UNIT- 1

CHAPTER -1

BASICS OF SIGNALS AND SYSTEMS

1.1 Introduction

1.2 Energy Signal and Power Signal

1.3 Transformations of the independent variable

1.3.1 Examples of Transformations of the Independent Variable

1.4 Periodic Signals

1.5 Even and Odd Signals

1.6 Exponential and sinusoidal signals

1.6.1 Continuous-time complex exponential and sinusoidal signals

1.6.2 Discrete-time complex exponential and sinusoidal signals

1.6.3 Sinusoidal Signals

1.7. The discrete-Time Unit Impulse and Unit Step Sequences

1.8 The Continuous-Time Unit Step and Unit Impulse Functions

1.9 Sampling property of the continuous-time unit impulse:

1.10 Continuous-Time and Discrete-Time Systems

1.10.1 Simple Examples of Systems

1.11 Interconnects of Systems

1.12 Basic System Properties

1.12.1 Systems with and without Memory

1.12.2 Invertibility and Inverse System

1.12.3 Causality

1.12.4 Stability

1.12.5 Time Invariance

1.12.6 Linearity

1.0 OBJECTIVE

- Understand continuous and discrete time signals.
- Understand continuous and discrete time systems.
- Classify the signals and Systems

1.1 INTRODUCTION

Signals are represented mathematically as functions of one or more independent variables. Here we focus attention on signals involving a single independent variable. For convenience, this will generally refer to the independent variable as time.

There are two types of signals: continuous-time signals and discrete-time signals.

Continuous-time signal: The variable of time is continuous. A speech signal as a function of time is a continuous-time signal.

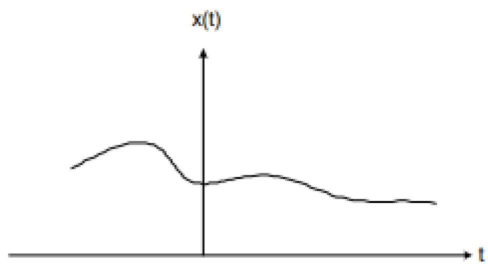


Figure 1.1: Graphical representation of Continuous-time signal

Discrete-time signal: the variable of time is discrete. The weekly Dow Jones stock market index is an example of discrete-time signal.

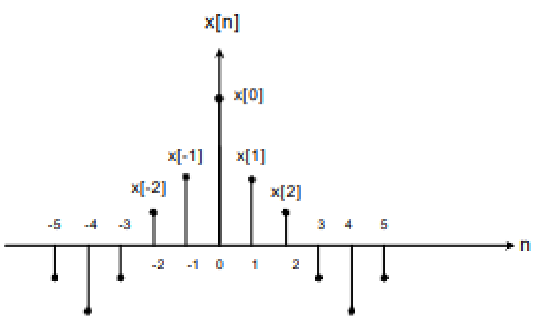


Figure 1.2 : Graphical representation of Discrete-time signals

To distinguish between continuous-time and discrete-time signals we use symbol t to denote the continuous variable and n to denote the discrete-time variable. And for continuous-time signals we will enclose the independent variable in parentheses (·), for discrete-time signals we will enclose the independent variable in bracket [·].

A discrete-time signal x [n]may represent a phenomenon for which the independent variable is inherently discrete. A discrete-time signal x [n]may represent successive samples of an underlying phenomenon for which the independent variable is continuous. For example, the processing of speech on a digital computer requires the use of a discrete time sequence representing the values of the continuous-time speech signal at discrete points of time.

1.2 Energy Signal and Power Signal

If v (t) and i(t)are respectively the voltage and current across a resistor with resistance R , then the instantaneous power is

$$p(t) = v(t)\, i(t) = \frac{1}{R} v^2(t)$$

The total energy expended over the time interval is $t_1 \leq t \leq t_2$

$$\int_{t1}^{t2} p(t)dt = \int_{t1}^{t2} \frac{1}{R} v^2(t)\, dt$$

and the average power over this time interval is

$$\frac{1}{t2-t1} \int_{t1}^{t2} p(t)dt = \frac{1}{t2-t1} \int_{t1}^{t2} \frac{1}{R} v^2(t)\, dt$$

For any continuous-time signal x (t)(or any discrete-time signal x [n], the total energy over the time interval $t_1 \leq t \leq t_2$ in a continuous-time signal x (t) is defined as

$$\int_{t1}^{t2} |x(t)|^2\, dt$$

where |x| denotes the magnitude of the (possibly complex) number x .

The time-averaged power is

$$\frac{1}{t2-t1} \int_{t1}^{t2} |x(t)|^2\, dt$$

Similarly the total energy in a discrete-time signal x [n] over the time interval $n_1 \leq n \leq n_2$ is defined as

$$\sum_{n1}^{n2} |x[n]|^2$$

The average power is

$$\frac{1}{n2-n1+1} \sum_{n1}^{n2} |x[n]|^2$$

In many systems, we will be interested in examining the power and energy in signals over an infinite time interval, that is, for $-\infty \le t \le +\infty$ or. $-\infty \le n \le +\infty$

The total energy in continuous time is then defined

$E_\infty = \text{Lim } (T \rightarrow \infty) \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$,

And in discrete time,

$E_\infty = \text{Lim } (N \rightarrow \infty) \sum_{-N}^{N} |x[n]|^2 = \sum_{-\infty}^{\infty} |x[n]|^2$

For some signals, the integral in continuous Equation or sum in discrete might not converge, that is, if x (t) or x [n] equals a nonzero constant value for all time. Such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

The time-averaged power over an infinite interval

$P_\infty = \text{Lim } (T \rightarrow \infty) \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$

$$P_\infty = \text{Lim } (N \rightarrow \infty) \frac{1}{2N+1} \sum_{n1}^{n2} |x[n]|^2$$

Three types of signals:

Type 1: signals with finite total energy, $E_\infty < \infty$ and zero average power,

$P_\infty = \text{Lim } (T \rightarrow \infty) \frac{E_\infty}{2T} = 0$

Type 2: with finite average power P_∞.

If $P_\infty > 0$, then $E_\infty = \infty$.

An example is the signal x [n] = 4 ,

it has infinite energy, but has an average power of $P_\infty = 16$.

Type 3: signals for which neither P_∞ and E_∞ are finite. An example of this signal is x(t)= t .

1.3 Transformations of the independent variable

In many situations, it is important to consider signals related by a modification of the independent variable. These modifications will usually lead to reflection, scaling, and shift.

1.3.1 Examples of Transformations of the Independent Variable

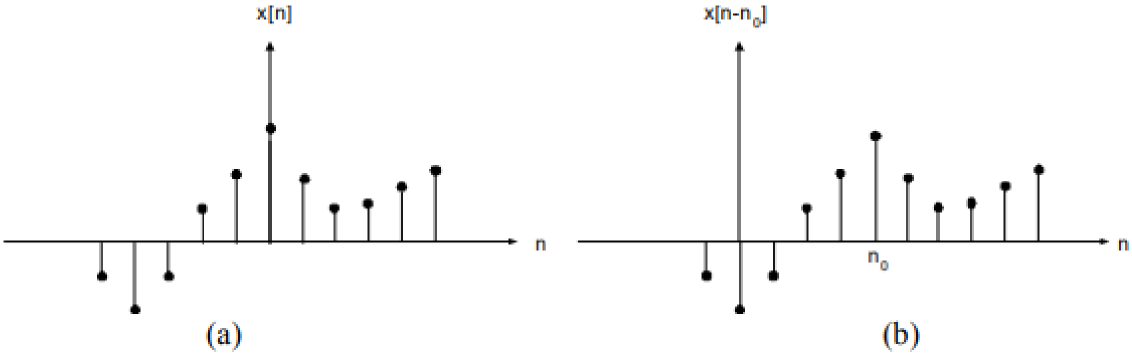


Figure.1.3 Discrete-time signals related by a time shift.

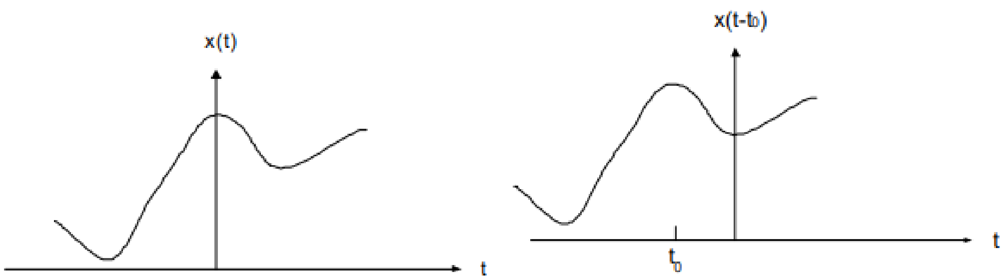


Fig. 1.4 Continuous-time signals related by a time shift.

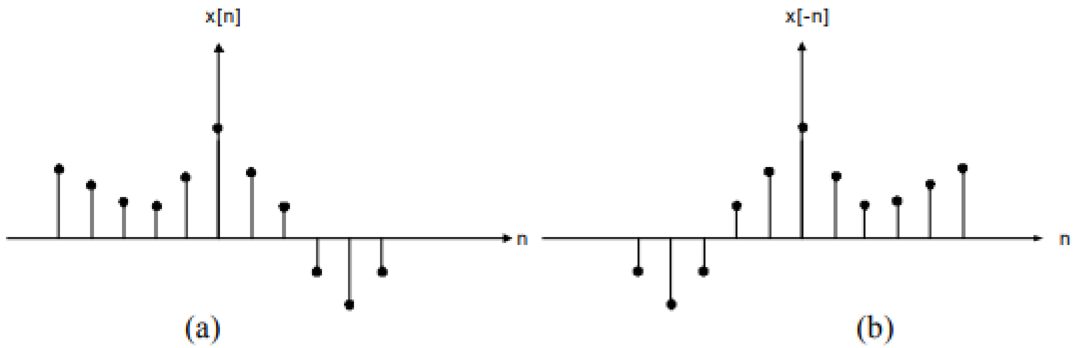


Fig. 1.5 (a) A discrete-time signal x [n]; (b) its reflection, x [-n] about n = 0

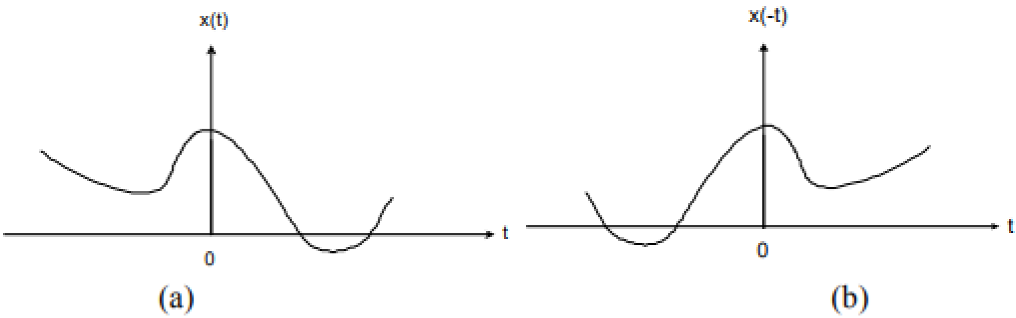


Fig. 1.6 (a) A continuous-time signal x(t) ; (b) its reflection, x (-t)about t = 0 .

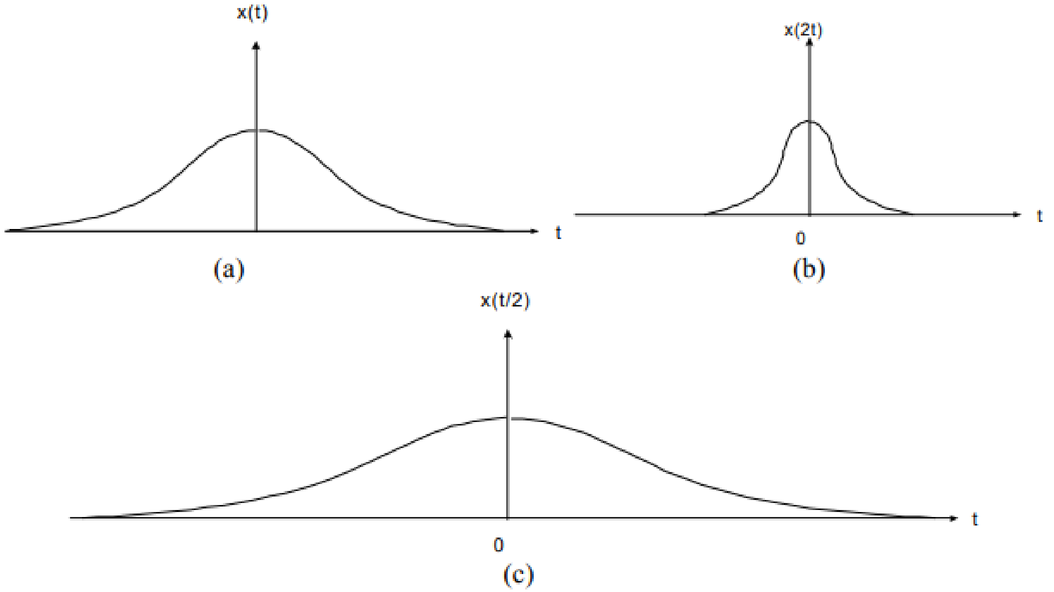


Fig. 1.7 Continuous-time signals related by time scaling.

1.4 Periodic Signals

A periodic continuous-time signal x (t) has the property that there is a positive value of T for which x (t) = x (t + T)   for all t

From Equation , we can deduce that if x (t) is periodic with period T,

then x (t) = x (t + mT) for all t and for all integers m .

Thus, x(t) is also periodic with period 2T, 3T, …. The fundamental period T0 of x(t) is the smallest positive value of T

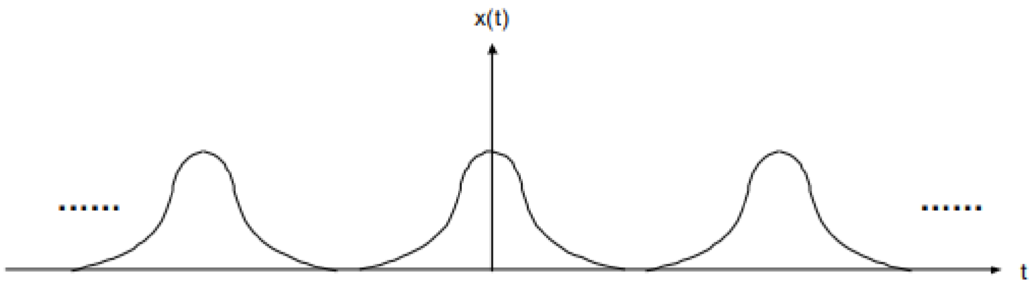


Fig. 1.8 Continuous-time periodic signal.

A discrete-time signal x [n] is periodic with period N ,

where N is an integer, if it is unchanged by a time shift of N,

x[n] = x [n + N] for all values of n.

If Equation holds, then x [n] is also periodic with period 2N , 3N , …. The fundamental period N_0 is the smallest positive value of N for which Equation holds.

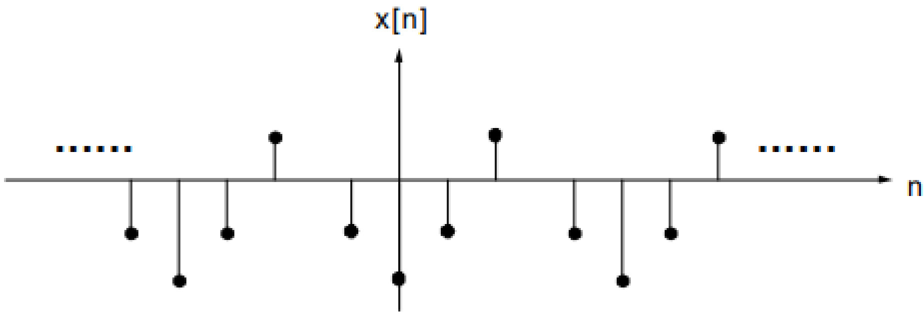


Fig. 1.9 Discrete-time periodic signal.

1.5 Even and Odd Signals

In addition to their use in representing physical phenomena such as the time shift in a radar signal and the reversal of an audio tape, transformations of the independent variable are extremely useful in examining some of the important properties that signal may possess.

Signal with these properties can be even or odd signal, periodic signal:

An important fact is that any signal can be decomposed into a sum of two signals, one of which is even and one of which is odd.

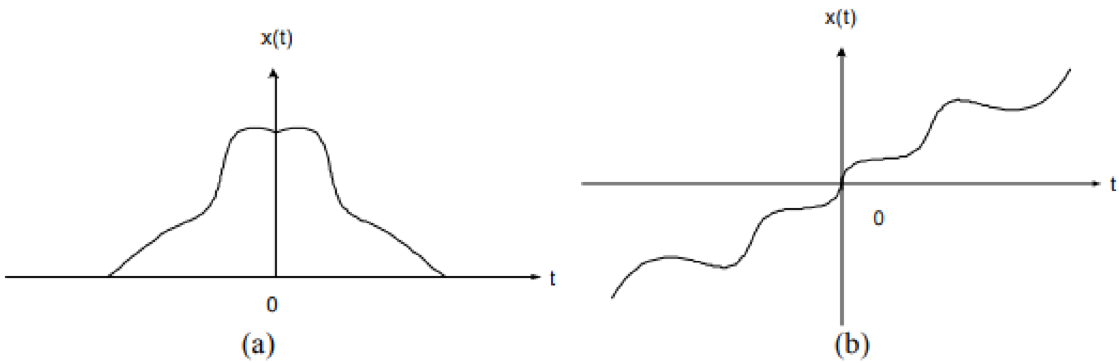


Fig. 1.10 An even continuous-time signal; (b) an odd continuous-time signal.

$$EV\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

which is referred to as the even part of x(t) .

Similarly, the odd part of x (t) is given by

$$OD\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

Exactly analogous definitions hold in the discrete-time case.

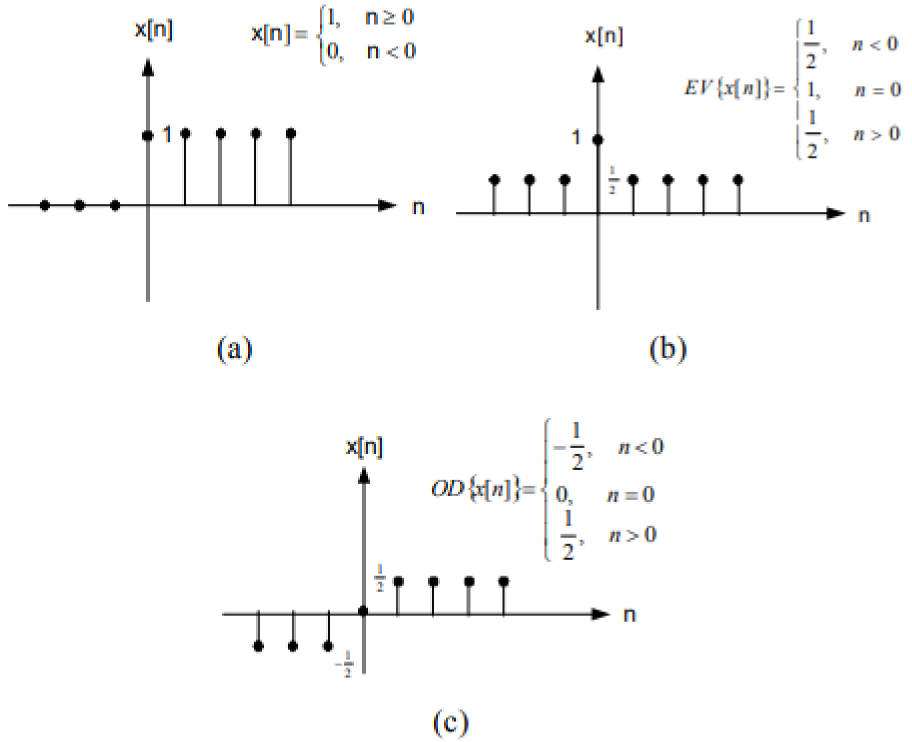


Fig.1.11 The even-odd decomposition of a discrete-time signal

1.6 Exponential and sinusoidal signals

1.6.1 Continuous-time complex exponential and sinusoidal signals

The continuous-time complex exponential signal

$$x(t)= Ce^{at}$$

Where C and a are in general complex numbers.

