \left\{x \in D; f(x) > C\right\}$

$$= \left\{x \in E; f(x) > C\right\} \cap D$$

For $f|_D|_E = \left\{x \in E; f|_{D|_E}(x) > C\right\}$

$$= \left\{x \in {}^E|_D; f(x) > C\right\}$$
$$= \left\{x \in E; f(x) > C\right\} \cap {}^E|_D$$

Converse

$= \left\{x \in \in; f(x) > C\right\} = \left\{x \in D; f(x) > C\right\} \cup \left\{x \in E/D; f(x) > C\right\}$

$\Rightarrow \left\{x \in D; f(x) > C\right\}$ is measurable and $\left\{x \in E/D; f(x) > C\right\}$ is measurable.

As union of measurable set is measurable

$\Rightarrow f$ is measurable.

3.3 PROPERTIES OF MEASURABLE FUNCTION

Let f and g be measurable function on E that are finite a.e. on E show that

1) (Linearity) for any $'\alpha'$ and $'\beta'$, $\alpha F + \beta g$ is measurable on F.

2) (Product) fg is measurable on E.

Solution :

Let $E_0 = \left\{x \in E : f(x) = \pm\infty\right\}$ and $g(x) = \pm\infty$ then as f and g are finite a.e. on E we have $m(E_0) = 0$

$\therefore$ the restriction $(f+g)|_{E_0}$ is measurable.

$\therefore$ any extension of $'f + g'$ as an extended real valued function to all of E is also measurable.

Without loss by generality, we may assume that 'f' and 'g' are finite all over E.

Now we first show that 'αf' is measurable for some $\alpha \in \mathbb{R}$.

If $\underline{\underline{\alpha = 0}}$ then αf is a zero function then for any $C \in \mathbb{R}$.

$$\{x \in E : (\alpha F)(x) > C\} = \{x \in E : \alpha f(x) > C\}$$
$$= \begin{cases} \phi \text{ if } C \geq 0 \\ E \text{ if } C < 0 \end{cases}$$

$\therefore \phi$ and E are measurable $\Rightarrow (x \in E; (\alpha F)(x) > C)$ is measurable

$\Rightarrow \alpha F$ is measurable.

Suppose $\underline{\underline{\alpha \neq 0}}$

$$\{x \in E : (\alpha F)(x) > C\} = \{x \in E : \alpha f(x) > C\}$$

$$= \begin{Bmatrix} \{x \in E; f(x) > C/\alpha\} \alpha > 0 \\ \{x \in E; f(x) < C/\alpha\} \alpha < 0 \end{Bmatrix} \text{..............................(*)}$$

$\because f$ is measurable and C & α are red numbers.

$\therefore (*)$ is measurable

$\Rightarrow \{x \in E_i (\alpha f)(x) > C\}$ is measurable

$\Rightarrow (\alpha f)(x)$ is measurable

$\Rightarrow \alpha f$ is measurable .... (1)

We now show that $(f + g)$ is measurable.

Let $C \in \mathbb{R}$

If $(f + g)(x) < C$

$\Rightarrow f(x) + g(x) < C$
$\Rightarrow f(x) < C - g(x)$

$\because Q$ is dense in $\mathbb{R}$, then is an $r \in Q$ such that $f(x) < r < C - g(x)$

$$\therefore \{x \in E; (f + g)(x) < C\} = \bigcup_{r \in Q} \{x \in E; f(x) < r\} \cap \{x \in E : g(x) < C - r\}$$

$\because Q$ is countable and $\{x \in E : f(x) < r\}$ is measurable & $\{x \in E : g(x) < C - r\}$ is measurable

$\therefore$ countable union of measurable set is measurable

$\Rightarrow \{x \in E : (f+g)(x) < C\}$ is measurable

$\Rightarrow f+g$ is measurable .... (2)

From (1) & (2)

$(\alpha f + \beta g)$ is measurable.

2) tpt (fg) is measurable

Note that $fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$

$\because f, g$ are measurable $\Rightarrow f+g, \alpha f$ is measurable it is enough tst square of measurable function is measurable.

Let $C \geq 0$

Then
$\{x \in E; f^2(x) > C\} = \{x \in E; f(x) > \sqrt{C}\} \cup \{x \in E; f(x) < C - \sqrt{C}\}$

Which is union of two measurable set.

$\therefore$by definition, f^2 is measurable,

If $C < 0$

$\{x \in E; f^2(x) > C\} = E$ which is measurable.

$\Rightarrow$In both the case f^2 is measurable

$\Rightarrow (fg)$is measurable.

* **Composition function** (fog)

Example 3:

Let g be measurable real valued function defined on E and f a continuous real valued function defined on all of $\mathbb{R}$ show that the composition fog is a measurable function on E.

Solution :

Given; Let 'g' be measurable function and 'f' be continuous function on $\mathbb{R}$.

Let $g; E \to \mathbb{R}$ be measurable and $f : \mathbb{R} \to R$ be a continuous

Let $C \in \mathbb{R}$

tst :*fog* is measurable

Note that $\{x \in E; (fog)(x) > C\}$

$\therefore (fog)^{-1}\left((C,\infty) = g^{-1}\left(f^{-1}\right)(C,\infty)\right)$

$\because (C,\infty)$ is open subset and f is continuous $\Rightarrow f^{-1}(C,\infty)$ is open in $\mathbb{R}$.

$\therefore f^{-1}(C,\infty) = 0$ for some open subset O of $\mathbb{R}$.

$\because O$ is open in $\mathbb{R}$, we can write

$$O = \bigcup_{i=1}^{\infty}(a_i, b_i)$$

$$\therefore g^{-1}\left(f^{-1}(C,\infty)\right) = g^{-1}\left(\bigcup_{i=1}^{\infty}(a_i,b_i)\right)$$

$$= \bigcup_{i=1}^{\infty}\left(g^{-1}(a_i,b_i)\right)$$

$$= \bigcup_{i=1}^{\infty}\left(\{x \in E : g(x) > a_i\} \cap \{x \in E : g(x) > b_i\}\right)$$

$\Rightarrow \{x \in E : g(x) > a_i\}$ is measurable and $\{x \in E : g(x) > b_i\}$ is measurable.

$\Rightarrow$ countable union of measurable set is measurable set.

$\{x : (fog)(x) > C\}$ is measurable

$\therefore fog$ is measurable function on E.

Check your Progress :

If f is measurable, then show that

1) f^k is measurable for all integer $K \geq 1$

2) $f + \lambda$ is measurable for a given constant $\lambda \in \mathbb{R}$

3) λf is measurable for a given constant $\lambda \in \mathbb{R}$

4) $|f|$ is measurable

5) $\sup f_n(n), \inf f_n(n), \lim\limits_{n\to\infty} \sup_A f_n(n) \lim\limits_{n\to\infty} \inf f_n(n)$ are measurable.

Definition :

For a sequence $\{f_n\}$ of functions with common domain E,a function f on E and a subset A of E, we say that

1) The sequence $\{f_n\}$converges to 'f' point wise E, on A provided $\lim_{n\to\infty}\{f_n\}(n)=f(x)$ for all $x\in A$

2) The sequence $\{f_n\}$ converges to 'f' point wise a.e. on A provided it converges to F pointwise on A/B where $m(B)=0$

3) The sequence $\{f_n\}$ converges to 'f' uniformly on A provided for each $\in>0,\exists N\in\mathbb{N}$ such that $|f-f_n|<\in$ on a for all $n\geq N$.

Theorem :

Let $\{f_n\}$ be a sequence of measurable function on E that converges point-wise a.e. on E to the function f, show that f is measurable.

Proof :

Let E_0 be a subset of E with $m(E_0)=0$ and $f_n\to f$ on E/E_0.

$\therefore m(E_0)=0$ & we have 'f' is measurable on E iff $^{f}|_{E-E_0}$ is measurable.

$\therefore$By replacing E by $E-E_0$ we may assume that the $\{f_n\}$ converges to f on E

tst f is measurable

Let $C\in\mathbb{R}$

tst $\{x\in E; f(x)<C\}$ is measurable

$\{x\in E; f(x)<C\}=\{x\in E; \lim_{n\to\infty} f(x)<C\}$ but

$\lim_{n\to\infty} f(x)<C$ iff there are natural nos. n and k for which $f_j(x)<C-\frac{1}{n}\quad \forall j\geq k$

$$\therefore \{x\in E; f(x)<C\}=\bigcup\left(\bigcap\left\{x\in E; f_j(x)<C-\frac{1}{n}\right\}\right)$$

$$1\leq k,n<\infty$$

note that $\bigcap_{j=k}^{\infty}\left\{x\in E; f_j(x)<C-\frac{1}{n}\right\}$ is measurable.

Countable union of measurable set is measurable

$\Rightarrow \{x\in E; f(x)<C\}$ is measurable.

Simple Functions :

Definitions :

A real-valued functions ϕ defined on a measurable set E is said to be simple if it is measurable and takes only a finite number of values.

If ϕ is simple, has domain E and takes the distinct values $C_1 ...., C_n$ then $\phi=\sum_{k=1}^{n} C_k \chi_{E_k}$ on E, where $E_k=\{x\in E; \phi(x)=C_k\}$.

This particular expression of ϕ is a linear combination of characteristic functions is called the canonical representation of the simple function ϕ.

Theorem : The simple Approximation Lemma

Let 'f' be a measurable real valued function on E. Assume 'f' is bound on E. Then for each $\in>0$, there are simple function $\phi_{\in}$ and Ψ_E defined on E which have the following approximation properties:

$\phi_E \le f \le \Psi_E$ and $0\le \Psi_E-\phi_E<E$ on E.

Proof :

Suppose $f: E\to R$ is bounded measurable f_n

$\therefore f$ is bounded, $\exists M>0$ such that $|f(x)|<M \quad \forall x\in E$

Let (c,d) be an open interval s.t. $f(E)\le(c,d)$ ($\therefore f$ is bounded)

Let $\in>0$

Consider the partition

$C=y_0<y,<....<y_{n=d}$ of $[c,d]$ with $y_k-y_{k-1}<\in, 1\le k\le n$

Define $\phi_E=\sum_{k=1}^{n} y_{k-1}\chi_{E_k}, \Psi_E=\sum_{k=1}^{n} y\,\chi_{E_k}$ where $E_k=f^{-1}([y_{k-1}, y_k])$

Note that $E_k = f^{-1}([y_{k-1}, y_k])$

$$= \{x \in E; f(x) \in [y_{k-1}, y_k]\}$$
$$= \{x \in E; y_{k-1} \le f(x) < y_k\}$$
$$= \{x \in E; f(x) \ge y_{k-1}\} \cap \{x \in E : f(x) < y_k\}$$

which is measurable. ($\therefore f$ is measurable)

$\therefore \chi_{E_k}$ are measurable, $1 \le k \le n$

$\Rightarrow \phi_{\in}$ & Ψ_E are measurable and takes only finite number of values

$\therefore \phi_{\in}$ & Ψ_E are simple functions.

Let $x \in E \Rightarrow f(x) \in (c,d)$

$\therefore \exists k$ s.t. $y_{k-1} \le f(x) < y_k$

$\therefore \phi_E(x) = y_{k-1} \le f(x) < y_k = \Psi_E(x)$ ………………… (1)

$\Rightarrow \phi_E(x) \le f(x) \le \Psi_E(x)$

Also by (1) $0 \le \Psi_E(x) - \phi_E(x) = y_k - y_{k-1} < \in$

Theorem : The Simple Approximation Theorem

An extended real valued function 'f' on a measurable set E is measurable if and only if there is a sequence $\{\phi_n\}$ of simple functions on E which converges point-wise on E to f and has the property that $|\phi_n| \le |f|$ on E for all 'n'.

If 'f' is non negative, we way choose $\{\phi_n\}$ to be increasing.

Proof :

Suppose f is measurable

Case (1) Assume $f \ge 0$

Let $n \in \mathbb{N}$, Define $E_n = \{x \in E; f(x) < n\}$

Then $f|_{E_n}$ is a bounded function.

$\therefore$By simple Approximation Lemma for $\in = \frac{1}{n}, \exists$ simple functions $\phi_{\in}$ & Ψ_E such that $\phi_{\in} \le f|_{E_n} \le \Psi_n$ and $0 \le \Psi_n - \phi_n < 1/n$.

We extend ϕ_n on E defining $\phi_n(x)=n$ if $f(x)\geq n$ construct the sequences $\{\phi_n\}$.

We now show that $\phi_n \to f$ pointwise on E

(1) If 'f' is finite

$\therefore \exists N \in \mathbb{N}$ such that $f(x) < \mathbb{N}$

$\Rightarrow x \in E_N$
$\therefore \phi_N(x) \leq f(x) \leq \Psi_N(x)$
$\Rightarrow f(x) - \phi_N(x) \leq \Psi_N(x) - \phi_N(x) < \frac{1}{N}$
$\Rightarrow f(x) - \phi_N(x) < \frac{1}{n} \forall\, n \geq \mathbb{N}$

$\Rightarrow \phi_n(x) \to f(x)$ as $n \to \infty$

(2) If $f = \infty$

$f(x) > N$ for any $N \in \mathbb{N}$

$\Rightarrow \phi_n(x) = n$
$\Rightarrow \lim_{n\to\infty} \phi_n(x) = \infty = f$

Case (2) 'f' is any measurable function

Define $f_{(x)}^{-1} = \max\{f(x), 0\}$

$f^{-1}(x) = \min\{f(x), 0\}$
$\Rightarrow f(x) = f^{+}(x) + \left(f^{1}(x)\right)$

$\because f^{+}$ and $-f^{-}$ are non-negative measurable function.

$\therefore$ By Case (1), $\exists$ a sequence of simple functions $\{\phi_n\} \& \{\psi_n\}$ s.t. $\phi_n \to f^{+}$ pointwise and $\Psi_n \to f^{-}$ pointwise.