Real exponential signals

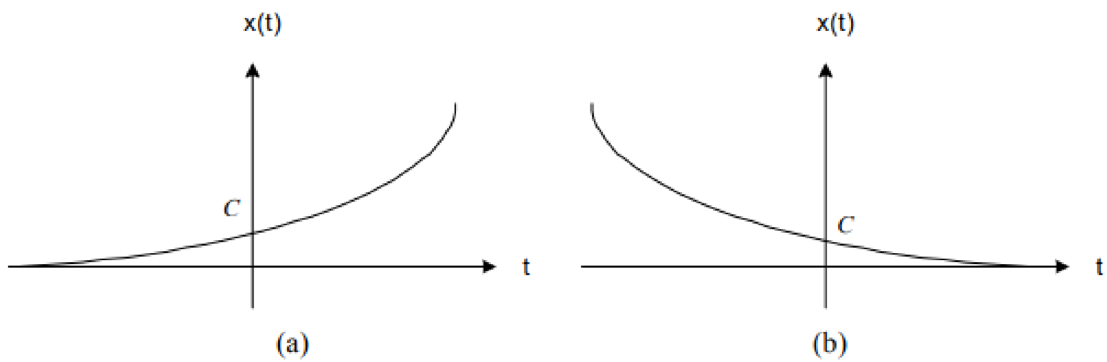


Fig. 1.12 The continuous-time complex exponential signal at $x(t)= Ce^{at}$

, (a) $a > 0$; (b) $a < 0$.

Periodic complex exponential and sinusoidal signals

If a is purely imaginary,

we have $x(t) = e^{j\omega_0 t}$

An important property of this signal is that it is periodic. We know x (t) is periodic with period T if

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T}$$

For periodicity, we must have

$e^{j\omega_0 T} = 1$

For $\omega_0 \neq 0$, the fundamental period T_0 is

$$T_0 = \frac{2\pi}{|\omega_0|}$$

Thus, the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period.

A signal closely related to the periodic complex exponential is the sinusoidal signal

$x(t) = A\cos(\omega_0 t + \Theta)$

With seconds as the unit of t, the units of Θ and ω_0 are radians and radians per second. It is also known $\omega_0 = 2\pi f_0$, where f_0 has the unit of circles per second or Hz.

The sinusoidal signal is also a periodic signal with a fundamental period of T_0 .

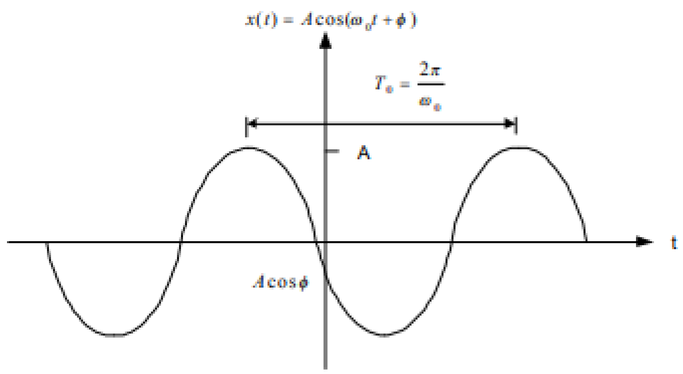


Fig. 1.13 Continuous-time sinusoidal signal.

Using Euler's relation, a complex exponential can be expressed in terms of sinusoidal signals with the same fundamental period:

$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$

Similarly, a sinusoidal signal can also be expressed in terms of periodic complex exponentials with the same fundamental period:

$$A\cos(\omega_0 t+\phi)=\frac{A}{2}e^{j\phi}e^{j\omega_0 t}+\frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

A sinusoid can also be expresses as

$$A\cos(\omega_0 t+\phi)=A\,\mathrm{Re}\left\{e^{j(\omega_0 t+\phi)}\right\}$$

And

$$A\sin(\omega_0 t+\phi)=A\,\mathrm{Im}\left\{e^{j(\omega_0 t+\phi)}\right\}$$

Periodic signals, such as the sinusoidal signals provide important examples of signal with infinite total energy, but finite average power. For example:

$$E_{period}=\int_0^{T_0}\left|e^{j\omega_0 t}\right|dt=\int_0^{T_0}1dt=T_0$$

$$P_{period}=\frac{1}{T_0}\int_0^{T_0}\left|e^{j\omega_0 t}\right|dt=\int_0^{T_0}1dt=1$$

Since there are an infinite number of periods as t ranges from $-\infty$ to $+\infty$, the total energy integrated over all time is infinite. The average power is finite since

$$P_\infty=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|e^{j\omega_0 t}\right|^2dt=1$$

General complex Exponential signals

Consider a complex exponential at Ce^{at}, where $C=|C|\,e^{j\Theta}$ is expressed in polar and

$a = r + j\,w0$ is expressed in rectangular form.

Then

$$Ce^{at}=|C|e^{j\theta}e^{(r+j\omega_0)t}=|C|e^{rt}e^{j(\omega_0 t+\theta)}=|C|e^{rt}\cos(\omega_0 t+\theta)+j|C|e^{rt}\sin(\omega_0 t+\theta).$$

Thus, for $r = 0$, the real and imaginary parts of a complex exponential are sinusoidal.

For $r > 0$, sinusoidal signals multiplied by a growing exponential.

For $r < 0$, sinusoidal signals multiplied by a decaying exponential.

Damped signal – Sinusoidal signals multiplied by decaying exponentials are commonly referred to as damped signal.

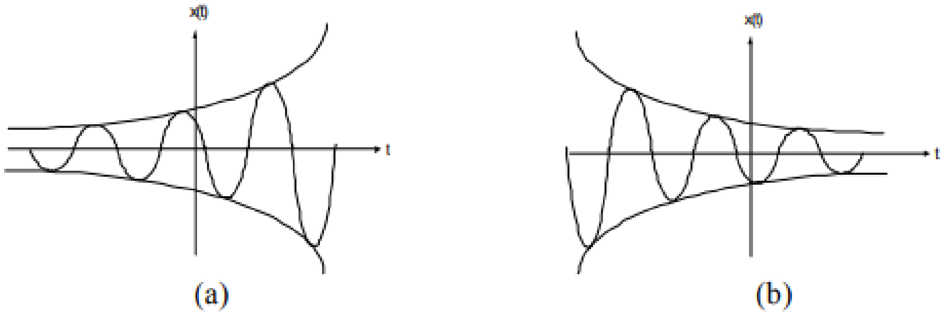


Fig. 1.14 (a) Growing sinusoidal signal; (b) decaying sinusoidal signal.

1.6.2 Discrete-time complex exponential and sinusoidal signals

A discrete complex exponential or sequence is defined by

$$x[n]= C\alpha^{n}$$

where C and αare in general complex numbers. This can be alternatively expressed

$$x\,[n]= C\,e^{\beta n}$$

where $\alpha = e^{\beta}$

Real Exponential Signals

If C and α are real, we have the real exponential signals

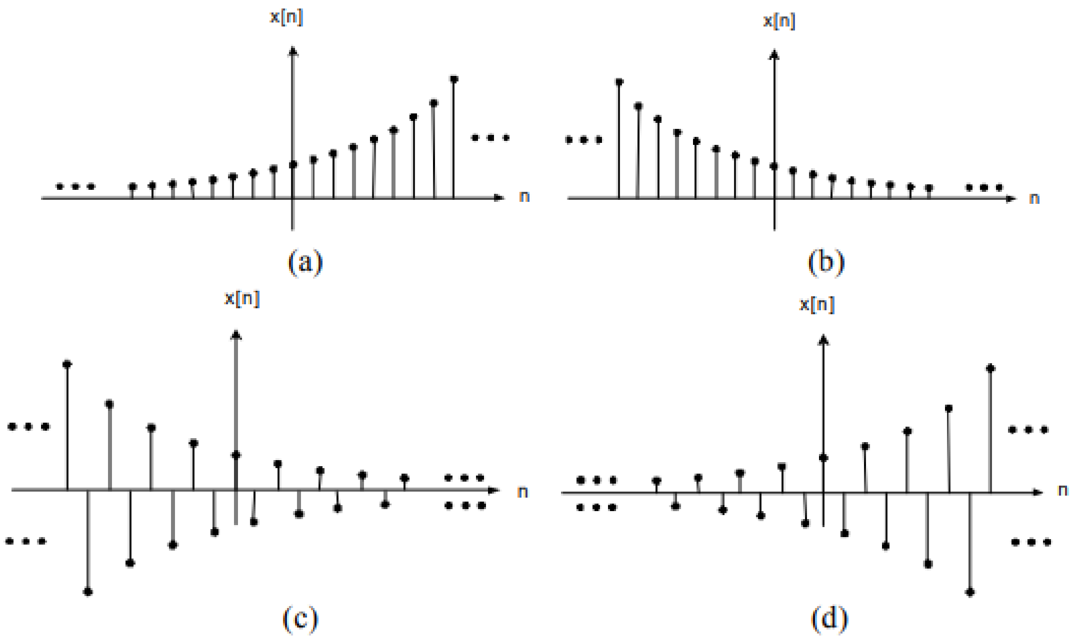


Fig. 1.15 Real Exponential Signal $x\,[n]= C\alpha^{n}$

: (a) α >1; (b) 0<α< 1 (c) -1< α< 0; (d) α < -1

1.6.3 Sinusoidal Signals

$$\mathbf{x[n] = e^{j\omega 0 n}}$$

$$\mathbf{e^{j\omega 0 n} = \cos\omega_0 n + j \sin \omega_0 n}$$

Similarly, a sinusoidal signal can also be expresses in terms of periodic complex exponentials with the same fundamental period:

$$A \cos (\omega_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega 0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega 0 n}$$

A sinusoid can also be expresses as

$$A \cos (\omega_0 n + \phi) = A \operatorname{Re}\{ e^{j(\omega 0 n + \phi)}\}$$

And $$A \sin (\omega_0 n + \phi) = A \operatorname{Im}\{ e^{j(\omega 0 n + \phi)}\}$$

The above signals are examples of discrete signals with infinite total energy, but finite average power. For example: every sample of $x[n] = e^{(-j\omega 0 n)}$ contributes 1 to the signal's energy. Thus the total energy $-\infty < n < +\infty$ is infinite, while the average power is equal to 1.

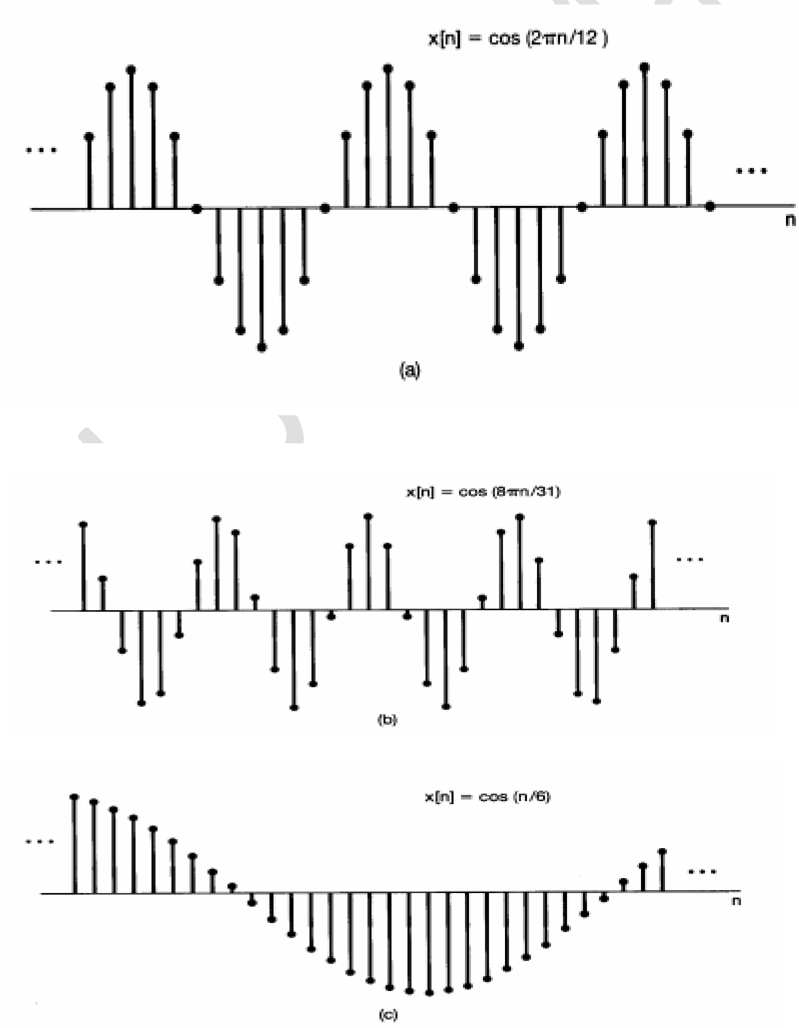


Fig.1.16 Discrete-time sinusoidal signal.

1.7. The discrete-Time Unit Impulse and Unit Step Sequences

Discrete-time unit impulse is defined as

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases},$$

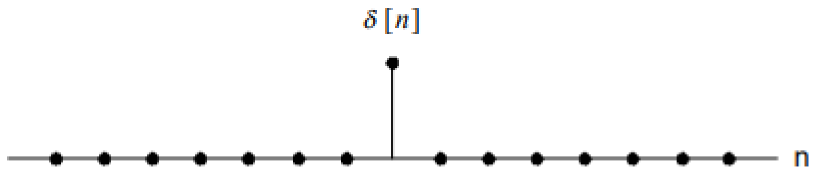


Fig. 1.17 Discrete-time unit impulse.

Discrete-time unit step is defined as

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases},$$

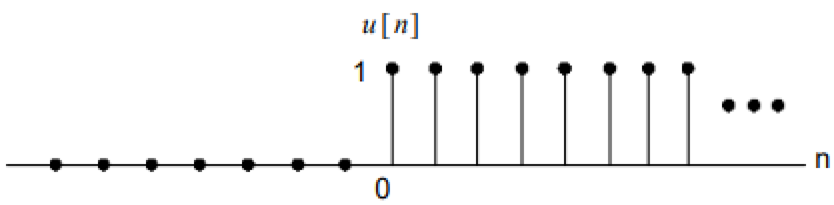


Fig. 1.18 Discrete-time unit step sequence.

The discrete-time impulse unit is the first difference of the discrete-time step

$$\delta[n] = u[n] - u[n-1]$$

The discrete-time unit step is the running sum of the unit sample:

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

It can be seen that for $n < 0$, the running sum is zero, and for $n \geq 0$, the running sum is 1.

If we change the variable of summation from m to k = n - m we have,

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k].$$

The unit impulse sequence can be used to sample the value of a signal at n = 0. Since it is nonzero only for n = 0, it follows that

$$x[n]\delta[n] = x[0]\delta[n].$$

More generally, a unit impulse

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

This sampling property is very important in signal analysis.

1.8 The Continuous-Time Unit Step and Unit Impulse Functions

Continuous-time unit step is defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases},$$

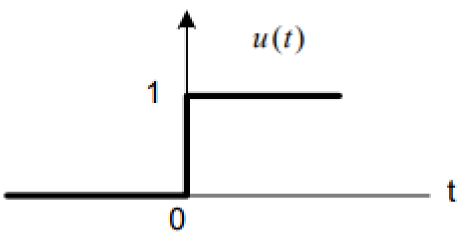


Fig. 1.19 Continuous-time unit step function

The continuous-time unit step is the running integral of the unit impulse

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau.$$

The continuous-time unit impulse can also be considered as the first derivative of the continuous time unit step,

$$\delta(t) = \frac{du(t)}{dt}.$$

Since u (t) is discontinuous at t = 0 and consequently is formally not differentiable. This can be interpreted, however, by considering an approximation to the unit step $u_\Delta(t)$, as illustrated in the figure below, which rises from the value of 0 to the value 1 in a short time interval of length Δ.

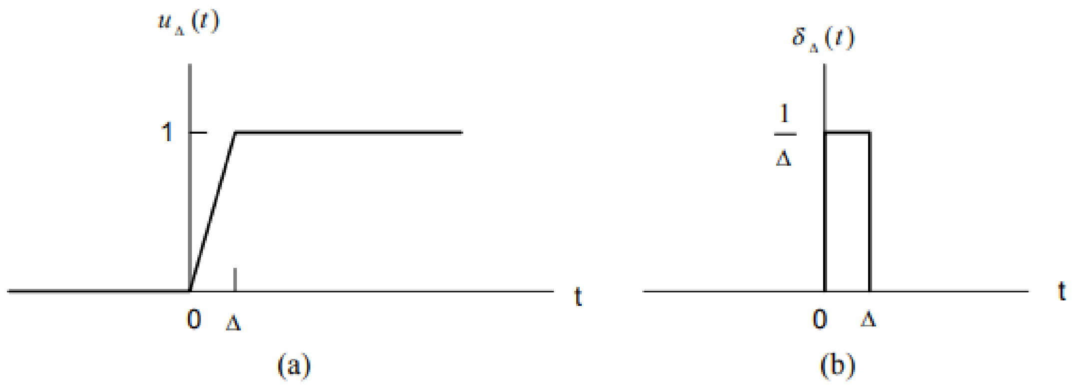


Fig. 1.20 (a) Continuous approximation to the unit step $u_\Delta(t)$; (b) Derivative of $u_\Delta(t)$.

The derivative is

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt},$$

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & otherwise \end{cases},$$

Note that It is a short pulse, of duration Δ and with unit area for any value of Δ. As Δ -> 0 , becomes narrower and higher, maintaining its unit area. At the limit,

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t),$$

$$u(t) = \lim_{\Delta \to 0} u_\Delta(t),$$

And

$$\delta(t) = \frac{du(t)}{dt}.$$

Graphically, it is represented by an arrow pointing to infinity at t = 0 , “1” next to the arrow represents the area of the impulse.

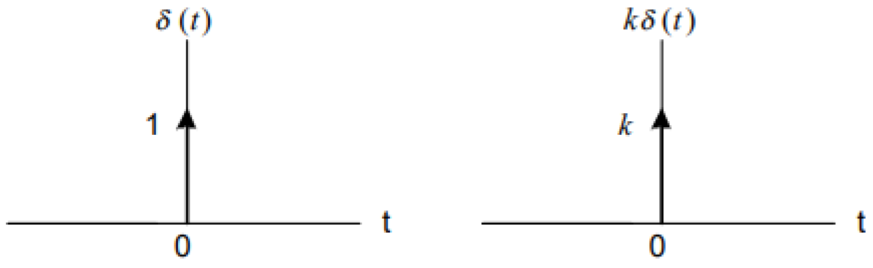


Fig. 1.21 Continuous-time unit impulse

1.9 Sampling property of the continuous-time unit impulse:

$$x(t)\delta(t) = x(0)\delta(t)$$

Or more generally,

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

Example:

Consider the discontinuous signal x (t)

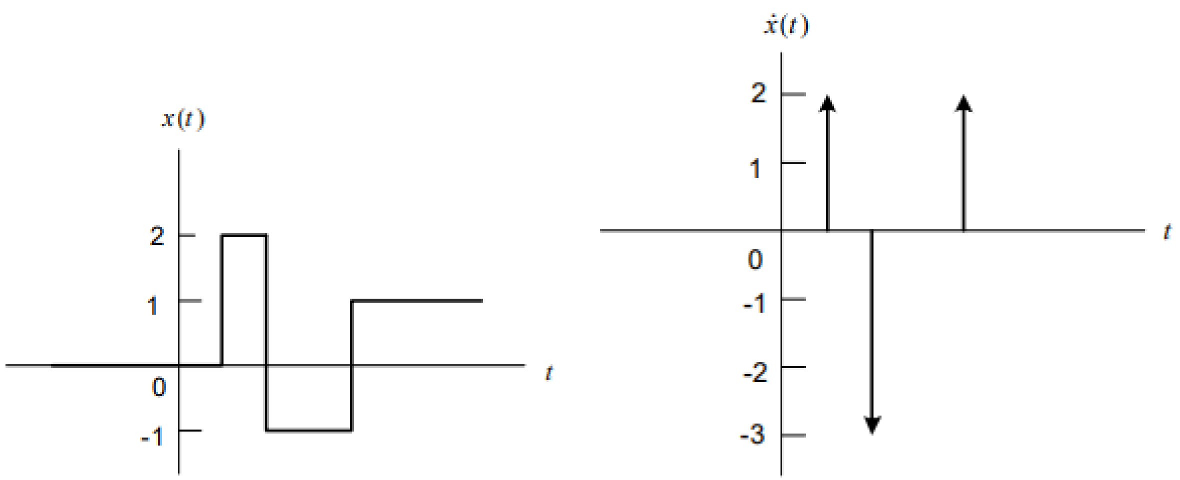
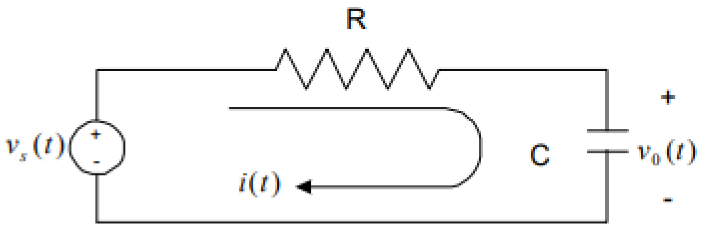


Fig. 1.22 The discontinuous signal and its derivative.

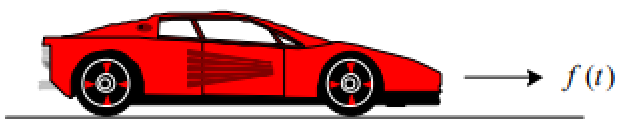
Note that the derivative of a unit step with a discontinuity of size of k gives rise to an impulse of area k at the point of discontinuity.

1.10 Continuous-Time and Discrete-Time Systems

A system can be viewed as a process in which input signals are transformed by the system or cause the system to respond in some way, resulting in other signals as outputs. Examples



(a)

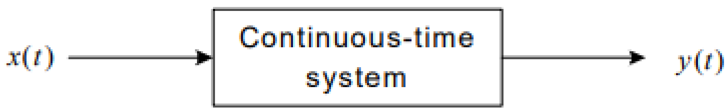


(b)

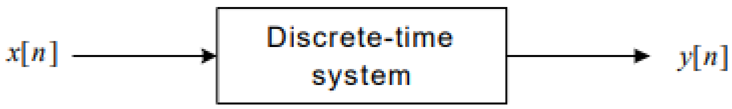
Fig. 1. 23 Examples of systems. (a) A system with input voltage $v_s(t)$ and output voltage $v_0(t)$.

(b) A system with input equal to the force f(t) and output equal to the velocity v(t) .

A continuous-time system is a system in which continuous-time input signals are applied and results in continuous-time output signals.



A discrete-time system is a system in which discrete-time input signals are applied and results in discrete-time output signals.



1.10.1 Simple Examples of Systems

Example 1: Consider the RC circuit in Fig. 23 (a).

The current i(t) is proportional to the voltage drop across the resistor:

$$i(t) = \frac{v_s(t) - v_C(t)}{R}.$$

The current through the capacitor is

$$i(t) = C\frac{dv_C(t)}{dt}.$$

Equating the right-hand sides of both the above equations, we obtain a differential equation describing the relationship between the input and output:

$$\frac{dv_C(t)}{dt} + \frac{1}{RC}v_C(t) = \frac{1}{RC}v_s(t),$$

Example 2: Consider the system in Fig. 23 (b), where the force f(t) as the input and the velocity v(t) as the output. If we let m denote the mass of the car and pv the resistance due to friction. Equating the acceleration with the net force divided by mass, we obtain

$$\frac{dv(t)}{dt} = \frac{1}{m}\left[f(t) - \rho v(t)\right] \quad \Rightarrow \quad \frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t).$$

It is first-order linear differential equations of the form:

$$\frac{dy(t)}{dt} + ay(t) = bx(t).$$

Example 3: Consider a simple model for the balance in a bank account from month to month. Let y [n] denote the balance at the end of nth month, and suppose that y[n] evolves from month to month according the equation:

$$y[n] = 1.01y[n-1] + x[n],$$

or

$$y[n] - 1.01y[n-1] = x[n],$$

where x [n] is the net deposit (deposits minus withdraws) during the nth month 1.01y[n -1] models the fact that we accrue 1% interest each month.

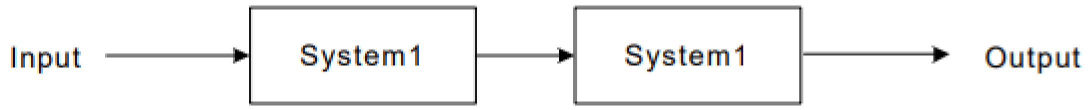
Some conclusions:

· Mathematical descriptions of systems have great deal in common;

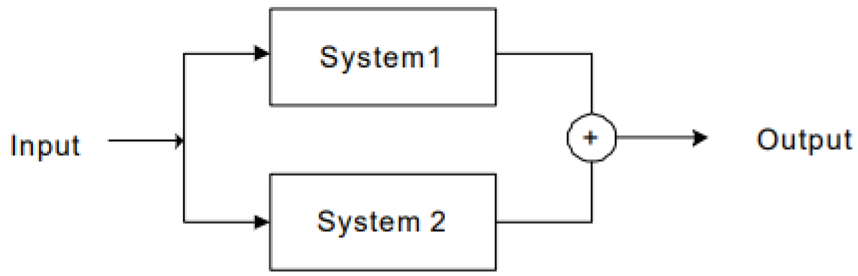
· A particular class of systems is referred to as linear, time-invariant systems.

· Any model used in describing and analyzing a physical system represents an idealization of the system.

1.11 Interconnects of Systems



(a)



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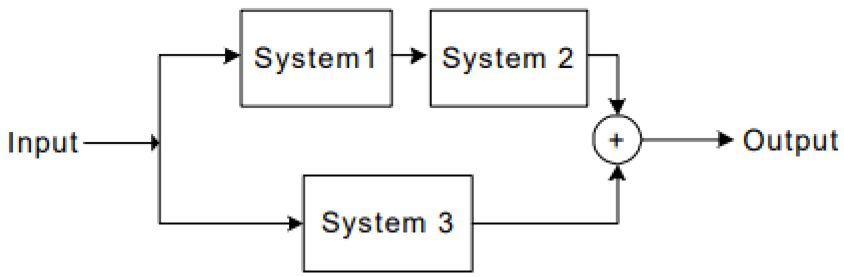


(c)

Fig. 1.24 Interconnection of systems. (a) A series or cascade interconnection of two systems; (b) A parallel interconnection of two systems;

(c) Combination of both series and parallel systems.

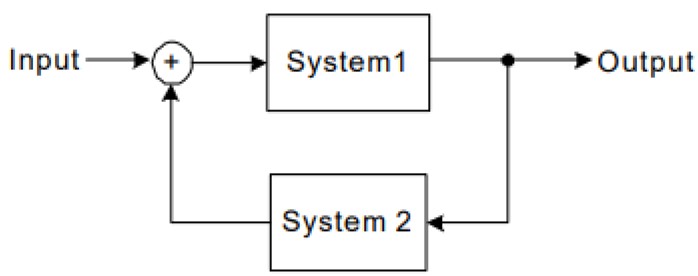
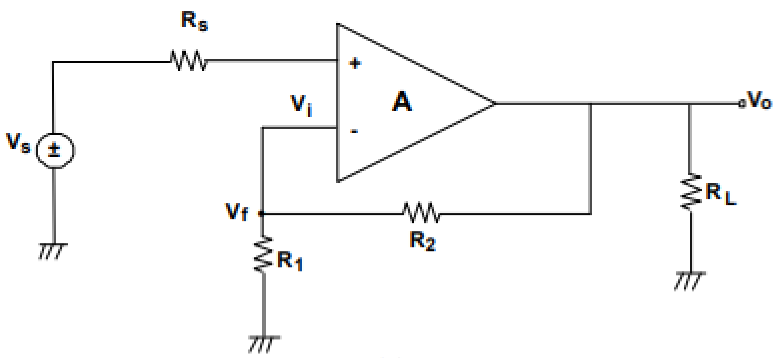
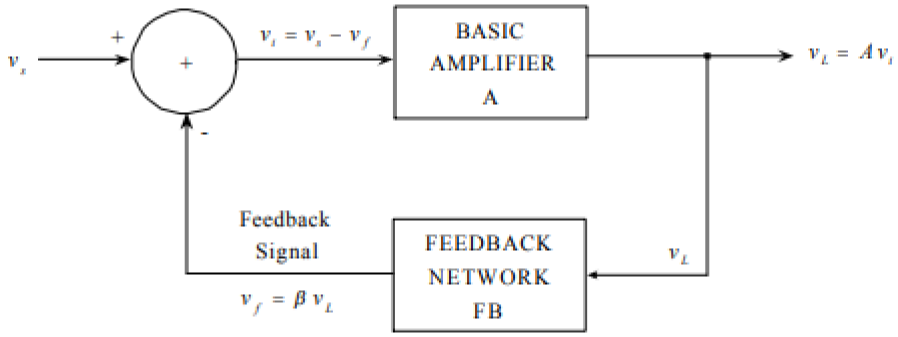


Fig. 1.25 Feedback interconnection.



(a)



(b)

Fig. 1.26 A feedback electrical amplifier.

1.12 Basic System Properties

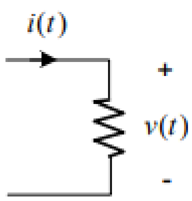
1.12.1 Systems with and without Memory

A system is memoryless if its output for each value of the independent variable as a given time is dependent only on the input at the same time. For example:

$$y[n] =(2\, x[n] - x^2[n])^2$$

is memoryless.

A resistor is a memoryless system, since the input current and output voltage has the relationship,



$$v(t) = R\, i(t),$$

where R is the resistance.

One particularly simple memoryless system is the identity system, whose output is identical to its input, that is

$$y(t)=x(t) \text{ or } y[n]=x[n]$$

An example of a discrete-time system with memory is an accumulator or summer.

$$y[n]=\sum_{k=-\infty}^{n} x[k]=\sum_{k=-\infty}^{n-1} x[k]+x[n]=y[n-1]+x[n],$$

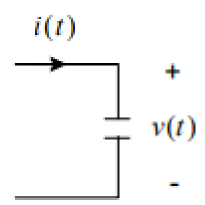
Or

$$y[n]-y[n-1]=x[n]$$

Another example is a delay

$$y[n]=x[n-1]$$

A capacitor is an example of a continuous-time system with memory



$$v(t)=\frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau,$$

where C is the capacitance

1.12.2 Invertibility and Inverse System

A system is said to be invertible if distinct inputs leads to distinct outputs.

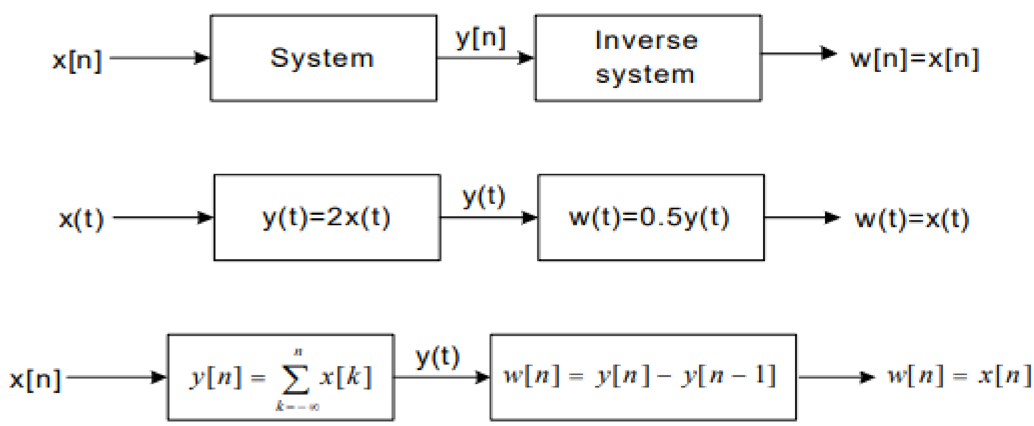


Fig. 1.27Concept of an inverse system.

Examples of non-invertible systems:

$y[n] = 0$,

the system produces zero output sequence for any input sequence.

$y(t) = x^2(t)$,

in which case, one cannot determine the sign of the input from the knowledge of the output. Encoder in communication systems is an example of invertible system, that is, the input to the encoder must be exactly recoverable from the output.

1.12.3 Causality

A system is causal if the output at any time depends only on the values of the input at present time and in the past. Such a system is often referred to as being nonanticipative, as the system output does not anticipate future values of the input.

The RC circuit in Fig. 23 (a) is causal, since the capacitor voltage responds only to the present and past values of the source voltage. The motion of a car is causal, since it does not anticipate future actions of the driver.

The following expressions describing systems that are not causal:

$$y[n] = x[n] - x[n+1],$$

and

$$y(t) = x(t+1)$$

All memoryless systems are causal, since the output responds only to the current value of input.

Example: Determine the Causality of the two systems:

(1) $y[n] = x[-n]$

(2) $y(t) = x(t)\cos(t+1)$

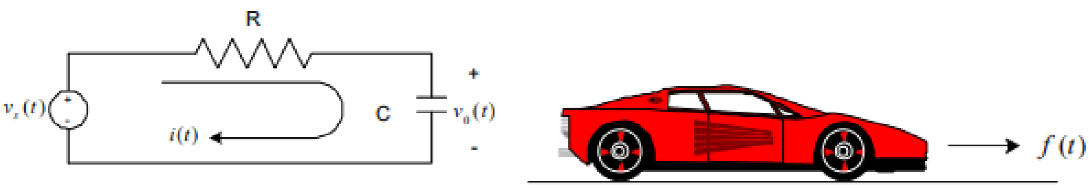
Solution: System (1) is not causal, since when $n < 0$, e.g. $n = -4$, we see that $y[-4] = x[4]$, so that the output at this time depends on a future value of input.

System (2) is causal. The output at any time equals the input at the same time multiplied by a number that varies with time.

1.12.4 Stability

A stable system is one in which small inputs leads to responses that do not diverge. More formally, if the input to a stable system is bounded, then the output must be also bounded and therefore cannot diverge.

Examples of stable systems and unstable systems:



(a) (b)

The above two systems are stable system.

The accumulator $y[n] = \sum_{K=-\infty}^{n} x[k]$ is not stable, since the sum grows continuously even if x [n] is bounded.

Check the stability of the two systems:

- S1; y(t) = tx (t) ;
- S2: $y(t)= e^{x(t)}$
- S1 is not stable, since a constant input x (t)= 1, yields y(t) = t , which is not bounded – no matter what finite constant we pick,| y(t)| will exceed the constant for some t.
- S2 is stable. Assume the input is bounded |x (t)| < B , or - B < x (t) < B for all t.

We then see that y (t) is bounded $e^{-B} < y(t) < e^{B}$

1.12.5 Time Invariance

A system is time invariant if a time shift in the input signal results in an identical time shift in the output signal. Mathematically, if the system output is y (t) when the input is x(t) , a timeinvariant system will have an output of $y(t-t_0)$ when input is $x(t-t_0)$.

Examples: ·

The system y (t) = sin[x (t)] is time invariant.

The system y [n] = n x[n] is not time invariant. This can be demonstrated by using counterexample. Consider the input signal $x_1[n] =\delta[n]$, which yields $y_1[n] =0$. However, the input $x_2[n]= \delta[n-1]$ yields the output $y2[n]=n\,\delta[n-1]$. Thus, while $x_2[n]$ is the shifted version of $x_1[n]$, $y_2[n]$ is not the shifted version of $y_1[n]$.

The system y (t) = x(2t) is not time invariant.

To check using counter example. Consider x1(t) shown in Fig. 1.30 (a), the resulting output y1(t) is depicted in Fig. 1.30 (b). If the input is shifted by 2, that is, consider $x_2(t)= x_1(t-2)$, as shown in Fig. 1.30 (c), we obtain the resulting output $y_2(t)= x_2(2t)$ shown in Fig. 1.30 (d). It is clearly seen that $y_2(t) \neq y_1(t-2)$, so the system is not time invariant.

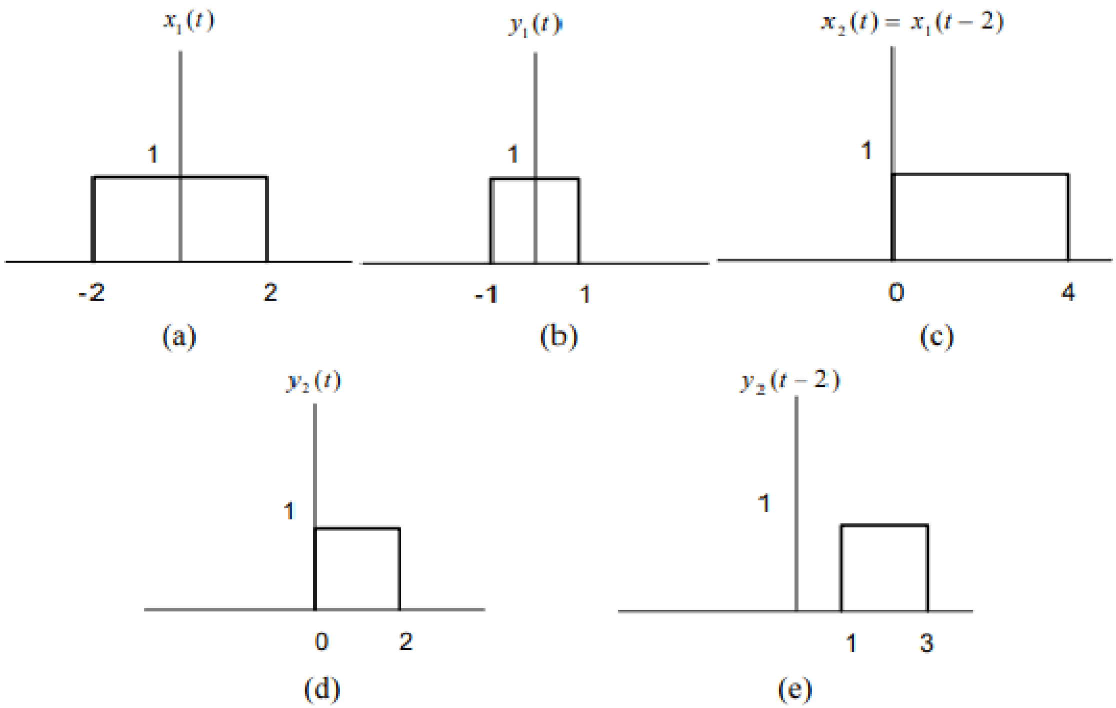


Fig. 1.28 Inputs and outputs of the system y(t) = x(2t)

1.12.6 Linearity

The system is linear if

- The response to x1(t)+ x2(t) is y1(t)+y2(t) - **additivity property**
- The response to $ax_1(t)$ is $ay_1(t)$ - **scaling or homogeneity property**.
- The two properties defining a linear system can be combined into a single statement:
- Continuous time: $ax_1(t)+bx_2(t) \rightarrow a\,y_1(t) + b\,y_2(t)$
- Discrete time: $ax_1[n] + b\,x_2[n] \rightarrow ay_1[n] + b\,y_2[n]$
- Here a and b are any complex constants.
- Superposition property: If $x_k[n]$, k = 1,2,3…… are a set of inputs with corresponding outputs $y_k[n]$, k = 1,2,3…. , then the response to a linear combination of these inputs given by

$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + a_3 x_3[n] + \ldots,$$

Is

$$y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + a_3 y_3[n] + \ldots,$$

which holds for linear systems in both continuous and discrete time.
For a linear system, zero input leads to zero output.

Examples:

- The system y (t) = t x(t) is a linear system.
- The system y(t) = x^2(t) is not a liner system. ·
- The system y [n][= Re{x [n] }, is additive, but does not satisfy the homogeneity, so it is not a linear system.
- The system y[n][= 2x [n] + 3 is not linear. y [n] = 3 if x [n] = 0 , the system violates the "zeroin/zero-out" property. However, the system can be represented as the sum of the output of a linear system and another signal equal to the zero-input response of the system. For system y [n]= 2x[n] + 3, the linear system is

$$x[n] \rightarrow 2\,x[n]$$

and the zero-input response is

$$y_0[n]=3$$

as shown in Fig. 1.29.