$\therefore \phi_n - \Psi_n \to f$ pointwise

$\therefore \phi_n$ and Ψ_n are simple function $\forall\, n$

$\Rightarrow \phi_n - \Psi_n$ a's also a simple function $\forall\, n$.

3.4 EGOROFF'S THEOREM

Theorem Statement (Assume E has finite measure)

Let $\{f_n\}$ be a sequence of measurable functions one that converges pointwise on E to the real valued function f. Then for each $\in > 0$ there is a closed set F contained in E for which $\{f_n\} \to f$ uniformly on F and $m(E/F) < \in$.

Proof :

Since $f_n \to f$ pointwise on E, for $\in > 0$, and $x \in E, \exists K \in \mathbb{N}$ such that $\left|f_j(x) - f(x)\right| < \in \ \forall\ j \geq K$ .... (1)

Since we want to get a region of uniform convergence, we accumulate all $x \in E$ for which the same N holds for a fixed E.

For any pair k & n define

$$E_k^n = \left\{x \in E : \left|f_j(x) - f(x)\right| < \frac{1}{n},\ \forall\ j \geq K\right\}$$

Not all E_k^n are empty otherwise it will contradict pointwise converges of $\{f_n\}\ \forall x \in E$.

$\because f_j$ and f are measurable $\Rightarrow E_k^n$ is measurable.

Note that from fixed n

$E_k^n \subseteq E_{k+1}^n$ and $\bigcup_{k=1}^{\infty} E_k^n = E$

$\therefore$ By the confinuity of measure.

$m(E) = \lim_{K \to \infty} m\left(E_k^n\right)$

$\because m(E)$ is finite, i.e. $m(E) < \infty$, for the above, $\in > 0$, such that $m(E) - M\left(E_k^n\right) < \frac{\in}{2^{n+1}}$

$\Rightarrow m\left(E/E_{k_n}^n\right) < \frac{\in}{2^{n+1}}$ by countable additivity).

By construction for each $x \in E_{k_n}^n$

$\left|f_j(x) - f(n)\right| < \frac{1}{n}\ \forall_j \geq k_n$ .... (2)

Let $\boxed{A=\bigcap E_{k_n}^{n}}$

We show that $f_n \to f$ uniformly on A

Let $\in > 0$ choose $n_0 \in \mathbb{N} \frac{1}{n_0} < \in$

By (2)

$\left|f_j(x)-f(n)\right| < \frac{1}{n_0} \ \forall_j \geq k_{n_0}$ on $E_{k_{n_0}}^{n_0}$

$\because A \subseteq E_{Kn_0}^{n_0}$

$\Rightarrow \left|f_j(x)-f(n)\right| < \frac{1}{n_0} < \in \ \forall_j \geq k_{n_0}$ on A

$\therefore f_n \to f$ uniformly on A.

Now $m(E/A) = m\left(E \cap A^C\right)$

$$= m\left(E \cap \left(\cup\left(E_{k_n}^{n}\right)^C\right)\right)$$

$$= m\left(\bigcup_{n=1}^{\infty}\left(E \cap \left(E_{k_n}^{n}\right)^C\right)\right)$$

$$\leq \sum_{n=1}^{\infty} m\left(E/E_{k_n}^{n}\right)$$

$$< \sum_{n=1}^{\infty} \frac{\in}{2^{n+1}} = \frac{\in}{2}$$

$\because E_{k_n}^{n}$ are measurable and countable intersection of measurable set is measurable.

$\Rightarrow A$ is measurable.

$\therefore \exists$ a closed subset F of A s.t. $m(A/F) < \in/2$

$$\therefore m(E/F) = m\left((E/A) \cup (A/F)\right)$$
$$= m(E/A) + m(A/F)$$
$$< \in/2 + \in/2 = \in$$

$\because f_n \to f$ uniformly on A & $F \subseteq A$

$\Rightarrow f_n \to f$ uniformly on F.

Examples 4 : Let f be a simple function defined on E. Then for each $\in > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which $f = g$ on F & $m(E/F) < \in$.

Solution:

Let f be a simple function defined on $E \subseteq \mathbb{R}$

Let f takes the values $a_1, ..., a_n$ be the distance values taken by 'f'.

$$\therefore f = \sum_{i=1}^{n} a_i \chi_{E_i}$$

Where $E_i = \{x \in E : F(x) = a_i\}$

Note that $E = \bigcup_{i=1}^{n} E_i$

$\because a_k 's$ are distinct $\Rightarrow E_k 's$ are disjoint

$\because f$ is measurable $\Rightarrow F_{ks}$' are measurable

Let $\in > 0$

For each $k, 1 \le k \le n$, E_k is measurable $\Rightarrow \exists$ closed subset F_k of E_k such that $m(E_k / F_k) < \frac{\in}{n}$

Let $F = \bigcup_{j=1}^{n} F_j$

$\Rightarrow F$ is closed

$$\begin{aligned}
m(E/F) &= m(E \cap F^C) \\
&= m\left(\left(\bigcup_{k=1}^{n} E_k\right) \cap F^C\right) \\
&= m\left(\bigcup_{k=1}^{n} (E_k \cap F^C)\right) \\
&= m\left(\bigcup_{k=1}^{n} \left(E_k \cap \left(\bigcap_{j=1}^{n} F_j^C\right)\right)\right) \\
&= m\left(\bigcup_{k=1}^{n} \left(\bigcap_{j=1}^{n} (E_k \cap F_j^C)\right)\right) \\
&= m\left(\bigcup_{k=1}^{n} (E_k \cap F_j^C)\right) \\
&= m\left(\bigcup_{k=1}^{n} (E_k / F_k)\right) \\
&\le \sum_{k=1}^{n} m(E_k / F_k) < \sum_{k=1}^{n} \frac{\in}{n}
\end{aligned}$$

$$< \frac{\in}{n} \cdot n$$

$$< \in$$

Define $g : F \to \mathbb{R}$ by $g(x) = a_i$ if $x \in F_i$

$\because E_i's$ are disjoint $\Rightarrow F_i's$ are disjoint g is well defined and $f = g$ on F we now show that 'g' is continuous on f then $F^1 = \bigcup_{i \neq k} F_i, F^1 \cap F_k = \phi$ and $x \in F_k$.

$\because \exists$ an open interval $I \subseteq F_K$ containing $'x'$ $I \cap F^1 = \phi$

$\therefore g(y) = a_k \ \forall y \in I$

$\therefore |g(y) - g(x)| = |a_k - a_k| = 0 < \in \ \forall_y \in I$

$\therefore g$ is continuous at x.

This is true for any $x \in F$

$\therefore g$ is continuous on F.

We can extend this continuous function 'g' on the closed set F to a continuous function on $\mathbb{R}$.

Let the new function be 'g' then 'g' is continuous on $\mathbb{R}$ and $g = F$ on f and $m(E/F) < \in$.

3.5 LUSIN'S THEOREM

Statement :

Let f be a real valued measurable function defined on E then for each $\in > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which $f = g$ on f and $m(E \mid f) < \in$.

Proof :

Let f be a real valued measurable function defined on E.

1) $m(E)$ is finite

$\therefore$ by simple Approximation theorem $\exists$ a sequence $\{\phi_n\}$ of simple function on E such that $\phi_n \to f$ and $|\phi_n| \leq |f|$ on $E \ \forall_n$.

$\therefore$ for each $n \in \mathbb{N}$ there is a continous function $'g_n'$ on $\mathbb{R}$ and a closed set f f_n conained in E for which $\phi_n = g_n$ on f_n & $m(E/F_n) < \frac{\in}{2^{n+1}}$.

$\because \phi_n \to f$ pointwise on E

By Egoroff's theorem

$\exists$ a closed set f_0 contained in E such that $\{\phi_n\} \to F$ uniformly on F_0 and $m(E/F_0) < \in\!/\!_2$.

Let $F = \bigcap_{h=0}^{\infty} F_n$

F is closed as countable intersection of closed sets.

Each ϕ_n is uniformly on $F(\because F \subseteq F_0)$

$\because \phi_n$ is continuous

$\Rightarrow f$ is continuous on F

i.e. $f/_F$ is continous.

We can extend $f/_F$ to a continuous function 'g' on $\mathbb{R}$.

Then $f = g$ on F

and $m(E/F) = m(E \cap F_n^C)$

$$= m\left(\bigcup_{n=0}^{\infty} E/F_n\right)$$

$$= m\left(\left(E|_{F_{n_0}}\right) \cup \left(\bigcup_{n=1}^{\infty} E/F_n\right)\right)$$

$$= m\left(\left(E/F_{n_0}\right) + \sum_{n=1}^{\infty} m(E/F_n)\right)$$

$$< \in\!/\!_2 + \sum_{n=1}^{\infty} \frac{\in}{2^{n+1}}$$

$$< \in\!/\!_2 + \in\!/\!_2$$

$$< \in$$

3.6 SUMMARY

In this chapter we have learned about

- Concept of measurable functions.
- Properties of measurable functions
- Simple functions & ith Approximation Theorem
- Egoroffs Theorem and LUSIN Theorem of Measurable function.

3.7 UNIT END EXERCISE

1. Pure that "every continuous function is measurable".
2. Show that the sum and Product of two simple function are simple function
3. Show that if $f,[0,\infty]\to\mathbb{R}$ is differentiable, than f^1 is measurable.
4. Prove that if f is a measurable function X, than the set $f^{-1}(\infty)=\{x\in X|f(x)=\infty\}$ is measurable.
5. Prove that if $f:[0,1]\to\mathbb{R}$ is continous atmost everywhere than f is measurable.
6. State and prove Egoroff's Theorem of measurable function.
7. State and Prove Lusin's Theorem of real valued measurable function.
8. If 'f' is measurable then show that $f^1(C)$ is measurable, $C\in\mathbb{R}$.
9. If f is measurable then show that $\dfrac{\lambda f}{(-f)}$ is measurable.
10. Show that χ_A is Measurable if and only if the set A is measurable.

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4

LEBESGUE INTEGRAL

Unit Structure :

4.0 OBJECTIVES

After going through this chapter you can able to know that

- Lebesgue integral
- Lebesgue integral of a simple function
- Lebesgue integral of a bounded measurable function
- The general Lebesgue integral

4.1 INTRODUCTION

We have already learned simple functions, measurable functions. Now here we are going to discuss. Lebesgue integral on this function. Lebesgue integral over come on the class of all Riemannintegrable functions & the limitation of operations. So now we defined the general notation of the Lebesgue integral on $\mathbb{R}^n$ step by step.

4.2 LEBESGUE INTEGRAL OF SIMPLE FUNCTION

Definition :

For a simple function ϕ with canonical representation $\phi(x)=\sum_{i=1}^{n} a_i X_{E_i}$ defined on a set of finite measure E, we define the integral of ϕ over E by $\int_E \phi = \sum_{i=1}^{n} a_i m(E_i)$.

Example 1 : Let $\{E_i\}_{i=1}^{n}$ be a finite disjoint collection of measurable subset of a set of finite measure E. For $1 \le i \le n,$Let $a_i \in \mathbb{R}$.

If $\phi = \sum_{i=1}^{n} a_i \chi_{Ei}$ on E, than $\int_E \phi = \sum_{i=1}^{n} a_i m(E_i)$.

Solution :

Let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ s.t. E_i's are pairwise disjoint which may not be in canonical form.

Let $\{b_j\}_{j=1}^{k}$ be distinct elements of $\{a_i, ...a_n\}$.

Define $F_j = \bigcup_{i \in I_j} E_i$ where $I_j = \{i : a_i = a_j\}$.

Note that F_j's are disjoint.

$\therefore m(F_j) = \sum_{i \in I_j} m(E_i)$

$\therefore \phi = \sum_{j=1}^{k} b_j \chi_{F_j}$ is a canonical representation of ϕ.

$\therefore$ By definition $\int_E \phi = \sum_{j=1}^{k} b_j m(F_j)$

$$= \sum_{j=1}^{k} b_j \left(\sum_{i \in I_j} m(E_i) \right)$$

$$\int_E \phi = \sum_{i=1}^{n} a_i m(E_i)$$

4.2.1 Theorem (Properties of integral simple function)

Let ϕ and Ψ be simple functions defined on a set of finite measure.

Then

1) Linearity : For any 'α'and 'β'

$$\int_E (\alpha\phi + \beta\Psi) = \alpha \int_E \phi + \beta \int_E \Psi$$

Proof :

Let $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ and $\Psi = \sum_{j=1}^{n} b_j \chi_{B_j}$ be canonival representation of ϕ and Ψ respectively.

$C_{ij} = A_i \cap B_j, 1 \le i \le n, 1 \le j \le m$

then $\phi = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i \chi C_{ij}$ and $\Psi = \sum_{i=1}^{n}\sum_{j=1}^{m} b_i \chi C_{ij}$ ………… (1)

$\therefore$ By definition $\int_E \phi = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i m(C_{ij})$ and $\int_\in \Psi = \sum_{i=1}^{n}\sum_{j=1}^{m} b_j m(C_{ij})$

By (1)

$$\alpha\phi + \beta\Psi = \sum_{i=1}^{n}\sum_{j=1}^{m} (\alpha a_i + \beta b_j)\chi C_{ij}$$

$\therefore$ By definition

$$\int_E \alpha\phi + \beta\Psi = \sum_{i=1}^{n}\sum_{j=1}^{m} (\alpha a_i + \beta b_j) m(C_{ij})$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha a_i\, m(C_{ij}) + \sum_{i=1}^{n}\sum_{j=1}^{m} \beta b_j\, m(C_{ij})$$

$$= \alpha\left(\sum_{i=1}^{n}\sum_{j=1}^{m} a_i m(C_{ij})\right) + \beta\left(\sum_{i=1}^{n}\sum_{j=1}^{m} b_i m(C_{ij})\right)$$

$$= \alpha\int_E \phi + \beta\int_E \Psi$$

2) Monotonicity

If $\phi \le \Psi$ on E then $\int_E \phi \le \int_E \Psi$

Proof :

Suppose $\phi \le \Psi$ on E

tst $\int_E \phi \le \int_E \Psi$

Let $f = \Psi - \phi \ge 0$

$\therefore$ By linearity property

$$\int_E \Psi \leq \int_E \phi = \int_E (\Psi - \phi) = \int_E f \geq 0$$

$$\therefore \int_E \Psi \geq \int_E \phi$$

3) Additivity :

For any two disjoint subset $A, B \subseteq E$ with finite measure, $\int_{A \cup B} \phi \geq \int_A \phi \int_B \phi$

Solution :

$$\int_{A \cup B} \phi = \int_E \phi \chi_{A \cup B}$$

$$= \int_E \phi(\chi_A + \chi_B)$$

$$= \int_E \phi \chi_A + \int_E \phi \chi_B$$

$$= \int_A \phi + \int_B \phi$$

4) Triangle inequality : If ϕ is a simple $|\phi|$ and $\left| \int_E \phi \right| \leq \int_E |\phi|$.