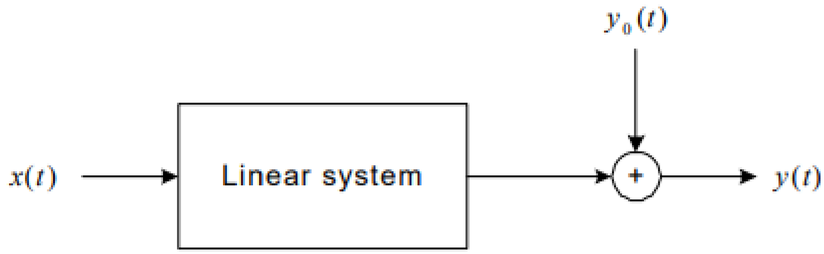


Fig. 1.29 Structure of an incrementally linear system. y_0(t) is the zero-input response of the system.

The system represented in Fig. 1.29 is called incrementally linear system. The system responds linearly to the changes in the input.

The overall system output consists of the superposition of the response of a linear system with a zero-input response.

SUMMARY

Signals are represented mathematically as functions of one or more independent variables.

There are two types of signals: continuous-time signals and discrete-time signals.

The variable of time is continuous in case of Continuous-time signal.

The variable of time is discrete in case of Discrete-time signal.

In many situations, it is important to consider signals related by a modification of the independent variable. These modifications will usually lead to reflection, scaling, and shift.

A periodic continuous-time signal x (t) has the property that there is a positive value of T for which x (t) = x (t + T)   for all t

Any signal can be decomposed into a sum of two signals, one of which is even and one of which is odd.

The sinusoidal signal is also a periodic signal with a fundamental period of T0 .

The continuous-time unit impulse can also be considered as the first derivative of the continuous time unit step.

The continuous-time unit step is the running integral of the unit impulse.

A continuous-time system is a system in which continuous-time input signals are applied and results in continuous-time output signals.

A discrete-time system is a system in which discrete-time input signals are applied and results in discrete-time output signals.

A system is memoryless if its output for each value of the independent variable as a given time is dependent only on the input at the same time.

A system is said to be invertible if distinct inputs leads to distinct outputs.

A system is causal if the output at any time depends only on the values of the input at present time and in the past. Such a system is often referred to as being nonanticipative, as the system output does not anticipate future values of the input.

A stable system is one in which small inputs leads to responses that do not diverge. More formally, if the input to a stable system is bounded, then the output must be also bounded and therefore cannot diverge.

A system is time invariant if a time shift in the input signal results in an identical time shift in the output signal. Mathematically, if the system output is y (t) when the input is x(t) , a time invariant system will have an output of $y(t-t_0)$ when input is $x(t-t_0)$.

The system is linear if

The response to x1(t)+ x2(t) is y1(t)+y2(t) - **additivity property**

The response to $ax_1(t)$ is $ay_1(t)$ - **scaling or homogeneity property**.

Books

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References

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UNIT-1

Chapter-2

FOURIER SERIES

2.0 OBJECTIVES

- Understand Trigonometric Fourier series components
- Periodic Fourier series components
- Understand the properties of Fourier transform

2.1INTRODUCTION TO FOURIER SERIES

We will now turn to the study of trigonometric series. You have seen that functions have series representations as expansions in powers of x, or x − a, in the form of Maclaurin and Taylor series. Recall that the Taylor series expansion is given by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

where the expansion coefficients are determined as

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

From the study of the heat equation and wave equation, we have found that there are infinite series expansions over other functions, such as sine functions. We now turn to such expansions and in the next chapter we will find out that expansions over special sets of functions are not uncommon in physics. But, first we turn to Fourier trigonometric series.

We will begin with the study of the Fourier trigonometric series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}.$$

We will find expressions useful for determining the Fourier coefficients {an, bn} given a function f(x) defined on [−L, L]. We will also see if the resulting infinite series reproduces f(x). However, we first begin with some basic ideas involving simple sums of sinusoidal functions.

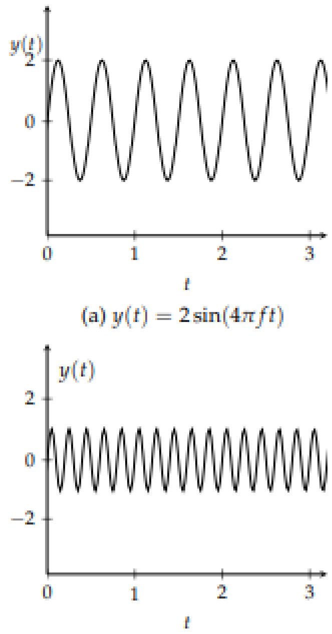
There is a natural appearance of such sums over sinusoidal functions in music. A pure note can be represented as

$$y(t) = A \sin(2\pi f t)$$

where A is the amplitude, f is the frequency in hertz (Hz), and t is time in seconds. The amplitude is related to the volume of the sound. The larger the amplitude, the louder the sound. In Figure 2.1 we show plots of two such tones with f = 2 Hz in the top plot and f = 5 Hz in the bottom one.

In these plots you should notice the difference due to the amplitudes and the frequencies. You can easily reproduce these plots and others in your favorite plotting utility.

As an aside, you should be cautious when plotting functions, or sampling data. The plots you get might not be what you expect, even for a simple sine function.



(a) $y(t) = 2\sin(4\pi f t)$

(b) $y(t) = \sin(10\pi f t)$

Figure 2.1: Plots of y(t) = A sin(2π f t) on [0, 5] for f = 2 Hz and f = 5 Hz.

In Figure 2.2 we show four plots of the function y(t) = 2 sin(4πt). In the top left you see a proper rendering of this function. However, if you use a different number of points to plot this function, the results may be surprising. In this example we show what happens if you use N = 200, 100, 101 points instead of the 201 points used in the first plot. Such disparities are not only possible when plotting functions, but are also present when collecting data. Typically, when you sample a set of data, you only gather a finite amount of information at a fixed rate. This could happen when getting data on ocean wave heights, digitizing music and other audio to put on your computer, or any other process when you attempt to analyze a continuous signal.

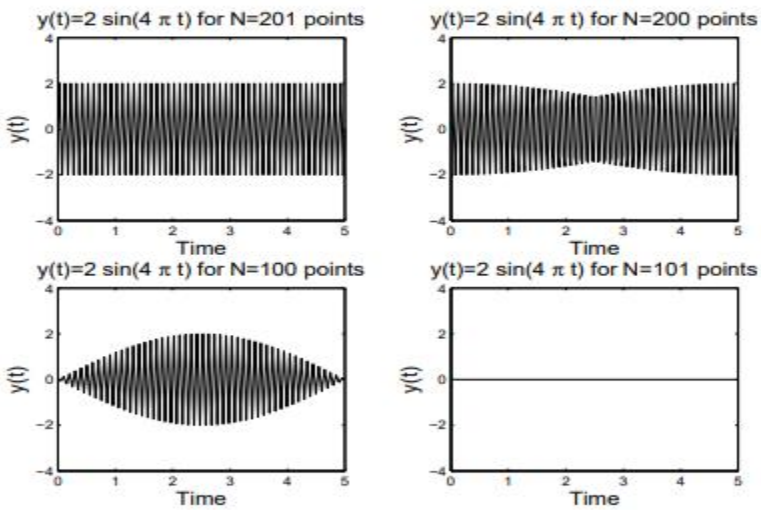
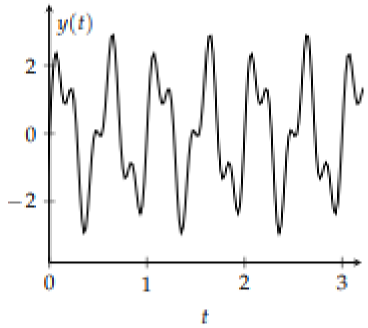
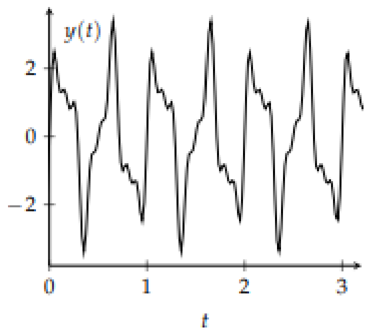


Figure 2.2: Problems can occur while plotting. Here we plot the function y(t) = 2 sin 4πt using N = 201, 200, 100, 101 points.

Next, we consider what happens when we add several pure tones. After all, most of the sounds that we hear are in fact a combination of pure tones with different amplitudes and frequencies. In Figure 2.3 we see what happens when we add several sinusoids. Note that as one adds more and more tones with different characteristics, the resulting signal gets more complicated. However, we still have a function of time.



(a) Sum of signals with frequencies $f = 2$ Hz and $f = 5$ Hz.



(b) Sum of signals with frequencies $f = 2$ Hz, $f = 5$ Hz, and $f = 8$ Hz.

Figure 2.3: Superposition of several sinusoids.

Given a function f(t), can we find a set of sinusoidal functions whose sum converges to f(t)?"

Looking at the superposition in Figure 2.3, we see that the sums yield functions that appear to be periodic. This is not to be unexpected. We recall that a periodic function is one in which the function values repeat over the domain of the function. The length of the smallest part of the domain which repeats is called the period. We can define this more precisely: A function is said to be periodic with period T if $f(t + T) = f(t)$ for all t and the smallest such positive number T is called the period.

2.2 GOAL - FOURIER ANALYSIS

Given a signal f(t), we would like to determine its frequency content by finding out what combinations of sines and cosines of varying frequencies and amplitudes will sum to the given function. This is called Fourier Analysis.

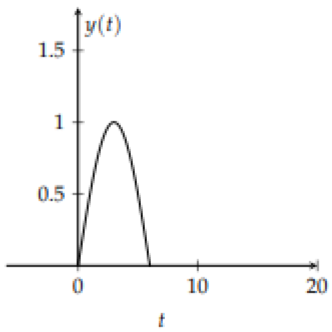
2.3 TRIGONOMETRIC FOURIER SERIES

As we have seen in the last section, we are interested in finding representations of functions in terms of sines and cosines. Given a function f(x) we seek a representation in the form

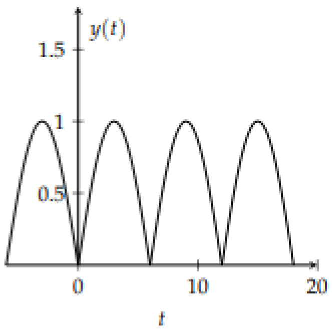
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right].$$

Notice that we have opted to drop the references to the time-frequency form of the phase. This will lead to a simpler discussion for now and one can always make the transformation $nx = 2\pi fnt$ when applying these ideas to applications.

The series representation in Equation is called a Fourier trigonometric series. We will simply refer to this as a Fourier series for now.



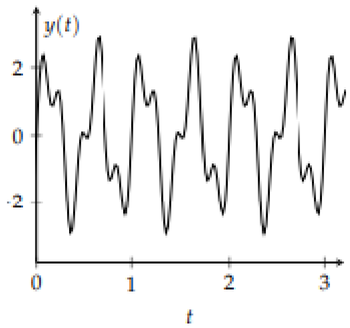
(a) Plot of function $f(t)$.



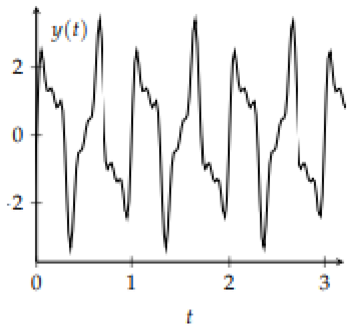
(b) Periodic extension of $f(t)$.

Figure 2.4: Plot of the function f(t) defined on [0, 2π] and its periodic extension.

The set of constants a_0, a_n, b_n, $n = 1, 2, \ldots$ are called the Fourier coefficients. The constant term is chosen in this form to make later computations simpler, though some other authors choose to write the constant term as a_0. Our goal is to find the Fourier series representation given f(x). Having found the Fourier series representation, we will be interested in determining when the Fourier series converges and to what function it converges.



(a) Sum of signals with frequencies $f = 2$ Hz and $f = 5$ Hz.



(b) Sum of signals with frequencies $f = 2$ Hz, $f = 5$ Hz, and $f = 8$ Hz.

Figure 2.5: Superposition of several sinusoids.

Looking at the superpositions in Figure 2.5, we see that the sums yield functions that appear to be periodic. This is not to be unexpected. We recall that a periodic function is one in which the function values repeat over the domain of the function. The length of the smallest part of the domain which repeats is called the period. We can define this more precisely: A function is said to be periodic with period T if $f(t + T) = f(t)$ for all t and the smallest such positive number T is called the period. For example, we consider the functions used in Figure 3.3. We began with $y(t) = 2 \sin(4\pi t)$. Recall from your first studies of trigonometric functions that one can determine the period by dividing the coefficient of t into 2π to get the period. In this case we have

$$T = \frac{2\pi}{4\pi} = \frac{1}{2}.$$

From our discussion in the last section, we see that The Fourier series is periodic. The periods of cos nx and sin nx are 2π n . Thus, the largest period, $T = 2\pi$, comes from the $n = 1$ terms and the Fourier series has period 2π. This means that the series should be able to represent functions that are periodic of period 2π. While this appears restrictive, we could also consider functions that are defined over one period. we can show a function defined on $[0, 2\pi]$. In the same figure, we show its periodic extension. These are just copies of the original function shifted by the period and glued together. The extension can now be represented by a Fourier series and restricting the Fourier series to [0, 2π] will give a representation of the original function. Therefore, we will first consider Fourier series representations of functions defined on this interval. Note that we could just as easily considered functions defined on [−π, π] or any interval of length 2π. We will consider more general intervals later in the chapter.

Fourier Coefficients Theorem 2.1. The Fourier series representation of f(x) defined on [0, 2π], when it exists, is given by equation with Fourier coefficients

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad n = 0,1,2,\ldots, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx, \quad n = 1,2,\ldots. \end{aligned}$$

These expressions for the Fourier coefficients are obtained by considering special integrations of the Fourier series. We will now derive the an integrals in equation. We begin with the computation of a0. Integrating the Fourier series term by term in Equation above, we have

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} \frac{a_0}{2}\,dx + \int_0^{2\pi} \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]\,dx.$$

We will assume that we can integrate the infinite sum term by term. Then

we will need to compute

$$\begin{aligned} \int_0^{2\pi} \frac{a_0}{2}\,dx &= \frac{a_0}{2}(2\pi) = \pi a_0, \\ \int_0^{2\pi} \cos nx\,dx &= \left[\frac{\sin nx}{n}\right]_0^{2\pi} = 0, \\ \int_0^{2\pi} \sin nx\,dx &= \left[\frac{-\cos nx}{n}\right]_0^{2\pi} = 0. \end{aligned}$$

From these results we see that only one term in the integrated sum does not vanish leaving

$$\int_0^{2\pi} f(x)\,dx = \pi a_0.$$

This confirms the value for a_0^2. Next, we will find the expression for an. We multiply the Fourier series above by cos mx for some positive integer m. This is like multiplying by cos 2x, cos 5x, etc. We are multiplying by all possible cos mx functions for different integers m all at the same time. We will see that this will allow us to solve for the an's.

We find the integrated sum of the series times cos mx is given by

$$\int_0^{2\pi} f(x)\cos mx\,dx = \int_0^{2\pi} \frac{a_0}{2}\cos mx\,dx + \int_0^{2\pi} \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n \sin nx\right]\cos mx\,dx.$$

Integrating term by term, the right side becomes

$$\int_0^{2\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\int_0^{2\pi}\cos mx\,dx + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi}\cos nx\cos mx\,dx + b_n\int_0^{2\pi}\sin nx\cos mx\,dx\right].$$

We have already established that $\int_0^{2\pi}\cos mx\,dx = 0$ which implies that the first term vanishes. Next we need to compute integrals of products of sines and cosines. This requires that we make use of some of the trigonometric identities listed . For quick reference, we list these here.

Useful Trigonometric Identities

$$\begin{aligned}
\sin(x \pm y) &= \sin x\cos y \pm \sin y \cos x \\
\cos(x \pm y) &= \cos x\cos y \mp \sin x \sin y \\
\sin^2 x &= \frac{1}{2}(1-\cos 2x) \\
\cos^2 x &= \frac{1}{2}(1+\cos 2x) \\
\sin x \sin y &= \frac{1}{2}(\cos(x-y) - \cos(x+y)) \\
\cos x \cos y &= \frac{1}{2}(\cos(x+y) + \cos(x-y)) \\
\sin x \cos y &= \frac{1}{2}(\sin(x+y) + \sin(x-y))
\end{aligned}$$

We first want to evaluate $\int_0^{2\pi}\cos nx\, cos\ mx\, dx$. We do this by using the

$$\begin{aligned}
\int_0^{2\pi}\cos nx\cos mx\,dx &= \frac{1}{2}\int_0^{2\pi}\left[\cos(m+n)x + \cos(m-n)x\right]dx \\
&= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{2\pi} \\
&= 0.
\end{aligned}$$

There is one caveat when doing such integrals. What if one of the denominators m ± n vanishes?

For this problem m + n ≠ 0, since both m and n are positive integers. However, it is possible for m = n. This means that the vanishing of the integral can only happen when m ≠ n. So, what can we do about the m = n case? One way is to start from scratch with our integration. (Another way is to compute the limit as n approaches m in our result and use L'Hopital's Rule.)

For n = m we have to compute $\int_0^{2\pi} \cos 2\, mx\, dx$. This can also be handled using a trigonometric identity. Using the half angle formula, with θ = mx, we find

$$\begin{aligned}\int_0^{2\pi} \cos^2 mx\, dx &= \frac{1}{2}\int_0^{2\pi} (1+\cos 2mx)\, dx \\ &= \frac{1}{2}\left[x + \frac{1}{2m}\sin 2mx\right]_0^{2\pi} \\ &= \frac{1}{2}(2\pi) = \pi.\end{aligned}$$

To summarize, we have shown that

$$\int_0^{2\pi} \cos nx \cos mx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases}$$

This holds true for m, n = 0, 1, ... [Why did we include m, n = 0?] When we have such a set of functions, they are said to be an orthogonal set over the integration interval. A set of (real) functions $\{\varphi_n(x)\}$ is said to be orthogonal on [a, b] if

$$\int_a^b \phi_n(x)\phi_m(x)\, dx = 0 \text{ when } n \neq m.$$

Furthermore, if we also have that

$$\int_a^b \phi_n^2(x)\, dx = 1,$$

these functions are called orthonormal.

The set of functions $\{cosnx)_{n=0}^{\infty}$are orthogonal on [0, 2π]. Actually, they are orthogonal on any interval of length 2π. We can make them orthonormal by dividing each function by √ π as indicated by Equation .

This is sometimes referred to normalization of the set of functions. The notion of orthogonality is actually a generalization of the orthogonality of vectors in finite dimensional vector spaces. The integral $\int_a^b f(x)\, f(x) dx$ is the generalization of the dot product, and is called the scalar product of f(x) and g(x), which are thought of as vectors in an infinite dimensional vector space spanned by a set of orthogonal functions.

we still have to evaluate $\int_0^{2\pi} \sin nx \cos mx\, dx$. We can use the trigonometric identity involving products of sines and cosines, Setting A = nx and B = mx,

That

$$\begin{aligned}\int_0^{2\pi} \sin nx \cos mx\, dx &= \frac{1}{2}\int_0^{2\pi} [\sin(n+m)x + \sin(n-m)x]\, dx \\ &= \frac{1}{2}\left[\frac{-\cos(n+m)x}{n+m} + \frac{-\cos(n-m)x}{n-m}\right]_0^{2\pi} \\ &= (-1+1)+(-1+1) = 0.\end{aligned}$$

So,

$$\int_0^{2\pi} \sin nx \cos mx \, dx = 0.$$

For these integrals we also should be careful about setting n = m. In this special case, we have the integrals

$$\int_0^{2\pi} \sin mx \cos mx \, dx = \frac{1}{2} \int_0^{2\pi} \sin 2mx \, dx = \frac{1}{2} \left[\frac{-\cos 2mx}{2m} \right]_0^{2\pi} = 0.$$

Finally, we can finish evaluating the expression in Equation. We have determined that all but one integral vanishes. In that case, n = m. This leaves us with

$$\int_0^{2\pi} f(x) \cos mx \, dx = a_m \pi.$$

Solving for a_m gives

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx \, dx.$$

Since this is true for all m = 1, 2, . . . , we have proven this part of the theorem. The only part left is finding the bn's This will be left as an exercise for the reader.

We now consider examples of finding Fourier coefficients for given functions. In all of these cases we define f(x) on [0,2 Π]

Example 2.1. f(x) = 3 cos 2x, x ∈ [0, 2π]. We first compute the integrals for the Fourier coefficients.

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} 3 \cos 2x \, dx = 0. \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} 3 \cos 2x \cos nx \, dx = 0, \quad n \neq 2. \\ a_2 &= \frac{1}{\pi} \int_0^{2\pi} 3 \cos^2 2x \, dx = 3, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} 3 \cos 2x \sin nx \, dx = 0, \forall n. \end{aligned}$$

The integrals for a0, an, n ≠ 2, and bn are the result of orthogonality. For a2, the integral can be computed as follows:

$$\begin{aligned} a_2 &= \frac{1}{\pi} \int_0^{2\pi} 3 \cos^2 2x \, dx \\ &= \frac{3}{2\pi} \int_0^{2\pi} [1 + \cos 4x] \, dx \\ &= \frac{3}{2\pi} \left[x + \underbrace{\frac{1}{4} \sin 4x}_{\text{This term vanishes!}} \right]_0^{2\pi} = 3. \end{aligned}$$

Therefore, we have that the only nonvanishing coefficient is $a_2 = 3$. So there is one term and $f(x) = 3 \cos 2x$.

Well, we should have known the answer to the last example before doing all of those integrals. If we have a function expressed simply in terms of sums of simple sines and cosines, then it should be easy to write down the Fourier coefficients without much work. This is seen by writing out the Fourier series,

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]. \\ &= \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + + a_2 \cos 2x + b_2 \sin 2x + . \end{aligned}$$

For the last problem, $f(x) = 3 \cos 2x$. Comparing this to the expanded Fourier series, one can immediately read off the Fourier coefficients without doing any integration. In the next example we emphasize this point.

Example 2.2. $f(x) = \sin^2 x$, $x \in [0, 2\pi]$.

We could determine the Fourier coefficients by integrating as in the last example. However, it is easier to use trigonometric identities. We know that

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x.$$

There are no sine terms, so $b_n = 0$, $n = 1, 2, \ldots$. There is a constant term, implying $a_0/2 = 1/2$. So, $a_0 = 1$. There is a cos 2x term, corresponding to $n = 2$, so $a_2 = -1\ 2$. That leaves $a_n = 0$ for $n \neq 0, 2$. So, $a_0 = 1$, $a_2 = -1\ 2$, and all other Fourier coefficients vanish

Example 2.3. $f(x) = 1$, $0 < x < \pi$, -1, $\pi < x < 2\pi$, .

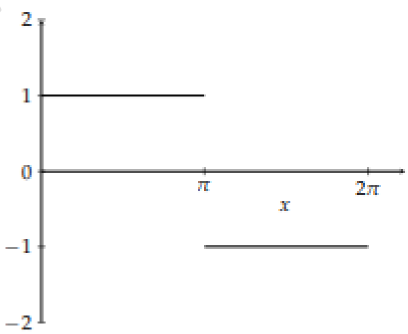


Figure 2.6: Plot of discontinuous function in Example 2.3

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(-1)\,dx \\
&= \frac{1}{\pi}(\pi) + \frac{1}{\pi}(-2\pi+\pi) = 0.
\end{aligned}
$$

$$
a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx
$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left[\int_0^{\pi}\cos nx\,dx - \int_{\pi}^{2\pi}\cos nx\,dx\right] \\
&= \frac{1}{\pi}\left[\left(\frac{1}{n}\sin nx\right)_0^{\pi} - \left(\frac{1}{n}\sin nx\right)_{\pi}^{2\pi}\right] \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left[\int_0^{\pi}\sin nx\,dx - \int_{\pi}^{2\pi}\sin nx\,dx\right] \\
&= \frac{1}{\pi}\left[\left(-\frac{1}{n}\cos nx\right)_0^{\pi} + \left(\frac{1}{n}\cos nx\right)_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[-\frac{1}{n}\cos n\pi + \frac{1}{n} + \frac{1}{n} - \frac{1}{n}\cos n\pi\right] \\
&= \frac{2}{n\pi}(1-\cos n\pi).
\end{aligned}
$$

We have found the Fourier coefficients for this function. Before inserting them into the Fourier series , we note that cos nπ = (−1) n . Therefore,

$$
1-\cos n\pi = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd.} \end{cases}
$$

So, half of the bn’s are zero. While we could write the Fourier series representation as

$$
f(x) \sim \frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n}\sin nx,
$$

we could let n = 2k − 1 in order to capture the odd numbers only. The answer can be written as

$$
f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1},
$$

Having determined the Fourier representation of a given function, we would like to know if the infinite series can be summed; i.e., does the series converge? Does it converge to f(x)? We will discuss this question later in the chapter after we generalize the Fourier series to intervals other than for x ∈ [0, 2π].

2.4 FOURIER SERIES OVER OTHER INTERVALS

In many applications we are interested in determining Fourier series representations of functions defined on intervals other than [0, 2π]. In this section we will determine the form of the series expansion and the Fourier coefficients in these cases. The most general type of interval is given as [a, b]. However, this often is too general. More common intervals are of the form [−π, π], [0, L], or

[−L/2, L/2]. The simplest generalization is to the interval [0, L]. Such intervals arise often in applications. For example, for the problem of a one dimensional string of length L we set up the axes with the left end at x = 0 and the right end at x = L. Similarly for the temperature distribution along a one dimensional rod of length L we set the interval to x ∈ [0, 2π]. Such problems naturally lead to the study of Fourier series on intervals of length L. We will see later that symmetric intervals, [−a, a], are also useful. Given an interval [0, L], we could apply a transformation to an interval of length 2π by simply rescaling the interval. Then we could apply this transformation to the Fourier series representation to obtain an equivalent one useful for functions defined on [0, L].

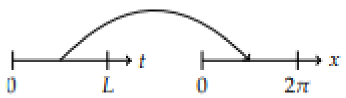


Figure 2.7: A sketch of the transformation between intervals x ∈ [0, 2π] and t ∈ [0, L]

We define x ∈ [0, 2π] and t ∈ [0, L]. A linear transformation relating these intervals is simply x = 2πt L as shown in Figure 2.7. So, t = 0 maps to x = 0 and t = L maps to x = 2π. Furthermore, this transformation maps f(x) to a new function g(t) = f(x(t)), which is defined on [0, L]. We will determine the Fourier series representation of this function using the representation for f(x) from the last section. Recall the form of the Fourier representation for f(x) in Equation

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right].$$

Inserting the transformation relating x and t, we have

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi t}{L} + b_n \sin \frac{2n\pi t}{L}\right].$$

This gives the form of the series expansion for g(t) with t ∈ [0, L]. But, we still need to determine the Fourier coefficients. Recall, that

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx.$$

We need to make a substitution in the integral of $x = 2\pi t L$. We also will need to transform the differential, $dx = 2\pi L\, dt$. Thus, the resulting form for the Fourier coefficients is

$$a_n = \frac{2}{L}\int_0^L g(t)\cos\frac{2n\pi t}{L}\,dt.$$

Similarly, we find that

$$b_n = \frac{2}{L}\int_0^L g(t)\sin\frac{2n\pi t}{L}\,dt.$$

We note first that when $L = 2\pi$ we get back the series representation that we first studied. Also, the period of $\cos 2n\pi t L$ is L/n, which means that the representation for $g(t)$ has a period of L corresponding to $n = 1$. At the end of this section we present the derivation of the Fourier series representation for a general interval for the interested reader.

At this point we need to remind the reader about the integration of even and odd functions on symmetric intervals. We first recall that $f(x)$ is an even function if $f(-x) = f(x)$ for all x. One can recognize even functions as they are symmetric with respect to the y-axis as shown in Figure 2.8

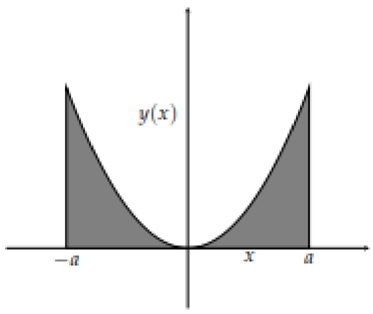


Figure 2.8: Area under an even function on a symmetric interval, [−a, a].

If one integrates an even function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

One can prove this by splitting off the integration over negative values of x, using the substitution $x = -y$, and employing the evenness of $f(x)$. Thus,

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \\ &= -\int_a^0 f(-y)\,dy + \int_0^a f(x)\,dx \\ &= \int_0^a f(y)\,dy + \int_0^a f(x)\,dx \\ &= 2\int_0^a f(x)\,dx.\end{aligned}$$

This can be visually verified by looking at Figure 2.8. A similar computation could be done for odd functions. f(x) is an odd function if f(−x) = −f(x) for all x. The graphs of such functions are symmetric with respect to the origin as shown in Figure 2.9. If one integrates an odd function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Odd Functions

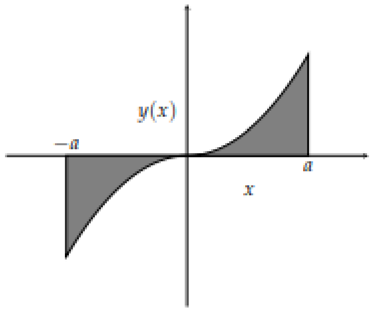


Figure 2.9: Area under an odd function on a symmetric interval, [−a, a].

Example 2.4.

Let f(x) = |x| on [−π, π] We compute the coefficients, beginning as usual with a0. We have, using the fact that |x| is an even function,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\,dx = \pi \end{aligned}$$

We continue with the computation of the general Fourier coefficients for f(x) = |x| on [−π, π]. We have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx.$$

Here we have made use of the fact that |x| cos nx is an even function. In order to compute the resulting integral, we need to use integration by parts ,

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du,$$

by letting u = x and dv = cos nx dx. Thus, du = dx and v = ∫ dv = $\frac{1}{n}$ sin nx.

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right].$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \end{aligned}$$

Fourier Series on $[-\frac{L}{2}, \frac{L}{2}]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right].$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \cos \frac{2n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \sin \frac{2n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \end{aligned}$$

Fourier Series on $[-\pi, \pi]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right].$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx. \quad n = 1, 2, \ldots. \end{aligned}$$

Continuing with the computation, we have

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} x \cos nx\, dx. \\ &= \frac{2}{\pi} \left[\frac{1}{n} x \sin nx \Big|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin nx\, dx \right] \\ &= -\frac{2}{n\pi} \left[-\frac{1}{n} \cos nx \right]_0^{\pi} \\ &= -\frac{2}{\pi n^2} (1 - (-1)^n). \end{aligned}$$

Here we have used the fact that $\cos n\pi = (-1)^n$ for any integer n. This leads to a factor $(1 - (-1)^n)$. This factor can be simplified as

$$1 - (-1)^n = \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}.$$

So, $a_n = 0$ for n even and $a_n = -\frac{4}{\pi n^2}$ for n odd. Computing the b_n's is simpler. We note that we have to integrate |x| sin nx from x = −π to π. The integrand is an odd function and this is a symmetric interval. So, the result is that $b_n = 0$ for all n. Putting this all together, the Fourier series representation of f(x) = |x| on [−π, π] is given as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ n\ odd}}^{\infty} \frac{\cos nx}{n^2}.$$

While this is correct, we can rewrite the sum over only odd n by reindexing. We let n = 2k − 1 for k = 1, 2, 3, ... Then we only get the odd integers. The series can then be written as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}.$$

Throughout our discussion we have referred to such results as Fourier representations. We have not looked at the convergence of these series. Here is an example of an infinite series of functions. What does this series sum to? We show in Figure 2.10 the first few partial sums. They appear to be converging to f(x) = |x| fairly quickly. Even though f(x) was defined on [−π, π] we can still evaluate the Fourier series at values of x outside this interval. In Figure 2.11, we see that the representation agrees with f(x) on the interval [−π, π]. Outside this interval we have a periodic extension of f(x) with period 2π. Another example is the Fourier series representation of f(x) = x on [−π, π] This is determined to be

$$f(x) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

As seen in Figure 2.12 we again obtain the periodic extension of the function. In this case we needed many more terms. Also, the vertical parts of the

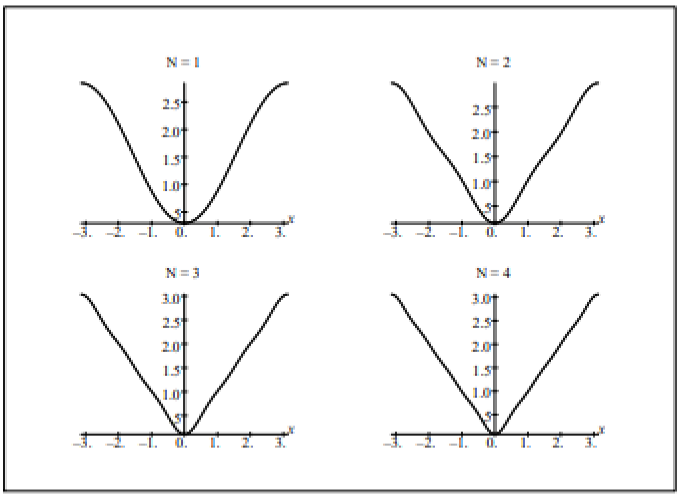


Figure 2.10: Plot of the first partial sums of the Fourier series representation for f(x) = |x|.

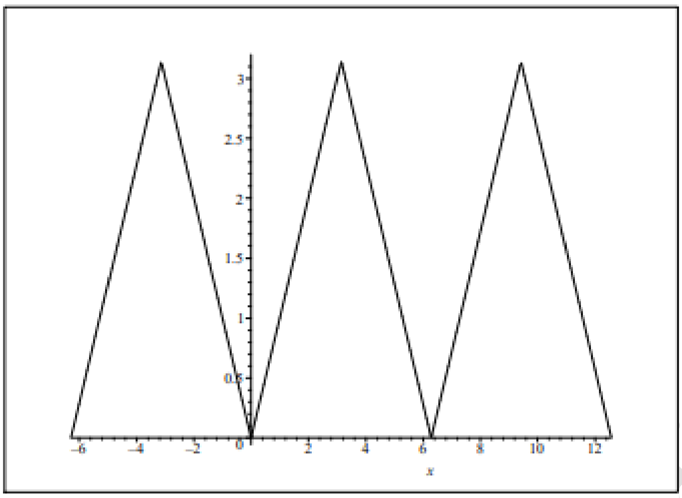


Figure 2.11: Plot of the first 10 terms of the Fourier series representation for f(x) = |x| on the interval [−2π, 4π].

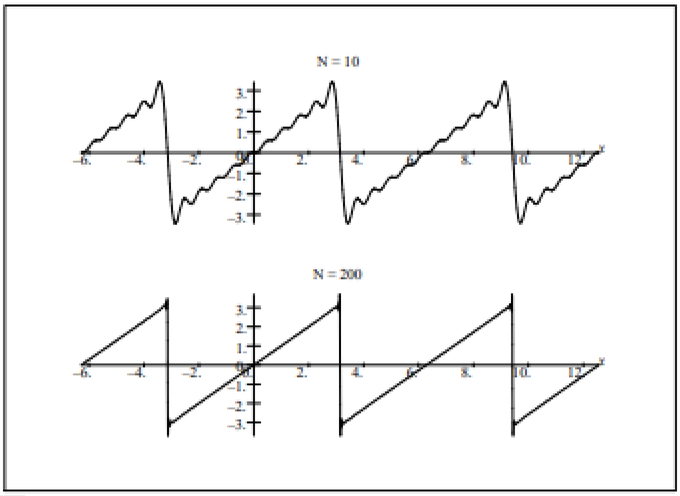


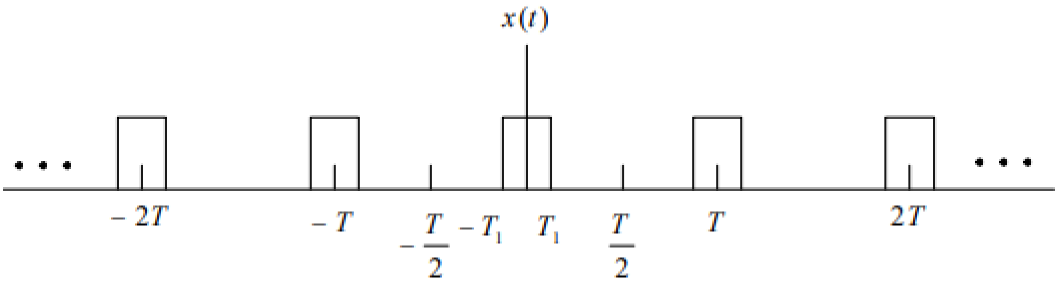
Figure 2.12: Plot of the first 10 terms and 200 terms of the Fourier series representation for f(x) = x on the interval [−2π, 4π].

2.5 Representation of Aperiodic Signals: The Continuous-Time Fourier Transform

2.5.1 Development of the Fourier Transform Representation of an Aperiodic Signal

Starting from the Fourier series representation for the continuous-time periodic square wave:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases},$$



The Fourier coefficients a_k for this square wave are

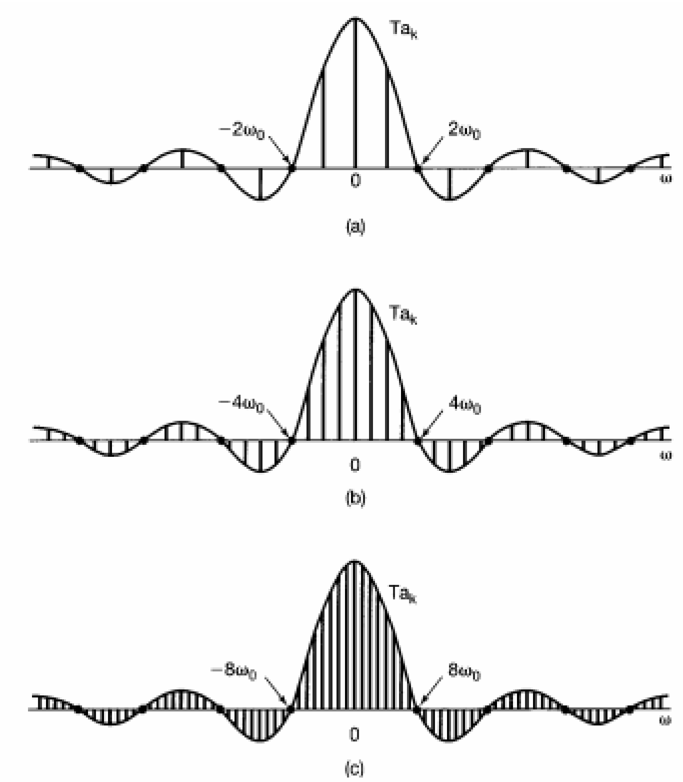
$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}.$$

or alternatively

$$Ta_k = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega = k\omega_0},$$

where $2\sin(\omega T_1)/\omega$ represent the envelope of Ta_k ·

- When T increases or the fundamental frequency $\omega 0 = 2\Pi / T$ decreases, the envelope is sampled with a closer and closer spacing. As T becomes arbitrarily large, the original periodic square wave approaches a rectangular pulse.
- Ta_k becomes more and more closely spaced samples of the envelope, as $T \rightarrow \infty$, the Fourier series coefficients approaches the envelope function.

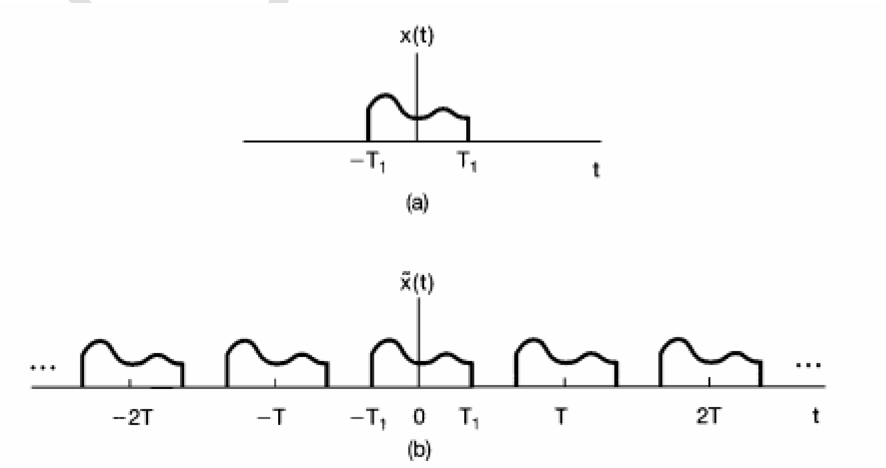


This example illustrates the basic idea behind Fourier's development of a representation for aperiodic signals.