Solution : Let ϕ be a simple function and $\phi = \sum_{i=1}^{n} a_i \chi_{Ai}$ be canonical representation of ϕ.

Then $|\phi| = \sum_{i=1}^{n} |a_i| \chi_{Ai}$ which is a simple function.

By Definition

$$\int_E \phi = \sum_{i=1}^{n} a_i m(Ai)$$

$$\therefore \left| \int_E \phi \right| = \left| \sum_{i=1}^{n} a_i m(Ai) \right|$$

$$\leq \sum_{i=1}^{n} |a_i m(Ai)| \text{ (by triangle inequality)}$$

$$\leq \sum_{i=1}^{n} |a_i| |m(Ai)|$$

$$\leq \sum_{i=1}^{n} |a_i| |m(Ai)|$$

$$\leq \int_E |a_i|$$

5) If $\phi = \Psi$ a.e. on E, then $\int_E \phi = \int_E \Psi$

Solution : Suppose $\phi = \Psi$ a.e. on F

Let $E_0 = \{x \in E; \phi(a) \neq \Psi(x)\}$

Then $m(E_0) = 0$ and on $E/E_0; \phi = \Psi$

Let $\phi = \sum_{i=1}^{n} a_i \chi_{Ai}$ and $\Psi = \sum_{j=1}^{n} b_j \chi_{Bj}$ be canonical representation of ϕ and Ψ representation.

$\therefore$By definition

$$\int_E \phi = \sum_{i=1}^{n} a_i m(Ai)$$

$$= \sum_{i=1}^{n} a_i m(A_i \cap E_0) \cup (A_i \cup E \mid E_0)$$

$$= \sum_{i=1}^{n} a_i m(A_i \cap E_0) + \sum_{i=1}^{n} a_i m(A_i \cap (E) E_0)$$

$$= O + \sum_{i=1}^{n} a_i m(A_i \cap (E/E_0))$$

$$\int_E \phi = \int_{E|E_0} \phi$$

Similarly

$$\int_E \Psi = \int_{E/E_0} \psi$$

$\because \phi = \Psi$ on E/E_0

$$\therefore \int_E \phi = \int_E \psi$$

* Lebesgue integral of a bounded measurable function on a set of finite measure.

We now extend the notion of integral of simple function to a bounded measurable function on a set of finite measure.

Let 'f' be a bounded real -valued function defined on a set of finite measure E. We define the lower and upper Lebesgue integral respectively, of 'f' over E to be $\sup\left\{\int_E \phi : \phi \text{ simple and } \phi \le f \text{ on } E\right\}$ and $\inf\left\{\int_E \Psi : \Psi \text{ simple and } f \le \Psi \text{ on } E\right\}$.

Since 'f' is bounded by the monotonicity property of the integral for simple functions, the lower and upper integral are finite and the lower integral $\le$ the upper integral.

4.3 DEFINITION

A bounded function 'f' on a domain E of finite measure is said to be Lebesgue integrable over E if its upper and lower Lebesgue integrals over E are equal. The common value of the upper and lower integrals is called the Lebesgue integrals or simply the integral, of 'f' over E and is denoted by $\int_E f$.

Example 2 : Show that a non negative bounded measurable function on a set E of finite measure is integrable E of finite measure is integrable over E.

Solution : Let 'f' be a bounded measurable function defined on E. where $m(E) < \infty$.

$\therefore$By simple Approximation Lemma

For $n \in \mathbb{N}, \exists$simple function ϕ_n and Ψ_n such that $\phi_n \le f \le \Psi_n$ and $0 \le \Psi_n - \phi_n < \frac{1}{n}$.

$$\therefore \int_E \Psi_n - \int_E \phi_n = \int_E \Psi_n - \phi_n < \int_E \frac{1}{n} = \frac{1}{n} m(E)$$

But, $\sup\left\{\int \phi; \phi \text{ simple}, \phi \le f\right\} \ge \int_E \phi_n$ and

$$\inf\left\{\int \Psi; \Psi \text{ simple}, f \le \Psi\right\} \le \Psi_n$$

$$0 \le \inf\left\{\int_E \Psi; \Psi\ simple, \Psi \ge f\right\} - \sup\left\{\int_E \phi; \phi\ simple, \phi \le f\right\}$$

$$\le \int_E \Psi_n - \int_E \phi_n < \frac{1}{n} m(E)$$

This is true for any $n \in \mathbb{N}$ and $m(E) < \infty$

$$\therefore \inf\left\{\int_E \Psi; \Psi\ simple, \Psi \ge f\right\}$$

$$= \sup\left\{\int_E \phi; \phi\ simple, \phi \le f\right\}$$

$\Rightarrow f$ is Lebesgue integrable over E.

Example :

Let 'f' be a bounded measurable function on a set E of finite measure. Show that if $\int_E f = 0$ then $f = 0$ a.e.

Solution : Suppose $\int_E f = 0$ and $f \ge 0$

tst $f = 0$ a.e.

Let $E_n = \left\{x \in E; f(x) > \frac{1}{n}\right\}$ then $\frac{1}{n}\chi_{E_n}(x) < f(x)$.

By monotonicity,

$$\int \frac{1}{n}\chi_{E_n}(x) < \int_E f = 0$$

$$\Rightarrow \frac{1}{n} m(E_n) < 0$$

$$\Rightarrow m(E_n) = 0$$

But $E_0 = \{x \in E; f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$

$$\therefore m(E_0)$$

$\Rightarrow f = 0$ a.e. over E.

4.3.1 Properties of integral of bounded function :

Theorem : Let 'f' and 'g' be bounded measurable functions defined on a set of finite measure E then

1) Linearity : for any $'\alpha'$ and β

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$$

Proof : Let f, g be bounded functions, $\alpha, \beta \in \mathbb{R}$

tst $\int_E \alpha f + \beta g = \alpha \int_E f + \beta \int_E g$

It is enough tst $\int_E \alpha f = \alpha \int_E f$ and $\int_E f + g = \int_E f + \int_E g$

If $\alpha = 0$ then $\alpha f = 0$

$\Rightarrow \int_E \alpha f = 0 = \alpha \int_E f$

Suppose $\alpha \neq 0$

$\therefore f$ is bounded $\Rightarrow \alpha f$ is bounded $\Rightarrow \alpha f$ is lebesgue integrable.

Let $\alpha > 0$

$\therefore \int_E \alpha f =$ upper lebesgue integrable of $'\alpha f'$

$$= \inf\left\{\int \Psi : \Psi \textit{ is simple} \,\&\, \Psi \geq \alpha - f\right\}$$

$$= \inf\left\{\alpha \int_E \left(\Psi/\alpha\right) : \Psi \textit{ simple} \,\&\, \Psi/\alpha \geq f\right\}$$

$$= \alpha \inf\left\{\int_E \left(\Psi/\alpha\right) : \Psi/\alpha \textit{ simple}, \Psi/\alpha \geq f\right\}$$

$$= \alpha \inf\left\{\int_E \phi : \phi \textit{ simple}, \phi \geq f\right\}$$

$$= \alpha \int_E f$$

Let $\alpha < 0$

Similarly for lower Lebesgue integral of αf

$$\therefore \int_E \alpha f = \alpha \int_E f$$

We now show that $\int_E f + g = \int_E f + \int_E g$

Let Ψ_1 and Ψ_2 be simple functions on E such that, $f \leq \Psi_1$ and $g \leq \Psi_2$ then Ψ_1 + Ψ_2 is a simple function and $f + g \leq \Psi_1 + \Psi_2$

$\therefore f$ and g are bounded $\Rightarrow f + g$ is bounded.

$\Rightarrow f + g$ is Lebesgue integrable

$\therefore$By definition

$$\int_E f + g = \inf\left\{\int \Psi; f + g \leq \Psi, \Psi \textit{ is simple}\right\}$$

$$\leq \int_E \Psi_1 + \Psi_2 = \int_E \Psi_1 + \int_E \Psi_2$$

This is true for any Ψ_1, Ψ_2 simple with $f \leq \Psi$, and $g \leq \Psi_2$

$\Rightarrow \int f + g$ is lower bound of

$$\left\{\int_E \Psi_1 + \int_E \Psi_2; \Psi_1 \geq f_1, \Psi_2 \geq g, \Psi_1, \Psi_2 \textit{ simple}\right\}$$

$$\Rightarrow \int_E f + g \leq \inf\left\{\int_E \Psi_1 + \int_E \Psi_2; \Psi_1 \geq f, \Psi_2 \geq g, \Psi_1, \Psi_2 \textit{ simple}\right\}$$

$$\leq \inf\left\{\int \Psi_1; \Psi_1 \geq f, \Psi_1 \textit{ simple}\right\} + \inf\left\{\int \Psi_2; \Psi_2 \geq g, \Psi_2 \textit{ simple}\right\}$$

$$\leq \int_E f + \int_E g$$

$$\therefore \int_E f + g \leq \int_E f + \int_E g$$

For the reverse inequality

Let ϕ_1 and ϕ_2 be simple function for which $\phi_1 \leq f$ & $\phi_2 \leq g$ on E then

$\phi_1 + \phi_2 \leq f + g$ and $\phi_1 + \phi_2$ is simple

$$\therefore \int_E f+g = \sup\left\{\int_E \phi ; f+g \geq \phi, \phi \; simple\right\}$$

$$\geq \int_E \phi_1 + \phi_2$$

$$\geq \int_E \phi_1 + \int_E \phi_2$$

This is true for any ϕ_1, ϕ_2 simple with $f \geq \phi_1$ & $g \geq \phi_2$

$\Rightarrow \int_E f+g$ is upper bound of

$$\left\{\int_E \phi_1 + \int_E \phi_2 ; \phi_1 \leq f, \phi_2 \leq g, \phi_1, \phi_2 \; simple\right\}$$

$$\Rightarrow \int_E f+g \geq \sup\left\{\int_E \phi_1 + \int_E \phi_2 ; \phi_1 \leq f, \phi_2 \leq g, \phi_1, \phi_2 \; simple\right\}$$

$$\geq \sup\left\{\int \phi_1 ; \phi_1 \leq f, \phi_1 \; simple\right\} + \sup\left\{\int_E \phi_2 ; \phi_2 \leq g, \phi_2 simple\right\}$$

$$\leq \int_E f + \int_E g$$

$$\therefore \int_E f+g \geq \int_E f + \int_E g$$

$$\therefore \int_E f+g = \int_E f + \int_E g$$

2) Monotonicity : If $f \leq g$ on E, then $\int_E f \leq \int_E g$

Proof

Suppose f and g are bounded mesurable function on a set E of finite measurable function and $f \leq g$

tst $\int_E f \leq \int_E g$

Let $h = f - g \geq 0$

$\Rightarrow h$ is non-negative bounded function.

$\therefore$By linearity

$$\int_E g - \int_E f = \int_E g - f = \int_E h$$

$\therefore h$ is bounded & $h \geq 0$

$\Rightarrow h \geq \Psi$ where $\Psi = 0$ simple function

But $\int_E h = \sup\left\{\int_E \Psi; simple, \Psi \leq h\right\}$

$$\Rightarrow \int_E h \geq \int_E \Psi = 0 * m(E) = 0$$

$$\therefore \int_E g - \int_E f = \int_E h \geq 0$$

$$\therefore \int_E g \geq \int_E f$$

3) Additivity : For any two disjoint subsets, $A, B \subseteq E$ with finite measure.

$$\int_{A\cup B} f = \int_A f + \int_B f$$

Proof :

Let 'f' be bounded measurable function on a set E of finite measure and A,B disjoint subsets of E.

tst $\int_{A\cup B} f = \int_A f + \int_B f$

$\because f$ is bounded measure.

$\Rightarrow f\chi_{A\cup B}, f\chi_A, \chi_B$ are bounded measurable functions.

$$\therefore \int_{A\cup B} f = \int_E \chi_{A\cup B} = \int_E f(\chi_A + \chi_B)$$

$$= \int_E f\chi_A + f\chi_B$$

$$= \int_E f\chi_A + \int_E f\chi_B$$

$$\int_{A\cup B} f = \int_A f + \int_B f$$

4) Triangle inequality : Let f be a bounded measurable function on a set of finite measure E, Then $\left|\int_E f\right| \leq \left|\int_E |f|\right|$.

Proof :

Let f be bounded measurable function on a set E of finite measurable

$\Rightarrow |f|$ is measurable and bounded on E.