Based on this idea, we can derive the Fourier transform for aperiodic signals.

Suppose a signal x(t) with a finite duration, that is, $x(t) = 0$ for $|t| > T_1$, as illustrated in the figure below.

- From this aperiodic signal, we construct a periodic signal $\dot{x}(t)$, shown in the figure below.



- As $T \rightarrow \infty$, $\sim x(t) = x(t)$, for any infinite value of t .
- The Fourier series representation of $\sim x(t)$ is

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} ,$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt .$$

Since ~x(t) = x(t) for |t| < T / 2 , and also, since x(t) = 0 outside this interval, so we have

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt .$$

- Define the envelope X(jw) of Ta_k as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt .$$

we have for the coefficients a_k ,

$$a_k = \frac{1}{T} X(jk\omega_0)$$

Then ~x(t) can be expressed in terms of X(jw), that is

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t} = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t}\omega_0 .$$

- As T → ∞ , ~x (t) = x (t) and consequently, Equation becomes a representation of x(t).
- In addition, ω_0 → 0 as T →∞ , and the right-hand side of Equation becomes an integral.
  We have the following Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \textit{Inverse Fourier Transform}$$

and

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \textit{Fourier Transform}$$

2.5.2 Convergence of Fourier Transform

If the signal x(t) has finite energy, that is, it is square integrable,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty ,$$

Then we guaranteed that X(jw) is finite or Equation converges.

If e(t) =~ x (t) - x (t) , we have

$$\int_{-\infty}^{\infty} |e(t)|^2 \, dt = 0 .$$

An alternative set of conditions that are sufficient to ensure the convergence:

Condition1: Over any period, x(t) must be absolutely integrable, that is

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty ,$$

Condition 2: In any finite interval of time, x(t) have a finite number of maxima and minima.
Condition 3: In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

2.5.3 Examples of Continuous-Time Fourier Transform

Example: consider signal x(t) e^{-at} u(t) = , a > 0 .

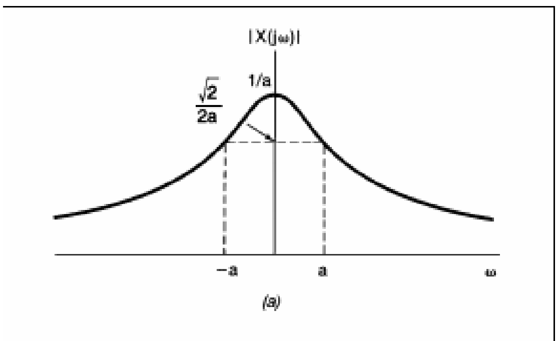
From Equation,

$$X(j\omega) = \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt = -\frac{1}{a + j\omega} e^{-(a+j\omega)t} \Big|_{0}^{\infty} = \frac{1}{a + j\omega}, \qquad a > 0$$

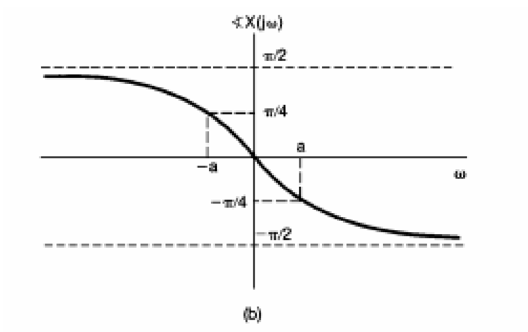
If a is complex rather then real, we get the same result if Re{a}> 0

The Fourier transform can be plotted in terms of the magnitude and phase, as shown in the figure below.

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}, \qquad \angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$



(a)

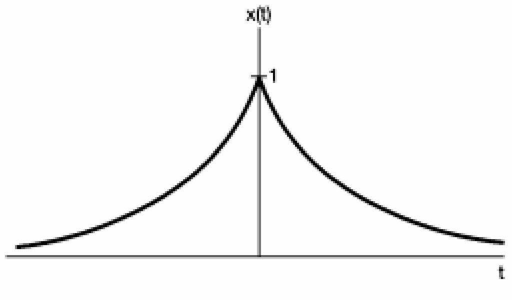


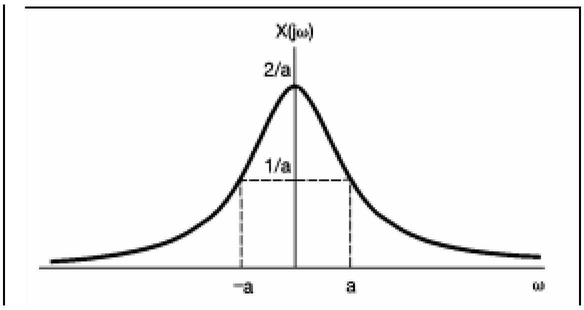
(b)

Example: Let   x (t)= e $^{-a|t|}$, a > 0

$$X(j\omega)=\int_{-\infty}^{\infty}e^{-a|t|}e^{-j\omega t}dt=\int_{-\infty}^{0}e^{at}e^{-j\omega t}dt+\int_{0}^{\infty}e^{-at}e^{-j\omega t}dt=\frac{1}{a-j\omega}+\frac{1}{a+j\omega}=\frac{2a}{a^2+\omega^2}$$

The signal and the Fourier transform are sketched in the figure below



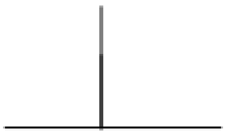


Example:

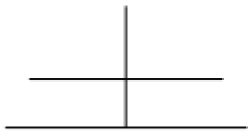
$$x(t)=\delta(t).$$

$$X(j\omega)=\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}dt=1.$$

$x(t)=\delta(t)$



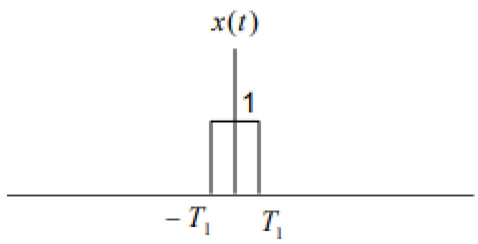
$X(j\omega)=1$

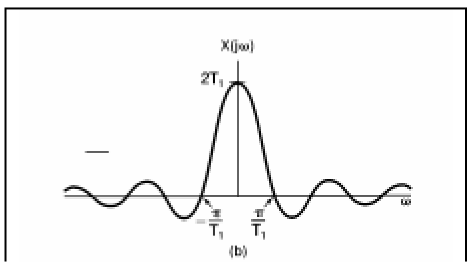


That is, the impulse has a Fourier transform consisting of equal contributions at all frequencies.

Example: Calculate the Fourier transform of the rectangular pulse signal

$$x(t)=\begin{cases}1, & |t|<T_1\\ 0, & |t|>T_1\end{cases}.$$





$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-T_1}^{T_1}1e^{-j\omega t}dt=2\frac{\sin\omega T_1}{\omega}.$$

The Inverse Fourier transform is

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\sin\omega T_1}{\omega} e^{j\omega t} d\omega ,$$

Since the signal x(t) is square integrable,

$$e(t) = \int_{-\infty}^{\infty} \left|x(t) - \hat{x}(t)\right|^2 dt = 0 .$$

xˆ(t) converges to x(t) everywhere except at the discontinuity, T1 t = ± , where xˆ(t) converges to ½, which is the average value of x(t) on both sides of the discontinuity.

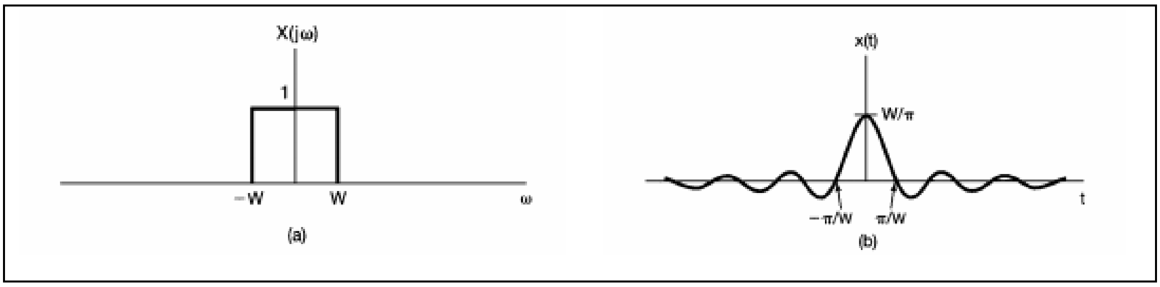
In addition, the convergence of xˆ(t) to x(t) also exhibits Gibbs phenomenon. Specifically, the integral over a finite-length interval of frequencies

$$\frac{1}{2\pi}\int_{-W}^{W} 2\frac{\sin\omega T_1}{\omega} e^{j\omega t} d\omega$$

As W →∞ , this signal converges to x(t) everywhere, except at the discontinuities. More over, the signal exhibits ripples near the discontinuities. The peak values of these ripples do not decrease as W increases, although the ripples do become compressed toward the discontinuity, and the energy in the ripples converges to zero.

Example: Consider the signal whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases} .$$



The Inverse Fourier transform is

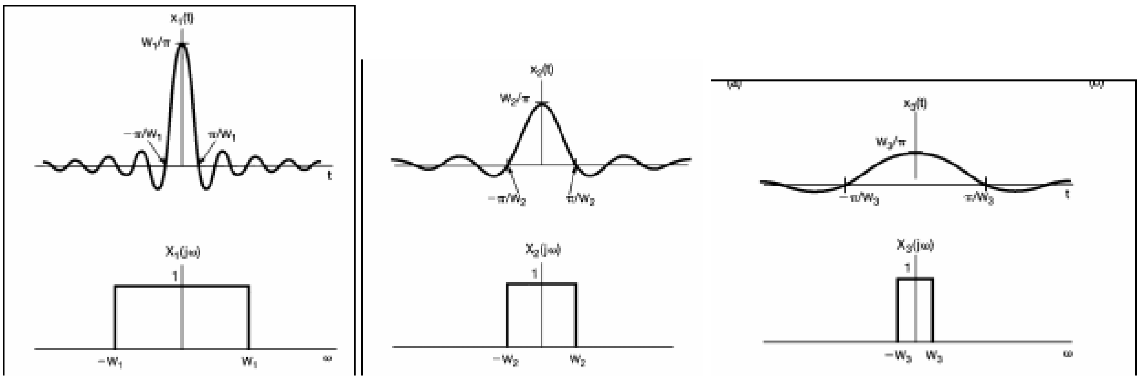
$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t} d\omega = \frac{\sin Wt}{\pi t} .$$

Comparing the results in the preceding example and this example, we have

$$\text{Square wave} \underset{FT^{-1}}{\overset{FT}{\rightleftarrows}} \text{Sinc function}$$

This means a square wave in the time domain, its Fourier transform is a sinc function. However, if the signal in the time domain is a sinc function, then its Fourier transform is a square wave. This property is referred to as Duality Property.

We also note that when the width of X(jw) increases, its inverse Fourier transform x(t) will be compressed. When W → ∞ , X(jw) converges to an impulse. The transform pair with several different values of W is shown in the figure below.



2.6 The Fourier Transform for Periodic Signals

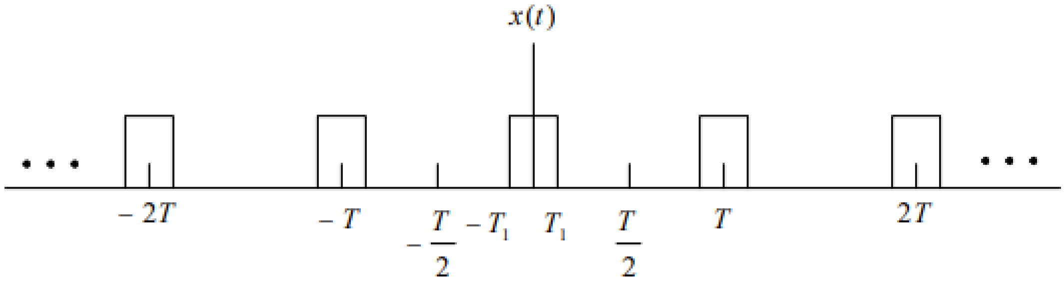
The Fourier series representation of the signal x(t) is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} .$$

It's Fourier transform is

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) .$$

Example: If the Fourier series coefficients for the square wave below are given



$$a_k = \frac{\sin k\omega_0 T_1}{\pi k},$$

The Fourier transform of this signal is

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\sin k\omega_0 T_1}{k}\,\delta(\omega - k\omega_0).$$

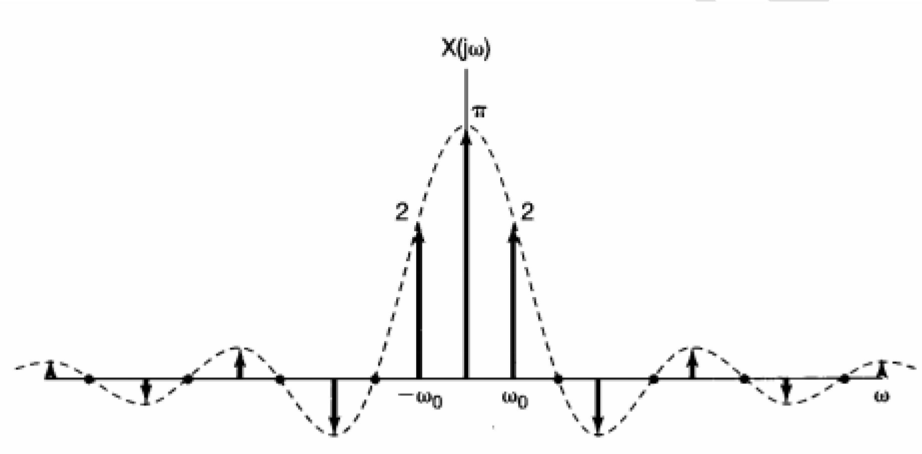
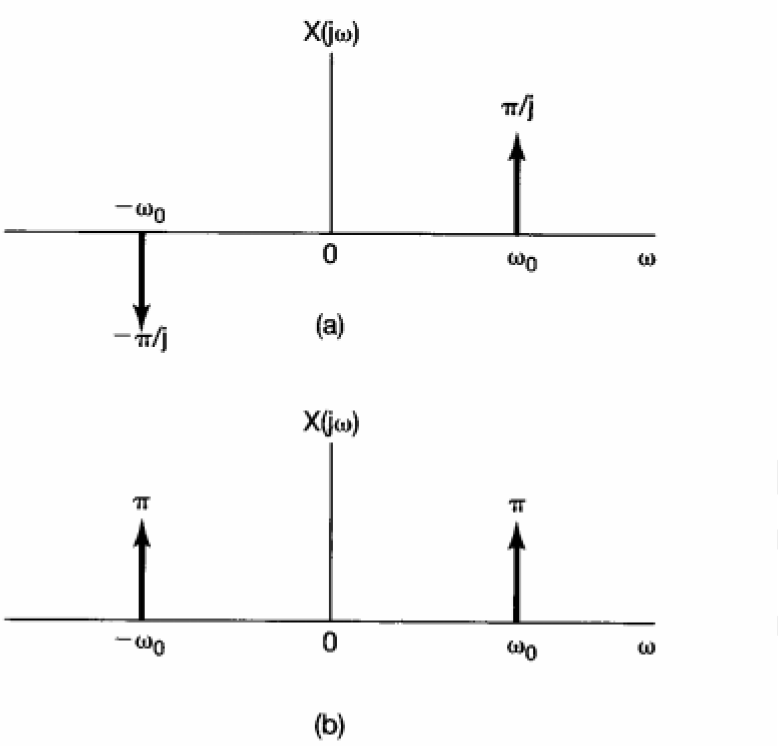


Figure : Fourier transform of a symmetric periodic square wave

Example:

The Fourier transforms for x (t) = sin ω_0t and x(t) = cosω_0t are shown in the figure below.



Fourier transforms of (a) $x(t) = \sin \omega_0 t$; (b) $x(t) = \cos \omega_0 t$.

Example: Calculate the Fourier transform for signal

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

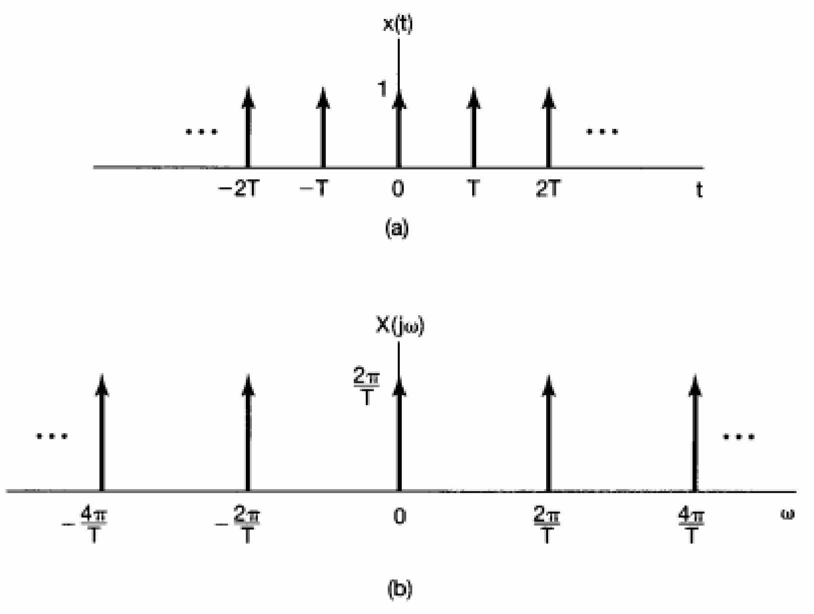
The Fourier series of this signal is

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-j\omega_0 t} = \frac{1}{T}.$$

The Fourier transform is

$$X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T_0}).$$

The Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period 2Π /T , as sketched din the figure below.



(a) Periodic impulse train; (b) its Fourier transform.

2.7 Properties of The Continuous-Time Fourier Transform

2.7.1 Linearity

$$\text{If } x(t) \xleftrightarrow{F} X(j\omega) \text{ and } y(t) \xleftrightarrow{F} Y(j\omega)$$

Then

$$ax(t) + by(t) \xleftrightarrow{F} aX(j\omega) + bY(j\omega)$$

2.7.2 Time Shifting

$$\text{If } x(t) \xleftrightarrow{F} X(j\omega)$$

Then

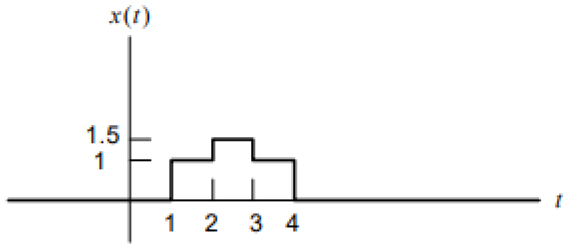
$$x(t - t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(j\omega)$$

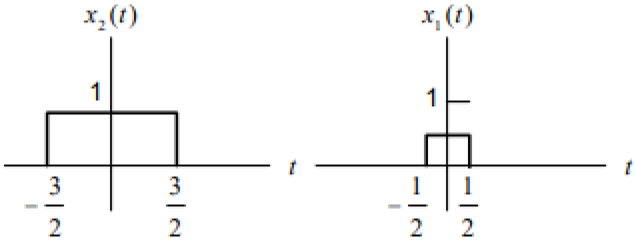
Or

$$F\{x(t - t_0)\} = e^{-j\omega t_0} X(j\omega) = |X(j\omega)| e^{j[\angle X(j\omega) - \omega t_0]}.$$

Thus, the effect of a time shift on a signal is to introduce into its transform a phase shift, namely, $-\omega_0 t$.

Example: To evaluate the Fourier transform of the signal x(t) shown in the figure below.