Note that

$\therefore -|f| \le f \le |f|$

$\therefore$ By monotonicity and linearity

$$-\int_E |f| \le \int_E f \le \int_E |f|$$

$$\Rightarrow \left|\int_E f\right| \le \int_E |f|$$

Example :

Let $\{f_n\}$ be a sequence of bounded measurable functions on a set of finite measure E. Show that if $f_n \to f$ uniformly on E, then $\lim_{n\to\infty} \int_E f_n = \int_E f$

Solution : Let $\{f_n\}$ be a sequence of bounded measurable function on a set E of finite and $f_n \to f$ uniformly on E

tst $\lim_{n\to\infty} \int_E f_n = \int_E f$

i.e. $\int_E f_n = \int_E f$

$\because f_n \to f$ uniformly on E

$\Rightarrow$ for a given $\in > 0$, $\exists_{n0} \in \mathbb{N}$

$\forall x \in E, |f_n(x) - f(x)| < \in / m(E) \; \forall n \ge n_0$

i.e. $|f_n - f| < \frac{\in}{m(E)} \forall n \ge n_0$ on E

For $n \ge n_0$

Now $\left|\int f_n - \int f\right| = \left|\int_E f_n - f\right|$

$$\leq \int_E |f_n - f|$$

$$< \int \frac{\in}{m(E)}$$

$$< \frac{\in}{m(E)} \cdot m(E) = \in$$

By definition

$\therefore \lim_{n\to\infty} \int f_n = \int_E f$.

Example 5 :

Show by an example that the pointwise convergence alone is not sufficient to the passage of the limit under the integral sign.

Solution : Example

Let $f = 0$, function on $E = [0,1]$

Let $\phi_k = K\,\chi\left[0, \frac{1}{k}\right] \to 0$ as $k \to \infty$

$\therefore \phi_k \to f$ pointwise

$$\int_E \phi_k = K.m\left(\left[0, \frac{1}{k}\right]\right)$$

$$= K.\frac{1}{k} = 1$$

$$\int_E f = 0$$

$$\therefore \int_E \phi_k \not\to \int_E f$$

Example 6 :

Let f be a bounded measurable function on a set of finite measure E. Assume g is bounded and $f - g$ a.e. on E,

Show that $\int_E f = \int_E g$

4.4 THE GENERAL LEBESGUE INTEGRAL

For an extended real-valued function 'f' on E, the positive part f^+ and the negative part f^- of f defined by

$f^+(x) = ma\times\{f(x), 0\}$ and

$f^+(x) = ma\times\{-f(x), 0\} \forall x \subset E$

Then f^+ and f^- are non-negative functions on f

$f = f^+ - f^-$ on E and $|f| = f^+ + f^-$ on E

Thus f is measurable iff f^+ and f^- are measurable.

Example 7 :

Let f be a measurable function on E, show that f^+ and f^- are integrable over E iff $|f|$ is integrable over E.

Ans. Suppose f^+ and f^- are integrable

$$\Rightarrow \int_E f^+ < \infty \ \& \int_E f^- < \infty$$

But $|f| = f^+ + f^-$

$$\Rightarrow \int_E |f| = \int_E f^+ + f = \int_E f^+ + \int_E f < \infty$$

$\therefore |f|$ is integrable

Conversely, suppose $|f|$ is integrable

$$\Rightarrow \int_E |f| < \infty$$

But $f^+ \le |f|$ & $f^- \le |f|$

$$\Rightarrow \int_E f^+ \le \int_E |f| < \infty \Rightarrow f^+ \text{ is integrable}$$

Similarly f^- is integrable.

Definition :

A measurable function f on E is said to be integrable over E if $|f|$ is integrable over E i.e. $\int_E |f| < \infty$. If 'f' is integrable over E, then we define the integral of 'f' over E by $\int_E f = \int_E f^+ - \int_E f^-$

Example :

Let 'f' be integrable over E. Show that f is finite a.e. on E and $\int_E f = \int_{E/E_0} f$ where $E_0 \subseteq E$ and $m(E_0) = 0$

Solution :

'f' is integrable on E

$\Rightarrow |f|$ is integrable

$\Rightarrow \int_E |f| < \infty$

Note that $|f|$ is non negative integrable function.

We now show that $|f|$ is finite a.e. on E.

Note that $\{x \in E; |f(x)| = \infty\}$

$= \cap \{x \in E; f(x) > x\}$

$\Rightarrow \{x \in E; |f(x)| = \infty\} \subseteq \{x \in E; f(x) > n\} \forall n$

But by chebychev's Lemma ………………………. (*)

$m(\{x \in E; |f(x) > n|\}) < \frac{1}{n} \int |f| \forall_n$

$\therefore |f|$ is integrable, $\int_E |f|$ is finite

i.e. $\int |f| < \infty$

$\Rightarrow m(\{x \in E; |f(n)| < n\}) = 0$

$\Rightarrow m(\{x \in E; |f(n)| = \infty\}) = 0$

$\Rightarrow |f(x)|$ is finite a.e. on E

$\therefore f \leq |f|$, we get

f is finite a.e. on E

Let $E_0 \subseteq E$ s.t. $m(E_0) = 0$

$\therefore$By definition

$$\int_E f = \int_E f^+ - \int_E f^-$$

$$= \int_{E|E_0} f^+ - \int_{E|E_0} f^-$$ ($\because f^+ \& f^-$ are non-negative integrable functions)

$$= \int_{E/E_0} (f^+ - f^-) = \int_{E/E_0} f$$

Example 9:

Define $f(x) = \frac{1}{x^{2/3}} \quad 0 < x < 1$

$= 0 \qquad x = 0$

Show that f is Lebesgue integrable on $[0,1]$ and $\int_0^1 \frac{1}{x^{2/3}} dx = 3$. Find also $f(x,2)$

Solution :

$$\frac{1}{x^{2/3}} \to \infty \text{ as } x \to 0$$

So f is unbounded in $[0,1]$ its Lebesgue integrability define

$$f(x,n) = \frac{1}{x^{2/3}} \text{ if } \frac{1}{x^{3/2}} \leq x \leq 1$$

$$= n \text{ if } O < x < 1/n^{3/2}$$

$$= 0 \text{ if } x = 0$$

Now $\int_0^1 f(x,n)\,dx = \int_0^{1|_{n^{3/2}}} f(x,n)\,dx + \int_{1|_{n^{3/2}}}^1 f(x,n)\,dx$

$$= \int_0^{1|_{n^{3/2}}} n dx + \int_{1|_{n^{3/2}}}^{1} \frac{1}{x^{2|3}} dx$$

$$= \frac{1}{\sqrt{n}} + 3\left[1 - \left(\frac{1}{n^{3/2}}\right)^{1/3}\right] = 3 - \frac{2}{\sqrt{n}} \forall n$$

by definition of the Lebesgue integral of on bounded functions

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_0^1 f(x,n)\,dx$$

$$= \lim_{n\to\infty}\left(3 - \frac{2}{\sqrt{n}}\right)$$

$$= 3$$

Lebesgue integrable define for $n = 2$

$$f(x,2) = \frac{1}{x^{2/3}} \text{ if } \frac{1}{z^{2/3}} \le x \le 1$$

$$= 2 \quad \text{if } 0 < x < \frac{1}{z^{2/3}}$$

$$= 0 \quad \text{if } x = 0$$

4.5 SUMMARY

In this chapter we have learned about

- Introduction concept of Lebesgue integral.
- Lebesgue integral of complex valued Measurable functions
- Lebesgue integral at a simple function.
- Lebesgue integral on bounded Measurable function general Lebesgue integral

4.6 UNIT END EXERCISE

1. Show that for a finite family $\{f_k\}_{n=1}$ of measurable functions with common domain E, the functions $Max\{f_1....f_n\}$ and $Min\{f_1....f_n\}$ also are measurable.
2. Show that the sum and product of two simple functions are simple.

3. For every non-negative and measurable function f on $[0,1]$ then show that $\int_{[0,1]} f\, dm = \inf \int_{[0,1]} \phi\, dm$.

4. Prove that a measurable function $f(x) L^1[0,1]$ if and only if $\sum_{n=1}^{\infty} 2^n m\left\{x \in [0,1]; |f(x)| \geq 2^n\right\} < \infty$

5. If $f \in L^1[0,1]$ find $\lim_{k\to\infty} \int_0^1 K \log\left(1 + \frac{|f(x)|^2}{K^2}\right) dx$

6. Let f be a Lebesgue integrable function on X use the positive and negative part of f to prove that $\left|\int_x f\, dx\right| \leq \int_x |f|\, dx$.

7. Let f be a non-negative measurable function on X and suppose that $f \leq M$ for some constant M prove that $\int_E f\, dx \leq \int_x |f|\, dx$ for

8. Calculate Lebesgue integral for the function $f(x) = \begin{cases} 1 \text{ where } x \text{ is rational} \\ 2 \text{ where } x \text{ is irrational} \end{cases}$

9. Evaluate $\int_0^5 f(x)\, dx$ if

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & \{1 \leq x \leq 2\} \cup \{3 \leq x < 4\} \\ 2 & \{2 \leq x < 3\} \cup \{4 \leq x < 5\} \end{cases}$$

by using Riemann and Lebesgue definition of the integral.

10. Show that if f is a non-negative measurable function then $f = 0$ a.e. on a set A iff $\int_A f\, dx = 0$

11. If $f(x) = 1/x$ if $0 < x < 1$

$= 9$ then f is not Lebesgue integrable in $[0,1]$

12. Let F be a non-negative measurable function on χ and suppose that $f \leq M$ for some constant M. Prove that $\int_E f\, d\mu \leq m\, \mu(E)$ for any measurable $E \subseteq \chi$.

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5

CONVERGENCE THEOREMS

Unit Structure :

5.1 INTRODUCTION

In this section we analyze the dynamics of integrability in the case when sequences of measurable functions are considered. Roughly speaking a "convergence theorem" states that integrability is preserved under taking limits. In other words, if one has a sequence $(f_n)_{n=1}^{\infty}$ of integrable functions, and if 'f' is some kind of a limit of the $f_n's$ then we would like to conclude that 'f' itself is integrable, as well as the equality $\int f = \lim_{n\to\infty} \int f_n$ such results are employed in two instances.

i) When we want to prove that some function 'f' is integrable. In this case we would look for a sequence $(f_n)_{n=1}^{\infty}$, of integrable approximation for f.

ii) When we want to construct and integrable function in this case, we will produce first the approximates and then we will examine the existence of the limit.

The first convergence result, which is some how primote, but very useful in the following.

5.2 MEASURABLE FUNCTIONS

Theorem :

Let (X, A, μ) be a finite measure space, let $G(C-(o,\infty))$ and let $f_n : X \to [0,9], n \geq 1$ be a sequence of measurable functions satisfying.

1) $f_1 \geq f_2 \geq .... \geq 0$

2) $\lim_{n\to\infty} f_n(x) =, \forall x \in X$ Then one has the equality $\lim_{n\to\infty} \int_A f_n \, dx = 0$.

Proof :

Let for each $\in > 0$ and each integer $n \geq 1$, the set $A_K^C = \{x \in X_e ; f_n(x) \geq \in\}$ obviously, we have $A_n^\in \in A, \forall \in > 0, n \geq 1$ we are going to use the following case.

Claim I :

For every $\in > 0$, one has the equality $\lim_{n\to\infty} \mu(A_n^\in) = 0$.

Fix $\in > 0$, Let us first observe that (a) we have the inclusion

$A_1^C \supset A_2^C \supset$ .... (II)

Second using (b) we clearly have the equality $\bigcap_{k=1}^{\infty} A_k^\in = \phi$. Since μ is finite using continuity property we have

$$\lim_{n\to\infty} \mu(A_n^\in) = \mu\left(\bigcap_{n=1}^{\infty} A_n^\in\right) = \mu(\phi) = 0$$

Claim II :

For every $\in > 0$, and every integer $n \geq 1$, one has the inequality $0 \leq \int_X f_n du \leq a\mu(A_n^\in) + \in \mu(x)$.

Fix $\in$ and n and let us consider the elementary functions.

$h_n^\in = ax_{A_n^\in} + \in x_{A_n^{-\in}}$ where $B_n^\in = X / A^E$ obviously, since $\mu(x) < \infty$ the function $h_n^\in$ is elementary integrable. By construction we clearly have $0 \leq f_n \leq h_n^\in$, so using the properties of integration, we get

$$0 \le \int_X f_n dx \le \int_X h_n^{\in} dx = a\mu\left(A_n^{\in}\right) + \in \mu\left(B^{\in}\right)$$

$$\le a\mu\left(A^{\in}\right) + \in \mu\left(X\right)$$

Using claim I & III it follows immediately that

$$0 \le \liminf_{n\to\infty} \int_X f_n d\mu \le \limsup_{n\to\infty} \int_X f_n d\mu \le \in \mu\left(X\right)$$

Since the last inequality hold for arbitary $\in > 0$, we get $\lim_{n\to\infty} \int_X f_n du = 0$

5.3 LEBESGUE THEOREM ON BOUNDED CONVERGENCE

Statement :

Let $\{f_n\}$ be a sequence of functions measurable on a measurable subset $A \subseteq [a,b]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ then if there exists a constant M such that $|f_n(x)| \le M$ for all 'n' and for all 'x', we have $\lim_{n\to\infty} \int_A f_n(x)\, dx = \int_A f(x)\, dx$.

Proof :

$\therefore \lim_{n\to\infty} f_n(x) = f(x)$ and $|f_n(x)| \le M$

$\Rightarrow |f(x)| \le M$

The function 'f' is bounded and measurable

Hence Lebesgue integrable.

Now we shall show that

$$\lim_{n\to\infty} \int_A |f_n(x) - f(x)|\, dx = 0$$

For a given $\in > 0$, we define a partition A into disjoint measurable sets A_k's as follows :

$$A_k = \left\{x : |f_{k-1} - f| \ge \in, |f_n - f| < \in, \forall_n \ge k\right\} K = 1,2,3,.....$$

In particular,

$$A_1 = \{x : |f_n - f| < \in ; n = 1,2,3,.....\}$$
$$A_2 = \{x : |f_1 - f| \geq \in ; |f_n - f| n < \in ; n = 2,3,4,.....\}$$

Clearly,

$$A = \bigcup_{K=1}^{\infty} A_k = \left(\bigcup_{K=1}^{\infty} A_k\right) \cup \left(\bigcup_{K=n+1}^{\infty} A_k\right)$$
$$= P_n \cup Q_n$$
$$m_A = m(P_n \cup Q_n) = mP_n + mQ_n$$

Now $\int_A |f_n - f|\, dx = \int_{P_n} |f_n - f|\, dx + \int_{Q_n} |f_n - f|\, dx$ ………… (1)

For each 'n', we have

$|f_n - f| < \in$ on P_n and $|f_n - f| \leq |f_n| + |f| \leq 2m$ on Q_n

Thus, $\int_A |f_n - f|\, dx < \in mP_n + 2M\ mQ_n$

As $n \to \infty$, $\lim_{n\to\infty} mP_n = mA$ and $\lim_{n\to\infty} mQ_n = 0$

Thus, $\int_A |f_n - f|\, dx < \in mA$

$\in$ being an arbitrary value

$$\therefore \lim_{n\to\infty} \int_A f_n(x)\, dx = \int_A f(x)\, dx$$

Example 1 :

Verify Bounded Convergence.

Theorem for the sequence of functions $f_n = \dfrac{1}{\left(1 + x/n\right)^n}; O \leq x \leq 1, n \in \mathbb{N}$.

$$|f_n(x)| = \left|\frac{1}{\left(1 + x/n\right)^n}\right| \leq 1 \forall n \text{ and } \forall x$$

Each f_n being bounded and measurable, the limit function.

$$\therefore \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{\left(1+\frac{x}{n}\right)^2} = \frac{1}{e^x}$$

It is also bounded and measurable. Now

$$\int_0^1 \frac{dx}{\left(1+\frac{x}{n}\right)^n} = n\frac{\left(1+\frac{x}{n}\right)^{(-n+1)}}{(-n+1)}\Bigg|_0^1$$

$$= \frac{n}{(n-1)}\left(1-\frac{1+\frac{1}{n}}{\left(1+\frac{1}{n}\right)^n}\right)$$

$$\therefore \lim_{n\to\infty} \int_0^1 \frac{dx}{\left(1+\frac{x}{n}\right)^n}$$

$$= \lim_{n\to\infty} \frac{n}{n-1}\left(1-\frac{1+\frac{1}{n}}{\left(1+\frac{1}{n}\right)^n}\right)$$

$$= \lim_{n\to\infty} \frac{n}{(n-1)}\left(1-\frac{\frac{(n+1)}{n}}{\left(\frac{n+1}{n}\right)^n}\right)$$

$$= 1-\frac{1}{e}$$

$$= \frac{e-1}{e}$$

Similarly,

$$\int_0^1 \lim_{n\to\infty} \frac{1}{\left(1+\frac{x}{n}\right)^n} dx = \int_0^1 \frac{1}{e^x} dx = \int_0^1 e^{-x}\, dx$$

$$= \left[-e^{-x}\right]_0^1 = \left(1-\frac{1}{e}\right) = \frac{e-1}{e}$$

Hence Bounded convergence theorem is verified.

5.4 LIMITS OF MEASURABLE FUNCTIONS

If $f_n : \mathbb{R} \to [-\infty, \infty] (n;1,2,....)$ is an finite sequence of functions then we say that $f : \mathbb{R} \to [-\infty, \infty]$ is the pointwise limit of the sequence $(f_n)_n$ if we have $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in \mathbb{R}$.

For any sequence $f_n : \mathbb{R} \to [-\infty, \infty]$ we can define $\limsup_{n\to\infty} f_n$ as the function with value at 'x' given by

$$\limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(\sup_{k\geq n} f_k(x)\right)$$

Something that always makes sense because $\sup_{k\geq n} f_k(x)$ decreases n increases or atleast does not get any bigger as n increase. Suppose that $\{f_n\}$ is a sequence of real number. Let A be the set of numbers such that $f_n \to f$ for some subsequence f_{n_k} of f_n.

$\therefore f$ is called a limit point of f_n, so A is the set of all limit points of $\{f_n\}$. Then supremum and infimum of A are denoted by the following $\liminf_{n\to\infty} f_n = \inf A, \limsup_{n\to\infty} f_n = \sup A$.

5.5 FATOU'S LEMMA

Statement :

If $\{f_n\}$ is a sequence of non-negative measurable functions, then for any measurable set E.

$$\liminf_{n\to\infty} \int_E f_n \, dx \geq \int_E \left(\liminf_{n\to\infty} f_n\right) dx$$

Proof : We write $f(x) = \liminf_{n\to\infty} f(x)$

We recall that for any x, $\liminf f_n(x) = \inf \inf f_n$ where Ex is the set of all limit points of $f_n(x)$.

$\therefore f_n \to f$ pointwise convergence on E

$\Rightarrow f_n \to f$ pointwise on $E/E, m(E_1) = 0$

$\therefore f_n \not\to f$ pointwise on E_1

$\because E_1 \subseteq E$ and $m(E_1) = 0$

We may assume $f_n \to f$ pointwise on E

$f_n's$ are non-negative measurable and $f_n \to f$

$\Rightarrow f$ is non-negative and measurable.

Now to show that $\int_E f \leq \liminf_{n\to\infty} \int_E f_n$

Let h be a bounded measurable function of finite support such that $0 < h < f$

$\Rightarrow m(E_0) < \infty$ where $E_0 = \{x \in E; h(x) \neq 0\}$

$\because h$ is bounded choose M such that $h(x) \leq M$ on E for $n \in \mathbb{N}$ Define $h_n = \min\{h, f_n\}$.

Clearly $h_n \geq 0$ is measurable bounded function and $h_n \leq M$. We can now show that $h_n \to a$ pointwise on E_0.

For $x \in E_0\ h(x) \leq f(x)$

Case I :

$$h(x) < f(x)$$

$\Rightarrow f(x) - h(x) > 0$

$\because f_n \to f$ pointwise on E for $0 < \in < f(x) - h(x)$

$\exists n_0 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \in\ \forall_n \geq n_0$

$\Rightarrow f(x) - \in < f_n(x) < f(x) + \in$
$\therefore h(x) < f(x) - \in < f_n(x) \forall_n \geq n_0$
$\therefore h_n(x) = \min(h, f_n) = h(x) \forall n \geq n_0$

$\Rightarrow h_n \to h$ pointwise on E_0

Case II :

$$h(x) = f(x)$$

Then $h_n(x) = f_n(x)$ on $f(x) \forall n$

$\because f_n \to f$ pointwise on E_0

$\Rightarrow h_n \to f = h$ pointwise E_0

By bounded convergence Theorem

For the bounded sequence $\{h_n\}$ restricted to E_0

We have $\lim_{n\to\infty}\int_{E_0} h_n - \int_{E_0} h$

$$\therefore \lim_{n\to\infty}\int_E h_n = \lim_{n\to\infty}\int_{E_0} h_n = \int_{E_0} h = \int_E h$$

$$\left[\because h_n = 0, on\ E/E_0\,; h = 0\ on\ E/E_0\right]$$

$$\int_E h = \lim_{n\to\infty}\int_E h_n = \lim\inf \int_E h_n \le \lim\inf \int_E f_n$$

This is true for any bounded measurable function with finite support such that $0 \le h \le f$

$\therefore$ By definition of $\int_E f$

$$\therefore \int_E f \le \lim_{h\to\infty}\inf \int_E f_n$$

5.6 LEBESGUE INTEGRAL OF NON-NEGATIVE MEASURABLE FUNCTION

Definition :

Let f be a measurable function defined on E. The support of 'f' is defined as $\sup(f) = \{x \in E; f(x) \ne 0\}$.

Definition :

A measurable function f on E is said to vanish outside a set of finite measure if $\exists$ a subset E_0 of E for which $m(E_0) < \infty$ & $f = 0$ on E/E_0. It is convenient to say that a function that vanishes outside a set of finite measure has finite support.

$\therefore$We have defined the integral of a bounded measurable function 'f' over a set of finite measure E. But $m(E) = \infty$ and f is bounded and measurable on E with finite. Support we can define its integral over E by $\int_E f = \int_{E_0} f$ where $m(E_0) < \infty$ and $f = 0$ on E/E_0.

Definition :

For a non-negative measurable function f on E we define integral of 'f' over E by $\int_E = \sup\left\{\int_E h : h\right.$ bounded; measurable of finite support and $\left. 0 \le h < f\ on\ E\right\}$.

Chebychev's Inequality :

Statement :

Let f be a non-negative measurable function on $E \subseteq \mathbb{R}$ then for any $\lambda > 0$.

$$m\{x \in E; f(x) \geq \lambda\} \leq \frac{1}{\lambda}\int_E f$$

Proof :

Let $E_\lambda = \{x \in E : f(x) \geq \lambda\}$

Case I :

$m(\lambda_n) = \infty$ for each $n \in \mathbb{N}$ define $E_\lambda^n = E_\lambda \cap [-n, n]$. Then $\Psi_n = \lambda \chi_{E_\lambda^n}$.

Then Ψ_n is bounded measurable function

$$\therefore \lambda_m(E_\lambda^n) = \int_E \Psi_n \text{ and } \Psi_n \leq f$$

Note that $E_\lambda^n \leq E_\lambda^{n+1}$ and $\bigcup_{n=1}^{\infty} E_\lambda^n = E_\lambda$

$\therefore$By continuity of measure.

$$\infty = \lambda_m(E_\lambda) = \lim_{n\to\infty} \lambda_m(E_\lambda^n)$$
$$= \lim_{n\to\infty} \int_E \Psi_n$$

$\because \Psi_n$ is bounded on E and $\Psi_n \leq f$

$\therefore$by definition $\int_E f$, we get

$$\int_E \Psi_n \leq \int_E f$$
$$\infty = \lambda_m(E_\lambda) \leq \int_E f$$

Both side $= \infty$

$$\therefore m(E_\lambda) \leq \frac{1}{\lambda}\int_E f$$

Case II : $m(E_\lambda) \leq \infty$

Define $h = \lambda \chi_{E_\lambda}$ then h is bounded measurable function $h \leq f$

$\therefore$ by definition of $\int_E f$, we get $\lambda m(E_\lambda) = \int h \leq \int_E f$

$$m(E_\lambda) \leq \frac{1}{\lambda} \int_E f$$

$$\therefore m\{x \in E; f(x) \geq \lambda\} \leq \frac{1}{\lambda} \int_E f$$

5.7 THE MONOTONE CONVERGENCE THEOREM

Statement : Let $\{f_n\}$ be an increasing sequence on non-negative measurable functions on A. If $\lim_{n\to\infty} f_n(x) = f(x)$ then $\lim_{n\to\infty} \int_A f_n = \int_A f$.

Proof :

Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions and $\lim_{n\to\infty} f_n = f(x)$ i.e. it is convergent at pointwise to f on A.

Now to show that $\lim_{n\to\infty} \int_A f_n = \int_A f$.

$\because f_n \to f$ pointwise on A and $f_n \leq f_{n+1} \ \forall\, n \in \mathbb{N}$

$\Rightarrow f_n \leq f \ \forall_n$ on A

$\Rightarrow \int_A f_n \leq \int_A f$ on A

$\Rightarrow \sup \int_A f_n \leq \int_A f$

$\lim_{n\to\infty} \sup \int_A f_n \leq \int_A f$ .... (I)

By the Fatou's lemma

$\int_A f \leq \lim_{n\to\infty} \inf \int_A f_n$ .... (II)

From I & II we get

$$\int_A f = \liminf_{n\to\infty} \int_A f_n = \limsup_{n\to\infty} \int_A f_n$$

$$\therefore \lim_{n\to\infty} \int_A f_n = \int_A f$$

5.8 DOMINATED CONVERGENCE THEOREM

(Generalisation of Bounded Convergence Theorem)

Statement : Let $\{f_n\}$ be a sequence of measurable function on E. Suppose there is a function 'g' that is integrable over E and dominates $\{f_n\}$ on E in the sense that $|f_n| \leq g$ on E for all n. If $f_n \to f$ pointwise almost everywhere on E,then f is integrable over E and $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Proof :

$\because |f_n| \leq g \,\forall_n$ on E and $f_n \to f$ pointwise on E.

$$\Rightarrow |f| \leq g \leq |g|$$

$$\Rightarrow \int |f| \leq \int |g| < \infty$$

$\Rightarrow f$ is measurable

$\because |f_n| \leq g$ and $|f| \leq g \Rightarrow g - f_n \geq 0$ and $g - f_n \to g - f$ pointwise

$\therefore$By Fatou's lemma

$$\int g - f \leq \liminf \int g - f_n$$

$$\leq \liminf \int_E \delta - \int_E f_n$$

$$\leq \int_E \delta - \limsup \int_E f_n$$

$$\therefore \limsup \int_E f_n \leq \int_E f \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}$$

Similarly $g + f_n \geq 0$ & $g + f_n \to g + f$ pointwise on E.

$\therefore$ By Fatou's lemma,

$$\int_E g + f \leq \liminf \int_E g + f_n$$

$$\int_E g + \int_E f \le \int_E g + \liminf \int_E f_n$$

$$\int_E f \le \liminf \int_E f_n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(III)}$$

From I & II we get

$$\liminf \int_E f_n = \limsup \int_E f_n - \int_E f$$

$$\therefore \lim_{n\to\infty} \int_E f_n = \int_E f$$

Example 2 :

Check the convergence of

$$f_n(x) = \frac{1}{n}; |x| \le n$$
$$= 0 \quad ; |x| > n$$

Solution : Let $f_n(x) = \frac{1}{n} \; ; |x| \le n$

$$= 0 \quad ; |x| > n$$

Then $f_n(x) \to 0$ uniformly on $\mathbb{R}$ but $\int_{-\infty}^{\infty} f_n dx = 2; \;\; n = 1, 2, 3, \ldots..$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{n} = 0 \text{ where } |x| \le n$$

$$= 0 \text{ when } |x| > n$$

$\therefore \lim_{n\to\infty} f_n(x) = 0$ uniformly on the whole real time.

Now, $\left|f_{2m}(x) - f_m(x)\right| = \left|\frac{1}{2m} - \frac{1}{m}\right| = \left|\frac{1}{2m}\right| < \in$

Whenever $M > \frac{1}{2\in}$

Now $\int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{-n} 0\,dx + \int_{-n}^{n} \frac{1}{n}\,dx + \int_{n}^{\infty} 0\,dx = 2$.