The signal x(t) can be expressed as the linear combination

$$x(t) = \frac{1}{2} x_1(t - 2.5) + x_2(t - 2.5).$$

$x_1(t)$ and $x_2(t)$ are rectangular pulse signals and their Fourier transforms are

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega} \text{ and } X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}$$

Using the linearity and time-shifting properties of the Fourier transform yields

$$X(j\omega) = e^{-j5\omega/2}\left\{\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right\}$$

2.7.3 Conjugation and Conjugate Symmetry

$$\text{If } x(t) \stackrel{F}{\longleftrightarrow} X(j\omega)$$

Then

$$x^*(t) \stackrel{F}{\longleftrightarrow} X^*(-j\omega)$$

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\right]^* = \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}dt,$$

Replacing ω by -ω , we see that

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x^*(t)e^{-j\omega t}dt,$$

Since $X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\right]^* = \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}dt$,

Replacing ω by $-\omega$, we see that

$X^*(-j\omega) = \int_{-\infty}^{+\infty} x^*(t)e^{-j\omega t}dt$,

The right-hand side is the Fourier transform of x * (t).

If x(t) is real, from Equation we can get

$$X(-j\omega) = X^*(j\omega).$$

We can also prove that if x(t) is both real and even, then X(jw) will also be real and even.

Similarly, if x(t) is both real and odd, then X(jw) will also be purely imaginary and odd.

A real function x(t) can be expressed in terms of the sum of an even function $x_e(t)$ = Ev{x(t)}and an odd function x_o (t) = Od{x(t)}. That is

$$x(t) = x_e(t) + x_o(t)$$

Form the Linearity property,

$$F\{x(t)\} = F\{x_e(t)\} + F\{x_o(t)\},$$

From the preceding discussion, F{x_e(t)} is real function and F{x_o(t)} is purely imaginary. Thus we conclude with x(t) real,

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega)$$

$$Ev\{x(t)\} \overset{F}{\longleftrightarrow} \mathrm{Re}\{X(j\omega)\}$$

$$Od\{x(t)\} \overset{F}{\longleftrightarrow} j\,\mathrm{Im}\{X(j\omega)\}$$

Example: Using the symmetry properties of the Fourier transform and the result

$$e^{-at}u(t) \overset{F}{\longleftrightarrow} \frac{1}{a + j\omega}$$

to evaluate the Fourier transform of the signal x(t)=e $^{-|a|t}$, where a > 0 .

Since

$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) = 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] = 2Ev\{e^{-at}u(t)\},$$

So

$$X(j\omega) = 2\,\text{Re}\left(\frac{1}{a + j\omega}\right) = \frac{2a}{a^2 + \omega^2}$$

2.7.4 Differentiation and Integration

$$\text{If } x(t) \stackrel{F}{\longleftrightarrow} X(j\omega)$$

Then

$$\frac{dx(t)}{dt} \stackrel{F}{\longleftrightarrow} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{F}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

Example: Consider the Fourier transform of the unit step x(t) = u(t).

It is know that

$$g(t) = \delta(t) \stackrel{F}{\longleftrightarrow} 1$$

Also note that

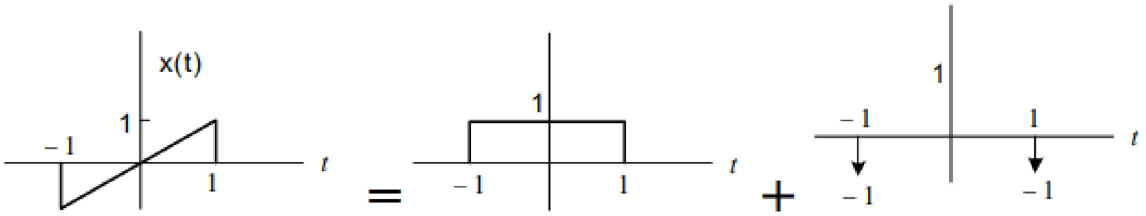
$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau$$

The Fourier transform of this function is

$$X(j\omega) = \frac{1}{j\omega} + \pi G(0)\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

where G(0) = 1.

Example: Consider the Fourier transform of the function x(t) shown in the figure below.



$$g(t) = \frac{dx(t)}{dt}$$

From the above figure we can see that g(t) is the sum of a rectangular pulse and two impulses.

$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}$$

Note that G(0) = 0 , using the integration property, we obtain

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega) = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}.$$

It can be found X(jw) is purely imaginary and odd, which is consistent with the fact that x(t) is real and odd.

2.7.5 Time and Frequency Scaling

$$x(t) \xleftrightarrow{F} X(j\omega),$$

Then

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X(\frac{j\omega}{a}).$$

From the equation we see that the signal is compressed in the time domain, the spectrum will be extended in the frequency domain.

Conversely, if the signal is extended, the corresponding spectrum will be compressed.

If a = -1, we get from the above equation,

$$x(-t) \xleftrightarrow{F} X(-j\omega).$$

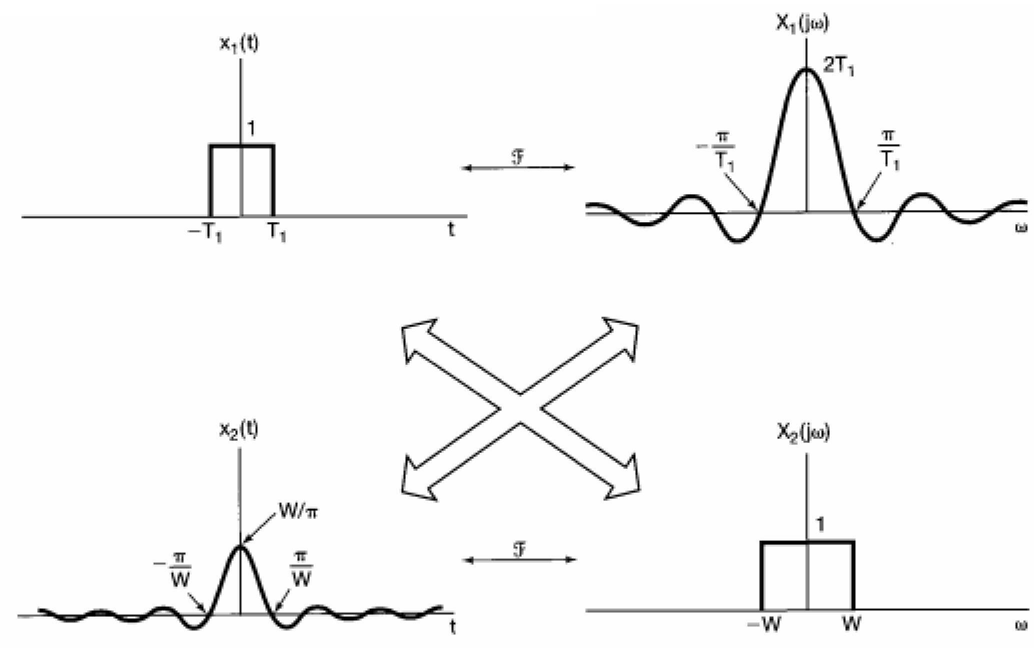
That is, reversing a signal in time also reverses its Fourier transform.

2.7.6 Duality

The duality of the Fourier transform can be demonstrated using the following example.

$$x_1(t) = \begin{cases} 1, & t < T_1 \\ 0, & t > T_1 \end{cases} \xleftrightarrow{F} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

$$x_2(t) = \frac{\sin WT_1}{\pi t} \xleftrightarrow{F} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$



The symmetry exhibited by these two examples extends to Fourier transform in general. For any transform pair, there is a dual pair with the time and frequency variables interchanged.

Example: Consider using duality and the result

$$e^{-|t|} \xleftrightarrow{F} X(j\omega) = \frac{2}{1+\omega^2}$$

to find the Fourier transform G(jw) of the signal

$$g(t) = \frac{2}{1+t^2}.$$

Since $e^{-|t|} \xleftrightarrow{F} X(j\omega) = \dfrac{2}{1+\omega^2}$, that is,

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}\,d\omega,$$

Multiplying this equation by 2Π and replacing t by - t , we have

$$2\pi e^{-|t|} = \int_{-\infty}^{\infty} \left(\frac{2}{1+\omega^2} \right) e^{-j\omega t} d\omega$$

Interchanging the names of the variables t and ω , we find that

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty} \left(\frac{2}{1+t^2} \right) e^{-j\omega t} d\omega \Rightarrow F^{-1}\left(\frac{2}{1+t^2} \right) = 2\pi e^{-|\omega|} .$$

Based on the duality property we can get some other properties of Fourier transform:

$$-jtx(t) \stackrel{F}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}$$

$$e^{j\omega_0 t} x(t) \stackrel{F}{\longleftrightarrow} X(j(\omega - \omega_0))$$

$$-\frac{1}{jt} x(t) + \pi x(0)\delta(t) \stackrel{F}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\eta) d\eta$$

2.7.7 Parseval's Relation

If $x(t) \stackrel{F}{\longleftrightarrow} X(j\omega)$,

We have

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Parseval's relation states that the total energy may be determined either by computing the energy per unit time 2 x(t) and integrating over all time or by computing the energy per unit frequency $|X(jw)|^2$ / 2Π and integrating over all frequencies. For this reason, 2 X (jw) is often referred to as the energy-density spectrum.

2.8 The convolution properties

$$y(t) = h(t) * x(t) \stackrel{F}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$

The equation shows that the Fourier transform maps the convolution of two signals into product of their Fourier transforms.

H(jw), the transform of the impulse response, is the frequency response of the LTI system, which also completely characterizes an LTI system.

Example: The frequency response of a differentiator.

$$y(t) = \frac{dx(t)}{dt}.$$

From the differentiation property,

$$Y(j\omega) = j\omega X(j\omega),$$

The frequency response of the differentiator is

$$H(j\omega) = \frac{Y(j\omega}{X(j\omega)}) = j\omega.$$

Example: Consider an integrator specified by the equation:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

The impulse response of an integrator is the unit step, and therefore the frequency response of the system:

$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

So we have

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega),$$

which is consistent with the integration property.

Example: Consider the response of an LTI system with impulse response

$$h(t) = e^{-at}u(t), \qquad a > 0$$

to the input signal

$$x(t) = e^{-bt}u(t), \qquad b > 0$$

To calculate the Fourier transforms of the two functions:

$$X(j\omega) = \frac{1}{b + j\omega}, \text{ and}$$

$$H(j\omega) = \frac{1}{a + j\omega}.$$

Therefore,

$$Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)},$$

using partial fraction expansion (assuming $a \neq b$), we have

$$y(t) = \frac{1}{b - a}\left[e^{-at}u(t) - e^{-bt}u(t)\right].$$

The inverse transform for each of the two terms can be written directly. Using the linearity property, we have

$$y(t) = \frac{1}{b - a}\left[e^{-at}u(t) - e^{-bt}u(t)\right].$$

We should note that when $a = b$, the above partial fraction expansion is not valid. However, with $a = b$, we have

$$Y(j\omega) = \frac{1}{(a + j\omega)^2},$$

Considering $\frac{1}{(a + j\omega)^2} = j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right]$, and

$$e^{-at}u(t) \stackrel{F}{\longleftrightarrow} \frac{1}{a + j\omega}, \text{ and}$$

$$te^{-at}u(t) \stackrel{F}{\longleftrightarrow} j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right],$$

so we have

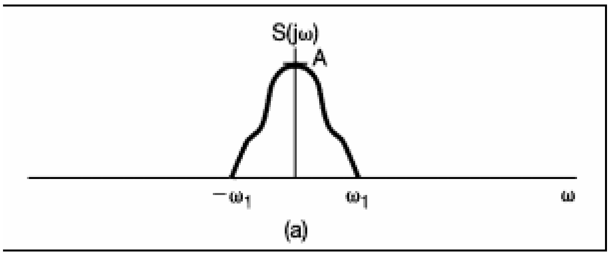
$$Y(t) = te^{-at}u(t).$$

2.9 The Multiplication Property

$$r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta$$

Multiplication of one signal by another can be thought of as one signal to scale or modulate the amplitude of the other, and consequently, the multiplication of two signals is often referred to as amplitude modulation.

Example: Let s(t) be a signal whose spectrum S(jw) is depicted in the figure below.



(a)

Also consider the signal

$$p(t) = \cos\omega_0 t\ ,$$

then

$$P(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0).$$

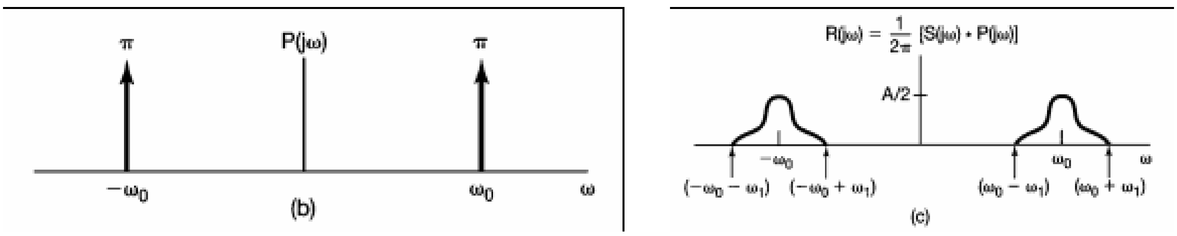
The spectrum of r(t) = s(t) p(t) is obtained by using the multiplication property,

$$R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\omega)P(j(\omega-\theta))d\theta$$

,

$$= \frac{1}{2}S(j\omega-\omega_0) + \frac{1}{2}S(j\omega+\omega_0)$$

which is sketched in the figure below.



(b) (c)

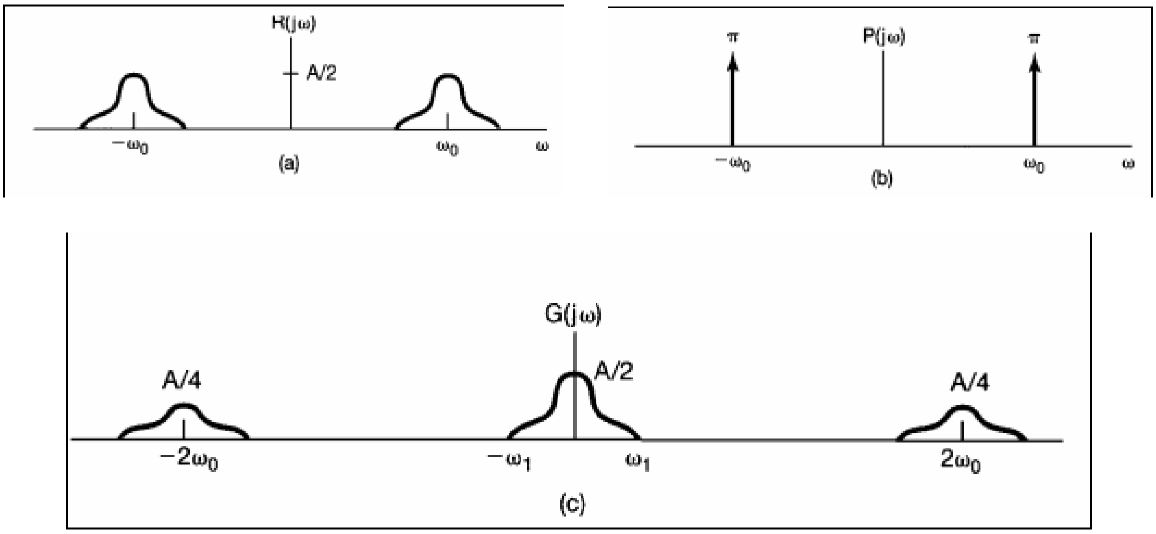
From the figure we can see that the signal is preserved although the information has been shifted to higher frequencies. This forms the basic for sinusoidal amplitude modulation systems for communications.

Example: If we perform the following multiplication using the signal r(t) obtained in the preceding example and p (t)= $\cos\omega_0 t$,

that is,

g(t) = r(t) p(t)

The spectrum of P(jw), R(jw) and G(jw) are plotted in the figure below



If we use a lowpass filter with frequency response H(jw) that is constant at low frequencies and zero at high frequencies, then the output will be a scaled replica of S(jw). Then the output will be scaled version of s(t)- the modulated signal is recovered.

2.10 Summary of Fourier Transform Properties and Basic Fourier Transform Pairs

PROPERTIES OF THE FOURIER TRANSFORM

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\vert a\vert}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega - \theta))d\theta$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \vert X(j\omega)\vert = \vert X(-j\omega)\vert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$	$\mathcal{R}e\{X(j\omega)\}$
	$x_o(t) = \mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$	$j\mathcal{I}m\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Tables of Fourier Properties and of Basic Fourier Transform Pairs

BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & T_1 < \|t\| \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \|t\| < T_1 \\ 0, & \|t\| > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \|\omega\| < W \\ 0, & \|\omega\| > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t),$ $\Re\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

System Characterized by Linear Constant-Coefficient Differential Equations An LTI system described by the following differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k},$$

which is commonly referred to as an Nth-order differential equation.

The frequency response of this LTI system

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)},$$

where X(jw), Y(jw) and H(jw) are the Fourier transforms of the input x(t), output y(t) and the impulse response h(t), respectively.

Applying Fourier transform to both sides, we have

$$F\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = F\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\},$$

From the linearity property, the expression can be written as

$$\sum_{k=0}^{N} a_k F\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k F\left\{\frac{d^k x(t)}{dt^k}\right\}.$$

From the differentiation property,

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega) \qquad \Rightarrow \qquad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

H(jw) is a rational function, that is, it is a ratio of polynomials in (jw).

Example: Consider a stable LTI system characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t), \text{ with } a > 0.$$

The frequency response is

$$H(j\omega) = \frac{1}{j\omega + a}.$$

The impulse response of this system is then recognized as

$$h(t) = e^{-at} u(t).$$

Example: Consider a stable LTI system that is characterized by the differential equation

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t).$$

The frequency response of this system is

$$H(j\omega)=\frac{(j\omega)+2}{(j\omega)^2+4(j\omega)+3}=\frac{j\omega+2}{(j\omega+1)(j\omega+3)}.$$

Then, using the method of partial-fraction expansion, we find that

$$H(j\omega)=\frac{1/2}{j\omega+1}+\frac{1/2}{j\omega+3}.$$

The inverse Fourier transform of each term can be recognized as

$$h(t)=\frac{1}{2}e^{-t}u(t)+\frac{1}{2}e^{-3t}u(t).$$

Example: Consider a system with frequency response of

$$H(j\omega)=\frac{j\omega+2}{(j\omega+1)(j\omega+3)}$$

and suppose that the input to the system is

$$x(t)=e^{-t}u(t),$$

find the output response.

The output in the frequency domain is give as

$$Y(j\omega)=H(j\omega)X(j\omega)=\left[\frac{j\omega+2}{(j\omega+1)(j\omega+3)}\right]\left[\frac{1}{j\omega+1}\right]=\frac{j\omega+2}{(j\omega+1)^2(j\omega+3))},$$

Using partial-fraction expansion, we have

$$Y(j\omega)=\frac{1/4}{j\omega+1}+\frac{1/2}{(j\omega+1)^2}+\frac{1/4}{(j\omega+3))},$$

By inspection, we get directly the inverse Fourier transform:

$$h(t)=\left[\frac{1}{4}e^{-t}+\frac{1}{2}te^{-t}-\frac{1}{4}e^{-3t}\right]u(t).$$

SUMMARY

A function is said to be periodic with period T if f(t + T) = f(t) for all t and the smallest such positive number T is called the period.

The Fourier series representation of f(x) defined on [0, 2π], when it exists, is given by equation with Fourier coefficients

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad n = 0,1,2,\ldots, \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx, \quad n = 1,2,\ldots.
\end{aligned}
$$

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} \frac{a_0}{2}\,dx + \int_0^{2\pi} \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]\,dx.$$

If one integrates an even function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx.$$

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}\right].$$

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^{L} f(x)\cos\frac{2n\pi x}{L}\,dx. \quad n = 0,1,2,\ldots, \\
b_n &= \frac{2}{L}\int_0^{L} f(x)\sin\frac{2n\pi x}{L}\,dx. \quad n = 1,2,\ldots.
\end{aligned}
$$

Fourier Series on $[-\frac{L}{2}, \frac{L}{2}]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}\right].$$

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\frac{2n\pi x}{L}\,dx. \quad n = 0,1,2,\ldots, \\
b_n &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\frac{2n\pi x}{L}\,dx. \quad n = 1,2,\ldots.
\end{aligned}
$$

Fourier Series on $[-\pi, \pi]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right].$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx. \quad n = 0,1,2,\ldots, \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \quad n = 1,2,\ldots.
\end{aligned}
$$

Over any period, x(t) must be absolutely integrable, that is

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty ,$$

In any finite interval of time, x(t) have a finite number of maxima and minima.