This emplies that uniform converges of $\{f_n(x)\}$ is not enough for $\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n$

This equality is Lebesgue integration.

In general, is only due to dominated convergence of the sequence $\{f_n(x)\}$.

$\therefore$However on the set of finite measure uniformly convergent sequence of bounded function are bounded convergent.

5.9 LEBESGUE INTEGRAL OF COMPLEX VALUED FUNCTIONS

If f is a complex valued function on $E \subseteq \mathbb{R}^n$ we may write as $f(x) = u(x) + i\vartheta(x)$ where u & v are real functions called the real and imaginary part of f.

A complex valued measurable function, $f : u + iv$ on E is said to be integrable if $\int_E |f(x)| = \int_E \sqrt{u(x^2) + v(x^2)} < \infty$ and the integral of 'f' is given by $\int_E f = \int_E u + i\int_E v$

Theorem :

Show that a complex valued function is integrable if and only if both of its real and imaginary parts are integrable.

Proof :

Suppose $f : u + iv$ is integrable

$\Rightarrow \int |f| < \infty$

$\Rightarrow \int \sqrt{u^2 + v^2} < \infty$

$u \leq |u| = \sqrt{u^2} \leq \sqrt{u^2 + v^2}$

$\Rightarrow \int |u| \leq \int \sqrt{u^2 + v^2} < \infty$

$\Rightarrow u$ is integrable

Similarly v is integrable

Conversely

Suppose u & v are integrable

$\Rightarrow \int |u| < \infty$ and $\int |V| < \infty$

By Minkowski's inequality

$$|f| = \sqrt{u^2 + v^2} \leq \sqrt{u^2} + \sqrt{v^2} = |u| + |v|$$
$$\Rightarrow \int |f| \leq \int |u| + \int |v| < \infty$$

$\therefore f$ is integrable.

Definition :

A measurable function $f : E \to C, E \subseteq \mathbb{R}^n$ is said to be an L^1 function if $\int_E |f| < \infty$.

Note : $L^1(\mathbb{R}^n) =$ {set of all complex valued function on $\mathbb{R}^n$ }

Definition : A family G of integrable function is dense in $L^1(\mathbb{R}^n)$ if for any $f \in \bigcup$ and $\in > 0 \, \exists \, g \in G$ so that $\int_E |f - g| < \in$

Example 3:

Show that the continuous function of compact support is dense in $L^1(\mathbb{R}^n)$.

Solution :

To show that : The continuous function of compact support is dense in $L^1(\mathbb{R}^n)$.

i.e. tst for any $f \in L^1$ and $\in > 0$.

$\exists$ a continuous function 'g' on $\mathbb{R}^n$ with compact support such that $\|f - g\|, < \in$ i.e. $\int |f - g| < \in$.

Let $f \in L^1(\mathbb{R}^n)$

We may assume 'f' is real valued becaue we may approximate its real and imaginary part independently.

In this can we write $f = f^+ - f^-$.

Where $f^+ \geq 0$ and $f^- \geq 0$

$\therefore$ It is enough to show the result $f \geq 0$.

$\therefore f \geq 0$ can be approximated by integrable simple functions.

It is enough to show that the result for an integrable simple functions.

$\therefore A_n$ integrable simple functions is a Linear combination of characteristic function.

It is enough to show for $f = \chi_E$ where E is a measurable set of finite measurable.

Let $\in > 0$

$\therefore$E is measurable $\exists$ a compact set k and an open set Ω of $\mathbb{R}^n$ such that $K \subseteq E \subseteq \Omega$ and $m(\Omega | k) < \in$

By Urysohn's Lemma

$\exists$ a continuous function $g : \Omega \to k$ such that $g \equiv 0$ on $\Omega | k$ & $g \equiv 1$ on K

$\therefore$g is continuous function with compact support

$\therefore |g - f| = |g - \chi_E| = 1 \;\; E | k$ and $|g - \chi_E| = 0$ on outside $E | k$

$$\therefore \int_{\mathbb{R}^n} |g - f| = \int_{E/k} 1 = m(E | k) \leq m(\Omega | k) < \in$$

$\therefore \exists$ continuous function of compact support such that $|g - f| < \in$.

$\therefore$ Continuous function of compact support is dense in $L^1(\mathbb{R}^n)$.

Example 4 :

Let $f \in L^1(\mathbb{R}^n)$ show that $\left| \int f \right| \leq \int |f|$

Solution : Let $f \in L^1$ to show that $\left| \int f \right| \leq \int |f|$

Let $z = \int f$

If $z = 0$ then clearly $\int |f| \geq 0 = z = |z| = \left| \int f \right|$

$\therefore \left| \int f \right| \leq \int |f|$

If $z \neq 0$

Define $\alpha = \dfrac{\overline{z}}{|z|}$

$\therefore |\alpha| = 1$ and $\alpha z = |z|$

$\therefore \left|\int f\right| = |z| = \alpha z = \alpha \int f = \int \alpha f$

Let $\alpha f = u + iv$

By definition

$\int \alpha f = \int u + i \int v$

$\therefore \left|\int f\right| = \int u + i \int v$

$\therefore \left|\int f\right| \in \mathbb{R} \Rightarrow \int v = 0$

$\therefore \left|\int f\right| = \int u$ ………………………………………….. (I)

$u \leq |u| \leq |\alpha f| = |\alpha||f| = |f|$

By Monotonicity property

$\int u \leq \int |f|$ …………………………………………….. (II)

By (I) and (II)

$\therefore \left|\int f\right| \leq \int |f|$ proved

Example 5 :

Show that $L^1(\mathbb{R}^n)$ is complete in its metric.

Solution :

Let $\{f_n\}$ be a Cauchy sequence in $L^1(\mathbb{R}^n)$ for $\in > 0, \exists\, n_0 \in \mathbb{N}$ such that $\|f_m - f_n\|_1 < \in \forall n, n \geq n_0$

$\therefore$ for each $k \in \mathbb{N}$

We can choose n_k such that for $m, n \geq n_k$ $\|f_m - f_n\|_1 < \frac{1}{2^k}$ and $n_k < n_{k+1}$

then the sequence f_{n_k} has the property that $\left\|f_{n_{k+1}} - f_{n_k}\right\| < \frac{1}{2^k}$.

Construct the series

$$f(n) = f_{n_1}(x) + f_{n_2}(x) - f_{n_1}(x) + f_{n_3}(x) - f_{n_2}(x) + ...$$

$$= f_{n_1}(x) + \sum_{K=1}^{\infty}\left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)$$

and $g(x) = |f_{n_1}(x)| + \sum_{K=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$

Let $S_k(g)$ denote the k^{th} partial sum of the series g then.

$$S_k(g) = |f_{n_1}(x)| + \sum_{i=1}^{k+1} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

Then $\{S_k(g)\}$ is a sequence of non-negative function converges pointwise to g.

$S_k(g) \le S_{k+1}(g) \, \forall \, n$

$\therefore$ By Monotone Convergence Theorem g is integrable and $\lim_{n\to\infty} \int S_k(g) = \int g$

Note that $|f| \le g$

$\Rightarrow \int (f) \le \int g < \infty$ ($\because$ g is integrable)

$\Rightarrow f$ is integrable

$\Rightarrow f$ is $L^1(\mathbb{R}^n)$

Let $S_k(f)$ denote the k^{th} partial sum of the series of f, then

$$S_k(f) = f_{n_1}(x) + \sum_{i=1}^{K-1} \left(f_{n_{i+1}}(x) - f_{n_i}(x)\right)$$
$$= f_{n_k}(x)$$

$\because S_k(f) \to f$ pointwise

$\Rightarrow f_{n_k} \to f$ pointwise

Now we show that $\Rightarrow f_{n_k} \to f$ in $L^1(\mathbb{R}^n)$

Note that $|f - f_{n_k}| \le g \, \forall \, k$

By Dominated convergence Theorem

$\lim_{n\to\infty} \int |f - f_{n_k}| = 0$

$\therefore \lim_{n\to\infty} \int \|f - f_{n_k}\|_1 = 0$

$\therefore f_{n_k} \to f$ in $L^1(\mathbb{R}^n)$

$\because f_n$ is Cauchy and has convergent subsequence f_{n_k} converges of f.

We get $f_n \to f$

$\therefore$ Every Cauchy sequence in L^1 is convergent.

$\therefore L^1$ is complete in its metric. Proved

5.10 REVIEW

In this chapter we have learnt following points.

- Limits of Measurable function
- Bounded convergence theorem of measurable function
- Monotone convergence theorem of measurable function.
- Fatou's lemma of measurable function
- Dominated convergence Theorem
- Complex valued measurable function
- Compactness of $L^1(\mathbb{R}^n)$

5.11 UNIT END EXERCISE

1. show by an example that the inequality in Fatou's lemma may be a strict inequality.

Example : Consider a sequence of function $(f_n)_{n\in\mathbb{N}}$ defined on $[0,1]$ by $f_n(x) = \frac{nx}{1+n^2x^2} x \in [0,1]$.

i) Show that (f_n) is uniformly bounded on $[0,1]$ and evaluate $\lim_{n\to\infty} \int_{[0,1]} \frac{nx}{1+n^2x^2} dx$

ii) Show that (f_n) doesnot converge uniformly on $[0,1]$

Solution :

1) For all $n \in \mathbb{N}$ for all $x \in [0,1]$ we have $1+n^2x^2 \geq 2nx \geq 0$ and $1+n^2x^2 > 0$

Hence $0 \leq f_n(x) = \frac{nx}{1+n^2x^2} \leq \frac{1}{2}$

Thus $f(x)$ is uniformly bounded on $[0,1]$

Since each f_n is continuous on $[0,1]$

$\therefore f$ is Riemann integrable on $[0,1]$

In this case Lebesgue integral and Riemann integral on $[0,1]$. Consider

$$\int_{[0,1]} \frac{nx}{1+n^2x^2}\,dx = \int_0^1 \frac{nx}{1+n^2x^2}\,dx$$

Put $1+n^2x^2 = t$

$$= \frac{1}{2x}\int_0^{1+n^2} 1/t\,dt$$

$$\int_{[0,1]} \frac{nx}{1+n^2x^2}\,dx = \frac{1}{2n}\log\left(1+n^2\right) = \frac{\log\left(1+n^2\right)}{2n}$$

Using L^1 Hospitalrule we get

$$\log_{n\to\infty} \frac{\log\left(1+n^2\right)}{2n} = 0$$

Hence $\lim\limits_{n\to\infty} \int_{[0,1]} \frac{nx}{1+n^2x^2}\,dx = 0$

ii) For each $x \in [0,1] \Rightarrow \lim\limits_{n\to\infty} \int_{[0,1]} \frac{nx}{1+n^2x^2} = 0$

Hence $f_n \to f$ pointwise on $[0,1]$

Now to show that f_n does not converges to $f = 0$ uniformly on $[0,1]$.

We find a sequence (x_n) in $[0,1]$.

Such that $x_n \to 0$ and $f_x(x_n) \not\to f(0) = 0$ as $n \to \infty$, taking $x_n = \frac{1}{n}$ then $f_n(x) = 1/2$.

Thus $\lim\limits_{n\to\infty} f_n(x_n) = \frac{1}{2} \neq f(0) = 0$

Example 2 :

Evaluate $\lim_{n\to\infty} \int_0^n \left(\frac{1+x}{n}\right)^n e^{-2x} dx$

Solution : We know that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \text{ and } \left(1+\frac{x}{n}\right)^n \le \left(1+\frac{x}{n+1}\right)^{n+1}.$$

Also we have $\left(1+\frac{x}{n}\right)^n \le e^x$

$$\therefore \left(1+\frac{x}{n}\right)^n e^{-2x} \le e^{-x}$$

$\therefore$ by Dominated convergence then to the function $\left(1+\frac{x}{n}\right)^n e^x$ with the dominating function e^{-x}

$$\therefore \lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x} dx$$

$$= \lim_{n\to\infty} \int_0^\infty 1_{[0,1]} x \left(1+\frac{x}{n}\right)^n e^{-2x} dx$$

$$= \int_0^\infty \lim_{n\to\infty} 1_{[0,1]} x \left(1+\frac{x}{n}\right)^n e^{-2x} dx$$

$$= \int_0^\infty e^{-x} dx$$

$$= 1$$

2) Show by an example that monotone convergence theorem does not hold for a decreasing sequence of functions.

3) Let $f_n(x) := \frac{x}{n^2}; 0 < x < n$

$= 0$; otherwise

Evaluate $\lim_{n\to\infty} \int_0^n f_n(x)\,dx$ and $\int_0^n \lim_{n\to\infty} f_n(x)\,dx$ are these equal?

4) $g(x) = 0 \quad 0 \le x \le 1/2$

$= 1 \quad 1/2 \le x \le 1$

$f_{2k}(x) = g(x), 0 \le x \le 1$

$f_{2k+1}(x) = g(1-x), 0 \le x \le 1$

To show that $\liminf_{n\to\infty} \int_0 f_n(x)\,dx > \int_0 \liminf_{n\to\infty} f_n(x)\,dx$

5) If $f_n; X \to [0,\infty]$ is measurable for $n = 1,2,....$ and $f(x) = \sum_{n=1}^{\infty} f_n(x) (x \in X)$ then show that $\int_X f\,dr = \sum_{n=1}^{\infty} \int_X f_n\,dr$.

6) Use the dominated convergence theorem to find $\lim_{n\to\infty} \int_1^{\infty} f_n(x)\,dx$ where $f_n(x) = \frac{\sqrt{x}}{1+nx^3}$.

7) If $a_n \le b_n$ for all n, then show that $\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n$.

8) State and prove bounded convergence theorem of measurable function.

9) Use convergence theorem to show that $f(t) = \int_{[0,\infty]} e^{-x} \cos(\pi t)\,d\,u(x)$ is continuous.

10) Use the dominated, convergence theorem to prove that $\lim_{n\to\infty} n \int_0^1 \sqrt{x} e^{n^2x^2}\,dx = 0$

11) Use the dominated convergence theorem to show that

$$\lim_{n\to\infty} \int_R \left(1+\frac{x^2}{n^2}\right)^{-\left(\frac{n+1}{2}\right)} dx = \int_R e^{-\frac{x^2}{2}}\,dx$$

❖❖❖❖

6

SPACE OF INTEGRABLE FUNCTIONS

Unit Structure

6.0 OBJECTIVE

After going through this chapter you will able to know:

- Signed measures is to generalize the concept of a traditional measure in measure theory to allow for negative values.
- The Hahn decomposition theorem is to provide a fundamental result in the theory of signed measures.
- Studying complex-valued Lebesgue measurable functions is to extend the notion of measurability and integration to functions whose range is the set of complex numbers.
- Studying the space L^1 (μ) of integrable functionsplays a central role in measure theory, functional analysis, and various fields of mathematics and applied sciences.

6.1 INTRODUCTION

Till now our measures have always assumed values that were greater than or equal to 0. In this chapter we will extend our definition to allow for both positive and negative values. signed measures extend the concept of measures by allowing them to take both positive and negative values.

This decomposition provides a clear separation of the positive and negative components of the signed measure, enabling a deeper understanding of the measure's behaviour on different subsets of the space. The sets *A* and *B* are unique up to null sets, meaning that any measurable subsets of *A* and *B* with measure zero can be added to or removed from *A* and *B* without affecting the positivity/negativity of the measure.

The space L^1 (μ) of integrable functions is a fundamental concept in measure theory and functional analysis. It provides a rich framework to study functions that are Lebesgue integrable with respect to a given measure μ on a measurable space. The integration theory based on the Lebesgue integral allows for a broader and more flexible class of functions compared to the traditional Riemann integral.

6.2 SIGNED MEASURES

A signed measure is a mathematical concept used in measure theory, which is a branch of mathematics that deals with the study of measures. Measures are used to assign a notion of size or volume to subsets of a given set. In traditional measure theory, measures are non-negative, meaning they take values in the real numbers and are non-negative for all sets. However, in certain applications and contexts, it becomes useful to work with more general measures that can take positive or negative values, and these are referred to as signed measures.

Definition: Le (X, A) beameasurablespace.Asignedmeasureon (X, A) is a function $\mu : A \to \mathbb{R}^*$ such that

i) μ takes on atmostone of the values − ∞ or ∞.

ii) μ(∅)=0.

iii) If $\{E_n\}_n^\infty = 1 \subseteq A$ is the sequence of pairwise disjoint set, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

Additionally, a signed measure can be decomposed into its positive and negative variations:

i) $\mu^{+E} = Max\{\mu(E), 0\}$ is positive variation.

ii) $\mu^{-E} = Max\{-\mu(E), 0\}$ is negative variation.

With these variations, the signed measure can be written as the difference of two non-negative measures:

$$\mu(E) = \mu^{+E} + \mu^{-\mathrm{E}}$$

It's important to note that a signed measure can take the value of positive infinity, negative infinity, or be finite.

6.3 HAHN DECOMPOSITION THEOREM

The Hahn decomposition theorem plays a significant role in understanding signed measures and provides a powerful tool for analysing their properties and behaviour. It highlights the duality between positive and negative parts of a signed measure, enabling deeper insights into the structure of measures and their applications in various mathematical contexts.

The Hahn decomposition theorem is valuable in several ways:

1. It allows us to analyze the positive and negative parts of a signed measure separately, which can be beneficial in various applications.

2. It is a key step in proving other important results in measure theory, like the Jordan decomposition theorem.

3. It is used to establish the Radon-Nikodym theorem, which is a fundamental result connecting measures and integrals in a more general setting.

To prove the Jordan decomposition of a signed measure, we first show that a measure space can be decomposed into disjoint subsets on which a signed measure is positive or negative, respectively. This is called the Hahn decomposition.

Definition: Let μ be a signed measure on $(X, \mathcal{A})$. A pair $\{P, N\}$ of elements in $\mathcal{A}$ for which P is positive N is negative, $P \cup N = X$ and $P \cap N = \emptyset$ is called a Hahn decomposition of X with respect to μ.

Radon-Nikodym theorem for signed measures:

Let $(\mathrm{X}, \mathrm{M}, \mu)$ be a σ- finite measure space and v a finite signed measure on measurable space (X, M) that is absolute continuous with respect to μ. Then there is a function f that is integrable over X with respect to μ and

$v(E) = \int_E f d\mu$ for all $\mathrm{E} \in \mathrm{M}$.

Function f is unique upto a set of μ measure zero.

Definition: Two measures μ and v on measurable space (X, M) are Mutually Singular if there are disjoint A and B in M. For which $X = A \bigcup B$ and $\mu(A) = v(B) = 0$

Definition: The decomposition of signed measure ν on measure space (X, M) into the difference of two (nonnegative) measures given in the Jordan Decomposition Theorem is called the Jordan decomposition of v

Lemma: Suppose that v is a signed measure on a measurable space (X, A). If $A \in \mathcal{A}$ and $0 < v(A) < \infty$, then there exists a positive subset $P \subset A$ such that $v(P) > 0$.

Proof: First, we show that if $A \in \mathcal{A}$ is a measurable set with $|v(A)| < \infty$, then $|v(B)| < \infty$ for every measurable subset $B \subset A$. This is because ν takes at most one infinite value, so there is no possibility of cancelling an infinite signed measure to give a finite measure. In more detail, we may suppose without loss of generality that $v : A \to [-\infty, \infty)$ does not take the value ∞. (Otherwise, consider $-v$.) Then $v(B) \neq \infty$; and if $B \subset A$, then the additivity of v implies that

$v(B) = v(A) - v(A \setminus B) \neq -\infty$

since $v(A)$ is finite and $v(A \setminus B \neq \infty$.

Now suppose that $0 < v(A) < \infty$.

Let $\delta_1 = \inf \{v(E) : E \in A$ and $E \subset A\}$.

Then $-\infty \leq \delta_1 \leq 0$, since $\emptyset \subset A$. Choose $A_1 \subset A$ such that $\delta_1 \leq v(A_1) \leq \delta_1/2$ if δ_1 is finite, or $\mu(A_1) \leq -1$ if $\delta_1 = -\infty$. Define a disjoint sequence of subsets $\{A_i \subset A : i \in N\}$ inductively by setting

$\delta_1 = \inf \{v(E) : E \in \mathcal{A}$ and $E \subset A \setminus \bigcup_{j=1}^{i-1} A_1\}$

and choosing $A_i \subset A \setminus \bigcup_{j=1}^{i-1} A_1$ such that

$\delta_1 \leq v(A_i) \leq 1/2\, \delta_1$

if $-\infty < \delta_1 \leq 0$, or $v(A_i) \leq -1$ if $\delta_1 = -\infty$.

Let $B = \bigcup_{i=1}^{\infty} A_i$, $P = A \setminus B$.

Then, since the A_i are disjoint, we have

$$v(B) = \sum_{i=1}^{\infty} v(A_i)$$

As proved above, $v(B)$ is finite, so this negative sum must converge. It follows that $v(A_i) \leq -1$ for only finitely many i, and therefore δ_1 is infinite for at most finitely many i. For the remaining i, we have

$\sum v(A_i) \leq 1/2 \sum \delta_i \leq 0$.

So $\sum \delta_i$ converges and therefore $\delta_1 \to 0$ as $i \to \infty$.

If $E \subset P$, then by construction $v(E) \geq \delta_1$ for every sufficiently large $i \in N$. Hence, taking the limit as $i \to \infty$, we see that $v(E) \geq 0$, which implies that P is positive. The proof also shows that,

since $v(B) \leq 0$, we have $v(P) = v(A) - v(B) \geq v(A) > 0$,

which proves that P has strictly positive signed measure.

Hahn decomposition theorem:

Statement: If v is a signed measure on a measurable space (X, A), then there is a positive set P and a negative set N for ν such that $P \cup N = X$ and $P \cap N = \emptyset$. These sets are unique up to v-null sets.

Proof: Suppose, without loss of generality, that $v(A) < \infty$ for every $A \in A$. (Otherwise, consider $-v$.)

Let $m = \sup\{v(A): A \in A$ such that A is positive for $v\}$,

and choose a sequence $\{A_i : i \in N\}$ of positive sets such that $v(A_i) \to m$ as $i \to \infty$. Then, since the union of positive sets is positive,

$P = \bigcup_{i=1}^{\infty} A_i$ is a positive set.

Moreover, by the monotonicity of v, we have $v(P) = m$. Since $v(P) \neq \infty$, it follows that

$m \geq 0$ is finite.

Let $N = X \setminus P$. Then we claim that N is negative for v.

If not, there is a subset $A' \subset N$ such that $v(A') > 0$, so by above Lemma, there is a positive set $P' \subset A'$ with $v(P') > 0$. But then $P \cup P'$ is a positive set with $v(P \cup P') > m$, which contradicts the definition of m.

Finally, if P', N' is another such pair of positive and negative sets, then $P \setminus P' \subset P \cap N'$,

so $P \setminus P'$ is both positive and negative for v and therefore null, and similarly for $P' \setminus P$.

Thus, the decomposition is unique up to v-null sets.

Remark: It is generally the case that the Hahn decomposition is not unique.

In fact, let $X = [0, 1]$ and let $\mathcal{A} = P(X)$. If $\mu_{1/2}$ is the point mass at 1/2, then if $P = \{1/2\}$ and $N = [0, 1] \setminus \{1/2\}$, then $\{P, N\}$ is a Hahn decomposition of [0, 1] with respect to μ. However, $P_1 = [0, 1/2]$ and $N_1 = (1/2, 1]$ is also a Hahn decomposition.

In fact, if $\{P, N\}$ is a Hahn decomposition of X with respect to μ and if $M \in \mathcal{A}$ is null, then $\{P \cup M, N \setminus M\}$ is a Hahn decomposition of X with respect to μ.

Furthermore, if $\{P_1, N_1\}$ and $\{P_2, N_2\}$ are Hahn decompositions of X with respect to μ, then

$M = P_1 \Delta P_2 = (P_1 \cap N_2) \cup (N_1 \cap P_2) = N_1 \Delta N_2$ is a null set.

Furthermore, since $E \cap P_1 \setminus P_2 \subseteq P_1 \Delta P_2$ and $E \cap P_2 \setminus P_1 \subseteq P_1 \Delta P_2$, it follows that

$\mu(E \cap P_1) = \mu(E \cap P_1 \cap P_2) = \mu(E \cap P_2)$ for each $E \in \mathcal{A}$.

Similarly, $\mu(E \cap N_1) = \mu(E \cap N_1 \cap N_2) = \mu(E \cap N_2)$ for each $E \in \mathcal{A}$.

What is true however, as we shall see, is that every Hahn decomposition induced a decomposition of μ into the difference of two positive measures and that any two Hahn decompositions induce the same decomposition of μ.

Theorem (Jordan decomposition): If v is a signed measure on a measurable space (X,A), then there exist unique measures $v^+, v^- : A \rightarrow$ [0, ∞], one of which is finite, such that

$v = v^+ - v^-$ and $v^+ \perp v^-$.

Proof: Let $X = P \cup N$ where P, N are positive, negative sets for v.

Then $v^+(A) = v(A \cap P)$, $v^-(A) = -v(A \cap N)$ is the required decomposition.

The values of $v^\pm$ are independent of the choice of P, N up to a v-null set, so thedecomposition is unique.

We call v^+ and v^- the positive and negative parts of v, respectively.

The total variation $|v|$ of v is the measure $|v| = v^+ + v^-$. We say that the signed measure v is σ-finite if $|v|$ is σ-finite.

6.4 COMPLEX VALUED LEBESGUE MEASURABLE FUNCTIONS ON $\mathbb{R}^d$

Complex valued Lebesgue measurable functions on $\mathbb{R}^d$ are an essential concept in measure theory and functional analysis.

Complex valued measurable functions: A function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is complex -valued measurable if both its real and imaginary parts, $\operatorname{Re}(f)$ and $lm(f)$ are measurable functions.

We then say that f is Lebesgue integrable if the function $|f(x)| = \sqrt{u(x)^2 + v(x)^2}$

(which is non-negative) is Lebesgue integrable in the sense defined previously.

It is clear that

$|u(x)| \leq |f(x)|$ and $|v(x)| \leq |f(x)|$

Also, if $a, b \geq 0$ it has $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$ so that

$|f(x)| \leq |u(x)| + |v(x)|$.

We can deduce from these straightforward inequalities that a complex-valued function is only integrable if its real and imaginary parts are integrable. Then, the Lebesgue integral of *f* is defined by

$$\int f(x)dx = \int u(x)dx + i\int v(x)dx$$

Finally, if *E* is a measurable subset of $\mathbb{R}^d$,and *f* is a complex-valued measurable function on *E*, we say that *f* is Lebesgue integrable on *E* if f_{XE} is integrable on $\mathbb{R}^d$, and we define $\int_E f = \int f_{XE}$.

The collection of all complex-valued integrable functions on a measurable subset $E\subset \mathbb{R}^d$ forms a vector space over $\mathbb{C}$. Indeed, if *f* and *g* are integrable, then so is *f* + *g*, since the triangle inequality gives $|(f+g)(x)| \leq |f(x)| + |g(x)|$, and monotonicity of the integral Then, at that point, the Lebesgue necessary offis characterized by

$$\int_E |f+g|\,d\mu \leq \int_E |f|\,d\mu + \int_E |g|\,d\mu < \infty$$

Also, it is clear that if $a \in \mathbb{C}$ and if f is integrable, then so is *af*. Finally, the integral continues to be linear over $\mathbb{C}$.

Approximation of Lebesgue integrable functions by continuous functions:

The main concept of the approximation is to demonstrate that a sequence of continuous functions may roughly approach any Lebesgue integrable function, both in terms of pointwise and integral convergence. Because continuous functions are easier to deal with and have a lot of attractive qualities that make them conducive to analysis, this finding is very relevant.

Formally, let *f* be a Lebesgue integrable function on a measurable set $E\subseteq \mathbb{R}^n$. Then, the approximation theorem states that for any ε > 0, there exists a continuous function g on $\mathbb{R}^n$ such that the Lebesgue measure of the set where *f* and *g* differ (i.e., {x ∈ E: |f(x) - g(x)| > ε}) is arbitrarily small. In mathematical terms, we can find a sequence (g_n) of continuous functions converging to f almost everywhere on *E*:

$$\lim_{n\to\infty} \int |f(x) - g_n(x)|\,d\mu = 0$$

where μ is the Lebesgue measure.

6.5 THE SPACE $L^1(\mu)$ OF INTEGRABLE FUNCTIONS

An important observation about the algebraic properties of integrable functions is their formation of a vector space. The fact that this vector space is complete within the appropriate norm is an essential analytic fact.

Definition: If $(X, \mathcal{A}, \mu)$ is a measure space, then the space $L^1(X)$ consists of the integrable functions $f: X \to \mathbb{R}$ with norm

$$\| f \|_{L^1} = \int |f| \, d\mu < \infty$$

where we identify functions that are equal a.e. A sequence of functions

$$\{f_n \in L^1(X)\}$$

converges in L^1, or in mean, to $f \in L^1(X)$ if

$$\| f - f_n \|_{L^1} = \int |f - f_n| \, d\mu \to 0 \qquad as\ n \to \infty$$

We also denote the space of integrable complex-valued functions $f{:}X \to \mathbb{C}$ by $L^1(X)$. For definiteness, we consider real-valued functions unless stated otherwise; in most cases, the results generalize in an obvious way to complex-valued functions.

Let us consider the particular case of $L^1(\mathbb{R}^d)$. As an application of the Borel regularity of Lebesgue measure, we prove that integrable functions on $\mathbb{R}^d$ may be approximated by continuous functions with compact support. This result means that $L^1(\mathbb{R}^d)$ is a concrete realization of the completion of $C_c(\mathbb{R}^d)$with respect to the $L^1(\mathbb{R}^d)$-norm, where $C_c(\mathbb{R}^d)$denotes the space of continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ with compact support. The support of *f* is defined by

$$supp\ f = \overline{\{x \in \mathbb{R}^d : f(x) \neq 0\}}$$

Thus, *f* has compact support if and only if it vanishes outside a bounded set.

Properties of L^1 integrable functions:

i) **Measurability:** L^1 integrable functions are required to be measurable. This means that the set $\{x: |f(x)| > M\}$ must be measurable for all $M > 0$.

ii) **Linearity**: If $f(x)$ and $g(x)$ are both L^1 integrable functions, then any linear combination of these functions, such as $af(x) + bg(x)$, where *a* and *b* are constants, is also L^1 integrable.

iii) **Triangle Inequality:** The integral of the absolute value of the sum of two L^1 integrable functions is less than or equal to the sum of their individual integrals:
$\int|f(x) + g(x)|\, dx \leq \int|f(x)|\, dx + \int|g(x)|\, dx$

iv) **Dominating Function:** If $|f(x)| \leq g(x)$ almost everywhere on the measurable set *E*, and $g(x)$ is L^1 integrable, then f(x) is also L^1 integrable.

v) **Convergence in Measure:** If a sequence of measurable functions $\{f_n(x)\}$ converges to $f(x)$ in measure, and each function $f_n(x)$ is L^1 integrable, then the limit function $f(x)$ is also L^1 integrable:

$$\lim_{n \to \infty} \int |f_n(x) - f(X)| \, dx = 0 \Rightarrow \int |f(x)| \, dx < \infty$$

vi) **Product of L^1 Functions:** If *f(x)* is an L^1 integrable function and *g(x)* is a bounded measurable function, then the product *f(x)g(x)* is also L^1 integrable.

vii) **Change of Measure**: If *f(x)* is an L^1 integrable function over a measurable set *E*, and there exists a measurable function *h(x)* such that $h(x) \geq 0$ and $\int h(x)\, dx < \infty$, then the integral of *f(x)* with respect to the measure defined by *h(x)* is also finite.

viii) **Approximation:** L^1 integrable functions can be approximated by simple functions (finite linear combinations of indicator functions).

ix) **Density in L^1:** Under appropriate conditions, continuous functions with compact support are dense in L^1 space.

Proposition: The space $L^1(\mathbb{R})$ is linear (over $\mathbb{C}$) and if $f \in L^1(\mathbb{R})$ the real and imaginary parts, *Re f*, *Im f* are Lebesgue integrable as are there positive parts and as is also the absolute value, $|f|$.

Proof: We first consider the real part of a function $f \in L^1(\mathbb{R})$. Suppose $f_n \in c_c(\mathbb{R})$ is an approximating sequence. Then consider $g_n = Re\, f_n$. This is absolutely summable,

since $\int |g_n| \leq \int |f_n|$ and

$$\sum_n f_n(x) = f_n \Rightarrow \sum_n g_n = \operatorname{Re} f(x)$$

Since the left identity holds a.e., so does the right and hence $Re\, f \in L^1(\mathbb{R})$.

The same argument with the imaginary parts shows that $Im\, f \in L^1(\mathbb{R})$. This also shows that a real element has a real approximating sequence and taking positive parts that a positive function has a positive approximating sequence.

Indeed, if $f, g \in L^1(\mathbb{R})$ have approximating series f_n and g_n then $h_n = f_n + g_n$ is absolutely summable,

$$\sum_n \int |h_n| \leq \sum_n \int |f_n| + \sum_n \int |g_n|$$

And

$$\sum_n f_n(x) = f(x), \sum_n g_n = g(x) \Rightarrow \sum_n h_n(x) = f(x) + g(x).$$

The first two conditions hold outside (probably different) sets of measure zero, *E* and *F*, so the conclusion holds outside $E \cup F$ which is of measure zero. Thus $f + g \in L^1(\mathbb{R})$. The case of cf for $c \in \mathbb{C}$ is more obvious.

The proof that $|f| \in L^1(\mathbb{R})$ if $f \in L^1(\mathbb{R})$ is similar but perhaps a little trickier. Again, let $\{f_n\}$ be a sequence as in the definition showing that $f \in L^1(\mathbb{R})$. To make a series for $|f|$ we can try the 'obvious' thing. Namely we know that

$$\sum_{j=1}^{n} f_j(x) \to f(x) \; if \; \sum_j |f_j(x)| < \infty$$

so certainly, it follows that

$$\left|\sum_{j=1}^{\infty} f_j(x)\right| \to |f(x)| \quad if \sum_j |f_j(x)| < \infty$$

So set

$$g_1(x) = |f_1(x)|;\; g_k(x) = \left|\sum_{j=1}^{k} f_j(x)\right| - \left|\sum_{j=1}^{k-1} f_j(x)\right| \forall x \in \mathbb{R}\,.$$

Then, for sure,

$$\sum_{k=1}^{N} g_k(x) = \left|\sum_{j=1}^{N} f_j(x)\right| \to |f(x)| \quad if \sum_j |f_j(x)| < \infty$$

So equality holds off a set of measure zero and we only need to check that $\{g_j\}$ is an absolutely summable series.

The triangle inequality in the 'reverse' form $||v| - |w|| \le |v - w|$ shows that, for $k > 1$,

$$|g_k(x)| = \left|\left|\sum_{j=1}^{k} f_j(x)\right| - \left|\sum_{j=1}^{k-1} f_j(x)\right|\right| \le |f_k(x)|$$

Thus,

$$\sum_k \int |g_k| \le \sum_k \int |f_k| < \infty$$

so the g_k's do indeed form an absolutely summable series and holds almost everywhere, so $|f| \in L^1(\mathbb{R})$.

Riesz-Fischer theorem:

The Riesz-Fischer theorem, also known as the Riesz representation theorem, is a fundamental result in functional analysis that establishes the connection between certain types of continuous linear functionals and elements of a Hilbert space. This theorem is named after the Hungarian mathematician Frigyes Riesz and the German mathematician Ernst Fischer, who both made significant contributions to its development.

Theorem: The vector space L^1 is complete in its metric.

Proof: Suppose $\{f_n\}$ is a Cauchy sequence in the norm, so that $\| f_n - f_m \| \to 0$ as $n, m \to \infty$.

The plan of the proof is to extract a subsequence of $\{f_n\}$ that converges to f, both pointwise almost everywhere and in the norm.

Under ideal circumstances we would have that the sequence $\{f_n\}$ converges almost everywhere to a limit f, and we would then prove that the sequence converges to f also in the norm.

Unfortunately, almost everywhere convergence does not hold for general Cauchy sequences. The main point, however, is that if the convergence in the norm is rapid enough, then almost everywhere convergence is a consequence, and this can be achieved by dealing with an appropriate subsequence of the original sequence.

Indeed, consider a subsequence $\{f_{n_k}\}_{K=1}^{\infty}$ of $\{f_n\}$ with the following property:

$$\| f_{n_{k+1}} - f_{nk} \| \le 2^{-k}, \textit{ for all } k \ge 1$$

The existence of such a subsequence is guaranteed by the fact that $\| f_{n_{k+1}} - f_{nk} \| \le \in$ whenever $n, m \ge N(\epsilon)$, so that it suffices to take $n_k = N(2^{-k})$. We now consider the series whose convergence will be seen below,

$$f(x) = f_{n1} + \sum_{k=1}^{\infty} \left(f_{n_{k+1}}(x) - f_{nk}(x) \right)$$

And

$$g(x) = |f_{n1}| + \sum_{k=1}^{\infty} \left| f_{n_{k+1}}(x) - f_{nk}(x) \right|$$

And note that

$$\int |f_{n1}| + \sum_{k=1}^{\infty} \int \left| f_{n_{k+1}}(x) f_{nk}(x) \right| \le \int |f_{n_1}| + \sum_{k=1}^{\infty} 2^{-k} < \infty .$$

So the monotone convergence theorem implies that g is integrable, and since $|f| \le g$, hence so is f. In particular, the series defining f converges almost everywhere, and since the partial sums of this series are precisely the f_{n_k} (by construction of the telescopic series), we find that

$$f_{nk}(x) \to f(x) \quad a.e.x$$

To prove that $f_{nk} \to f$ in L^1 as well, we simply observe that $|f - f_{nk}| \le g$ for all k, and apply the dominated convergence theorem to get $\| f_{nk} - f \|_{L^1} \to 0$ as $. k \to 0$

Finally, the last step of the proof consists in recalling that $\{f_n\}$ is Cauchy. Given $\in$, there exists N such that for all $n, m > N$ we have $\| f_n - f_m \| < \in /2$. If n_k is chosen so that $n_k > N$, and $\| f_{nk} - f \| < \in /2$, then the triangle inequality implies

$$\| f_n - f \| \le \| f_n - f_{nk} \| + \| f_{nk} - f \| < \in$$

whenever $n > N$. Thus $\{f_n\}$ has the limit f in L^1.

Hence proved.

6.6 LETS SUM UP

In this chapter we have learn:

- Signed measures are particularly useful when dealing with functions and distributions that have both positive and negative components or values. They find applications in various areas, such as the study of integration with respect to signed measures, Lebesgue-Stieltjes integration, complex analysis, and the study of distributions in functional analysis.
- The Hahn Decomposition Theorem addresses the decomposition of a measurable space into two disjoint sets, where the positive and negative components of a signed measure are concentrated separately.
- The space L^1 (μ) mathematicians and scientists gain access to a versatile class of functions that are essential for understanding the behaviour and properties of integrable functions with respect to a given measure μ.

6.7 UNIT END EXERCISES

1) A function f : $\mathbb{R} \longrightarrow \mathbb{R}$ is an element of $L^1(\mathbb{R})$ if and only if it is measurable and there exists $F \in L^1(\mathbb{R})$ such that $|f| \leq F$ almost everywhere.

2) Show that there are $f \in L^1(\mathbb{R}^d)$ and a sequence $\{f_n\}$ with fn $\in L^1(\mathbb{R}^d)$ such that
$\| f - f_n \|_{L^1} \to 0$, but $f_n(x) \to f(x)$ for no x.

3) Show that $f * g$ is integrable whenever f and g are integrable, and that
$\| f * g \|_{L^1(\mathbb{R}^d)} \leq \| f \|_{L^1(\mathbb{R}^d)} \| g \|_{L^1(\mathbb{R}^d)}$ '
with equality if f and g are non-negative.

4) State and prove Riesz-Fischer theorem.
5) Prove that if f and g are integrable functions on X, and $|g(x)| \leq |f(x)|$ for all x in X, then f and g are both in $L^1(\mu)$ if and only if f is in $L^1(\mu)$.

6) Let ν be a finite signed measure and let μ be a measure on (X, M). Then $\nu \ll \mu$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ when $\mu(E) < \delta$.

7) Show that if D is a Lebesgue measurable subset of $\mathbb{R}^d$, then
$L_D = \{E \in L_{\mathbb{R}^d} : E \subseteq D\}$.

8) Prove that "The space $Cc(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$. Explicitly, if $f \in L^1(\mathbb{R}^d)$, then for any $\in > 0$ there exists a function $g \in Cc(\mathbb{R}^n)$ such that $\| f - g \|_{L1} < \in$".

9) If (X, $\mathcal{A}$, μ) is a measure space and $v^+, v^- : \mathcal{A} \rightarrow [0, \infty]$ are measures, one of which is finite, then show that $v = v^+ - v^-$: is a signed measure.

10) Suppose that v is a signed measure on a measurable space (X, $\mathcal{A}$). If $A \in \mathcal{A}$ and $0 < v(A) < \infty$, then there exists a positive subset $P \subset A$ such that $v(P) > 0$.

6.8 LIST OF REFERENCE

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