In any finite interval of time, there are only a finite number of discontinuities.

The Fourier transform can be plotted in terms of the magnitude and phase, as

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}, \qquad \angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$

The impulse has a Fourier transform consisting of equal contributions at all frequencies.

The Fourier series representation of the signal x(t) is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} .$$

It's Fourier transform is

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) .$$

The Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period 2Π /T ,

Different properties of Fourier transforms are Linearity, Time Shifting, Conjugation and Conjugate Symmetry, Differentiation and Integration, Time and Frequency Scaling, Duality, Parseval's Relation, convolution properties, Multiplication Property

Question:

Q1) Explain Trigonometric Fourier Series with example.

Q2) Explain Exponential Fourier Series with example.

Q3) Explain Convergence Of Fourier Transform.

Q4) Explain Fourier Transform For Periodic Signals

Q5) Explain Properties of Fourier Transform .

Q6) Explain Convolution properties of Fourier Transform.

Q7) Explain Multiplication properties of Fourier Transform.

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Unit -2

Chapter -3

LAPLACE TRANSFORM

3.0 OBJECTIVES

- Understand Laplace transform of basic signals.
- To understand and apply properties of Laplace transform.
- To understand and apply unilateral Laplace transforms.

3.1 INTRODUCTION

Laplace transform represents continuous time signals in terms of complex exponentials i.e. e^{-st}

Continuous time systems are also analyzed more effectively using Laplace transform.

Laplace transform can be applied to the analysis of unstable systems also.

Types of Laplace Transform

i) Bilateral or two sided Laplace transform

ii) Unilateral or one sided Laplace transform

3.2 Definition of Laplace Transform

$$X(s)= \int_{-\infty}^{\infty} x(t) \; e^{-st} \; dt$$

Here the independent variable 's' is complex in nature and it is given as

$$s = \sigma + j\omega$$

Here σ is real part of 's' It is called attenuation constant.

jω is the imaginary part of 's' and it is called complex frequency.

The Laplace transform pair x(t) and X(s) is represented as,

$$x(t) \longleftrightarrow X(s)$$

The unilateral Laplace transform is given as

$$X(s)= \int_{0}^{\infty} x(t) \; e^{-st} \; dt$$

Laplace transform is mainly used for causal signals.

The Inverse Laplace transform is given as

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) \; e^{st} \; dt$$

3.3 Convergence of Laplace Transform

We know that Laplace Transform is basically the Fourier transform of x(t) $e^{-\sigma t}$.

If Fourier transform of x(t) $e^{-\sigma t}$ exists , then Laplace transform of x(t) exits.

For the Fourier transform to exists , x(t) $e^{-\sigma t}$ must be absolutely integrable.

$$\int_{-\infty}^{\infty} | x(t)\ e - \sigma t| \ | \ dt \ < \ \infty$$

The range of values of σ for which Laplace transform converges is called region of convergence or ROC.

Example 3.1:- Calculate the Laplace transform of following functions and plot their ROC

i) x(t)= e^{at} u(t)

Solution:-

i) $x(t) = e^{at}\, u(t)$

$$X(s) = \int_{-\infty}^{\infty} e^{at}\, u(t)\, e^{-st}\, dt$$

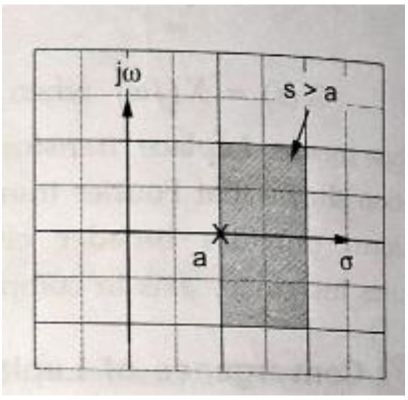
$$= \int_{0}^{\infty} e^{at}\, e^{-st}\, dt,$$

$$= \int_{0}^{\infty} e^{-(s-a)t}\, dt \quad = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty} = \lim_{t\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right] - \lim_{t\to 0}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]$$

$$\lim_{t\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right] = \frac{e^{-\infty}}{-(s-a)} = 0 \text{ if } (s-a) > 0$$

$$X(s) = 0 - \left[\frac{e^{0}}{-(s-a)}\right] \text{ for } (s-a) > 0 = \frac{1}{s-a} \text{ for } s > a.$$

$$e^{at}\, u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-a},\ ROC: s > a$$



The shaded area is called Region of convergence

Since Re(s) is real part of 's' i.e. σ .

Hence ROC : σ > a or Re(s) > a

ii) $x(t) = -e^{at}\ u(-t)$

Solution

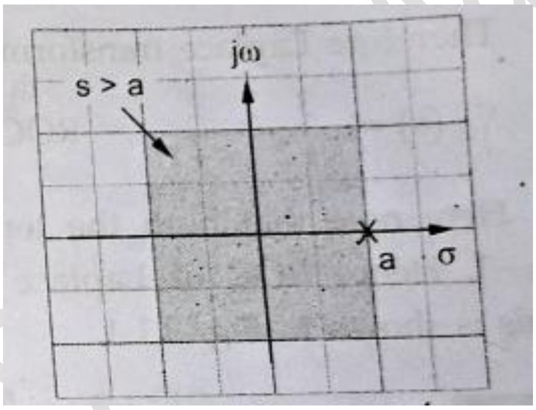
$$X(s) = \int_{-\infty}^{\infty} -e^{at}\, u(-t)\, e^{-st}\, dt = \int_{-\infty}^{0} -e^{at}\, e^{-st}\, dt, \quad \text{Since } u(-t) = 1 \text{ for } -\infty \le t \le 0$$

$$= -\int_{-\infty}^{0} e^{-(s-a)t}\, dt = \left[\frac{e^{-(s-a)t}}{s-a}\right]_{-\infty}^{0} = \lim_{t\to 0}\left[\frac{e^{-(s-a)t}}{s-a}\right] - \lim_{t\to -\infty}\left[\frac{e^{-(s-a)t}}{s-a}\right]$$

$$\lim_{t\to -\infty}\left[\frac{e^{-(s-a)t}}{s-a}\right] = \frac{e^{-\infty}}{s-a} = 0 \text{ if } s < a$$

$$\therefore \quad X(s) = \frac{e^{0}}{s-a} - 0 \text{ if } s < a$$

$$= \frac{1}{s-a}, \quad \text{ROC: } s < a$$



The shaded region shows ROC of $s < a$

Thus,

$$-e^{at}\, u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-a}, \quad \text{ROC } s < a$$

Example 3.2 Determine the Laplace transform of

i) $x_1(t) = e^{-2t}\, u(t) - e^{2t}\, u(-t)$

Solution:

$$e^{at}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a}, \quad \text{ROC}: s > a$$

with $a = -2$; $$e^{-2t}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \text{ROC}: s > -2$$

And $$-e^{at}\,u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a}, \quad \text{ROC}: s < a$$

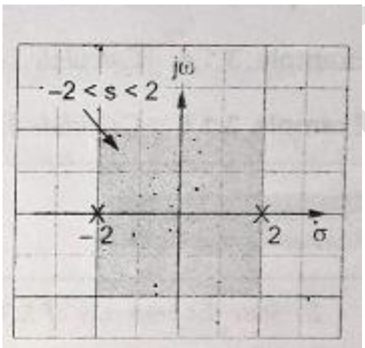
with $a = 2$; $$-e^{2t}\,u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-2}, \quad \text{ROC}: s < 2$$

From above result Laplace transform of $x_1(t)$ will be

$$X_1(s) = \frac{1}{s+2} + \frac{1}{s-2}, \quad \text{ROC } s > -2 \text{ and } s < 2$$

$$\text{or } -2 < s < 2$$

The figure shows ROC of -2 < s < 2



ii) $x_2(t) = 3\,e^{-2t}\,u(t) - e^{-t}\,u(t)$

Solution:

We have $$e^{at}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a}, \quad \text{ROC}: s > a$$

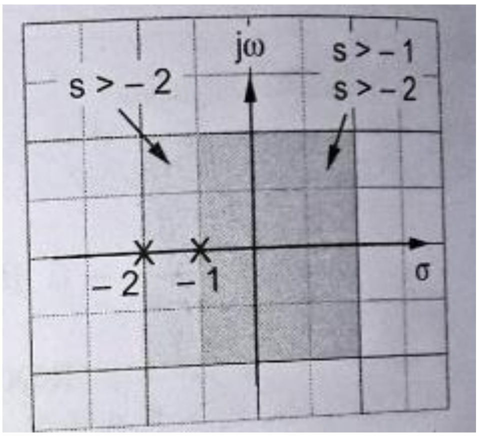
with $a = -2$; $$3e^{-2t}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{3}{s+2}, \quad \text{ROC}: s > -2$$

with $a = -1$; $$2e^{-t}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{2}{s+1}, \quad \text{ROC}: s > -1$$

Therefore Laplace transform of $x_2(t)$ become

$$X_2(s) = \frac{3}{s+2} - \frac{2}{s+1},$$

ROC : s > -2 and s > -1



Both the terms of converge for ROC of s > -1 .Hence ROC of Laplace transform will be s >-1

3.4 Properties of ROC

1. No poles lie in ROC.
2. ROC of the causal signal is right hand sided. It is of the form Re(s) > a.
3. ROC of the noncausal signal is left hand sided. It is of the form Re(s) < a.
4. The system is stable if its ROC includes jω axis of s-plane.

3.5 Properties of Laplace Transform

For all the properties we have,

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s),\ \text{ROC} : R_1 \text{ and } X_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s),\ \text{ROC} : R_2$$

3.5.1 Linearity

Statement: Laplace transform follows superposition principle ,i.e. it is linear

$$\mathcal{L}[a_1 x_1(t) + a_2 x_2(t)] = a_1 X_1(s) + a_2 X_2(s), \quad \text{ROC} : R_1 \cap R_2$$

Proof:

$$\mathcal{L}[a_1 x_1(t) + a_2 x_2(t)] = \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)] e^{-st}\, dt$$

$$= a_1 \int_{-\infty}^{\infty} x_1(t) e^{-st}\, dt + a_2 \int_{-\infty}^{\infty} x_2(t) e^{-st}\, dt$$

$$= a_1 X_1(s) + a_2 X_2(s) \quad \text{ROC} : R_1 \cap R_2$$

Here ROC : $R_1 \cap R_2$ indicates the intersection of R_1 and R_2

3.5.2 Time Shifting (Translation in Time Domain)

Statement: A time shift in the signal introduces frequency shift in frequency domain.

$$\mathcal{L}[x(t-t_0)] = e^{-st_0} X(s) \quad \text{ROC} : R$$

Proof:

$$\mathcal{L}[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-st}\, dt$$

By substituting in above equation we have

$$\mathcal{L}[x(t-t_0)] = \int_{-\infty}^{\infty} x(\tau)\, e^{-s(\tau+t_0)}\, d\tau = \int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau}\, e^{-st_0}\, d\tau = e^{-st_0} \underbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau}\, d\tau}_{X(s)}$$

$$= e^{-st_0} X(s) \;, \quad \text{ROC} : R$$

3.5.3 Shifting in s- Domain (Complex Translation)

Statement: A shift in the frequency domain is equivalent to multiplying the time domain signal by complex exponential.

$$\mathcal{L}\left[e^{s_0 t} x(t)\right] = X(s - s_0) \;, \quad \text{ROC} : R + Re(s_0)$$

Proof:

$$\mathcal{L}\left[e^{s_0 t} x(t)\right] = \int_{-\infty}^{\infty} e^{s_0 t} x(t) e^{-st}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-(s-s_0)t}\, dt$$

$$= X(s - s_0) \text{ with ROC} : R + Re(s_0)$$

3.5.4 Time Scaling

Statement: Expansion in time domain is equivalent to compression in frequency domain and vice versa

$$\mathcal{L}[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-st}\, dt$$

$$\text{ROC} : \frac{R}{a}$$

Proof:

$$\mathcal{L}[x(at)] = \int_{-\infty}^{\infty} x(\tau)\, e^{-s\frac{\tau}{a}}\, \frac{1}{a}\, d\tau = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-\left(\frac{s}{a}\right)\tau}\, d\tau$$

$$= \frac{1}{a} X\left(\frac{s}{a}\right) \quad \text{ROC} : \frac{R}{a}$$

Similar procedure can be repeated for Laplace transform of x(-at) . We get

$$\mathcal{L}[x(-at)] = \frac{1}{a} X\left(\frac{s}{-a}\right),\ \text{ROC} : \frac{R}{-a}$$

The above equations can be combined as follows:

$$\mathcal{L}[x(at)] = \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad , \quad \text{ROC} : \frac{R}{a}$$

As a special case with a= -1 we have

$$x(-t) \overset{\mathcal{L}}{\longleftrightarrow} X(-s) \quad , \quad \text{ROC} : \text{R}$$

This result shows that inverting the time axis inverts frequency axis as well as ROC.

3.5.5 Differentiation in Time Domain

Statement: Differentiation in time domain adds a zero to the system.

$$\frac{d}{dt} x(t) \overset{\mathcal{L}}{\longleftrightarrow} sX(s)\ ,\ \ \text{ROC} : R$$

Proof:

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds$$

Differentiate both sides of above equation with respect to ‘t’ i.e.

$$\frac{d}{dt} x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, s\, e^{st}\, ds = \underbrace{\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} [sX(s)]\, e^{st}\, ds}_{\text{Inverse Laplace of } sX(s)}$$

For multiple order derivative

$$\frac{d^n}{dt^n} x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} s^n X(s) \quad , \quad \text{ROC containing } R$$

3.5.6 Differentiation in s- Domain

Statement: Differentiation in s-domain corresponds to multiplying the time domain sequence by -t

$$-t\, x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds} X(s) \; , \quad \text{ROC} : R$$

Proof:

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Differentiating above equation with respect to ‘s’

$$\frac{d}{ds} X(s) = \int_{-\infty}^{\infty} x(t)\, (-t)\, e^{-st}\, dt = \underbrace{\int_{-\infty}^{\infty} [-t\, x(t)]\, e^{-st}\, dt}_{\text{Laplace transform of } -t\, x(t)}$$

$$\therefore \; -t\, x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds} X(s)$$

For multiple order differentiation in s-domain,

$$(-t)^n\, x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{d^n}{ds^n} X(s) \quad , \quad \text{ROC} : R$$

3.5.7 Convolution in Time Domain

Statement: The Laplace transform of convolution of two functions is equivalent to multiplication of their Laplace transforms.

$$x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)\, X_2(s)$$

ROC : containing $R_1 \cap R_2$

Proof:

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

Taking Laplace transform of both the sides,

$$\mathcal{L}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right\} e^{-st}\, dt$$

Changing the order of integration,

$$\mathcal{L}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\, d\tau \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-st}\, dt$$

$$\therefore\ \mathcal{L}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\, d\tau \int_{-\infty}^{\infty} x_2(\lambda)\, e^{-s(\lambda+\tau)}\, d\lambda$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, d\tau \int_{-\infty}^{\infty} x_2(\lambda)\, e^{-s\lambda} \cdot e^{-s\tau}\, d\lambda = \underbrace{\int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau} d\tau}_{X_1(s)} \ \underbrace{\int_{-\infty}^{\infty} x_2(\lambda) e^{-s\lambda} d\lambda}_{X_2(s)}$$

$$= X_1(s)\, X_2(s)\ ,\ \text{ROC} : R_1 \cap R_2$$

3.5.8 Integration in Time domain

Statement: Time domain integration adds a pole to the system.

$$\int_{-\infty}^{t} x(\tau) d\tau \quad \longleftrightarrow \quad \frac{X(s)}{s}\ , \text{ROC : } R \cap [\ \text{Re}(s) > 0\ ]$$

Proof:

$$x(t) * u(t) = \int_{-\infty}^{\infty} u(t-\tau)\, x(\tau)\, d\tau$$

Hence above equation becomes

$$x(t) * u(t) = \int_{-\infty}^{t} 1 \ x(\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t)$$

Taking Laplace transform of both sides,

$$\mathcal{L}\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\} = \mathcal{L}\{x(t) * u(t)\} \quad x(t) \xleftrightarrow{\mathcal{L}} X(s) \text{ and } u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$$

$$= X(s)\frac{1}{s}, \text{ By convolution property}$$

$$= \frac{X(s)}{s}, \quad \text{ROC} : R \cap [Re(s) > 0]$$

For multiple order of integration,

$$\mathcal{L}\left\{\int_{-\infty}^{t}\int_{-\infty}^{t} \ldots \int_{-\infty}^{t} x(t)\, dt_1\, dt_2 \ldots dt_n\right\} = \frac{X(s)}{s^n}$$

3.5.9 Integration in s- Domain

Statement:

Frequency domain integration corresponds to dividing the time domain signal by t

$$\frac{x(t)}{t} \xleftrightarrow{\mathcal{L}} \int_{s}^{\infty} X(s)\, ds, \quad \text{ROC} : \text{R}$$

Proof:

$$\int_{s}^{\infty} X(s)\, ds = \int_{s}^{\infty}\left[\int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt\right] ds$$

Changing the order of integration and rearranging the terms,

$$\int_{s}^{\infty} X(s)\,ds = \int_{-\infty}^{\infty} x(t)\left[\int_{s}^{\infty} e^{-st}\,ds\right] dt = \int_{-\infty}^{\infty} x(t)\left[\frac{e^{-st}}{-t}\right]_{s}^{\infty} dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{e^{-\infty} - e^{-st}}{-t}\right] dt = \int_{-\infty}^{\infty} x(t)\,\frac{e^{-st}}{t}\,dt \quad , \quad \text{ROC : R}$$

$$= \int_{-\infty}^{\infty} \frac{x(t)}{t}\,e^{-st}\,dt \quad , \quad \text{ROC : } R$$

$$= \mathcal{L}\left[\frac{x(t)}{t}\right]$$

3.6 Examples of Laplace Transform

Example 3.3 : Obtain the Laplace transform and ROC of following signals :

i) x(t) = u(t)

Solution:

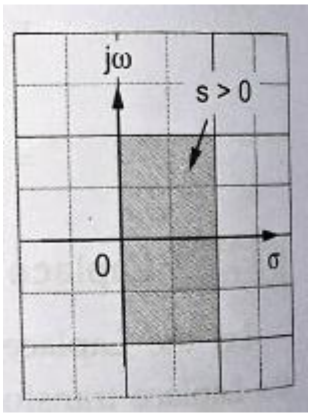
$$\mathcal{L}\,[u(t)] = \int_{-\infty}^{\infty} u(t)\,e^{-st}\,dt = \int_{0}^{\infty} e^{-st}\,dt, \text{ since } u(t) = 1 \text{ for } 0 \le t \le \infty$$

$$= \left[\frac{e^{-st}}{-s}\right]_{0}^{\infty} = \frac{e^{-s\times\infty} - e^{0}}{-s}$$

Here $e^{-s \times \infty} = e^{-\infty} = 0$

If $s > 0$. Then above equation will be,

$$\mathcal{L}\,[u(t)] = \frac{1}{s}, \text{ ROC : } s > 0 \text{ or } \sigma > 0$$



ROC

Thus

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}, \quad \text{ROC} : s > 0 \quad \text{or} \quad \sigma > 0$$

ii) x(t) = δ(t)

Solution:

$$\mathcal{L}\,[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt$$

Here use,

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, x(t)\; dt = x(t_0)$$

With $t_0 = 0$, the above equation becomes

$$\mathcal{L}\,[\delta(t)] = e^0 = 1.$$

This is convergent for all values of s.

$$\delta(t) \xleftrightarrow{\mathcal{L}} 1, \ \textbf{ROC} : \text{Entire s-plane}$$

iii) x(t)= r(t)

Solution :

$$\mathcal{L}[r(t)] = \int_{-\infty}^{\infty} r(t) e^{-st} dt = \int_{-\infty}^{\infty} t\, u(t) e^{-st} dt, \text{ since } r(t) = t\, u(t)$$

$$= \int_{0}^{\infty} t\, e^{-st} dt$$

Integrating by parts,

$$\mathcal{L}[r(t)] = \int_{-\infty}^{\infty} r(t) e^{-st} dt = \int_{-\infty}^{\infty} t\, u(t) e^{-st} dt, \text{ since } r(t) = t\, u(t)$$

$$= \int_{0}^{\infty} t\, e^{-st} dt$$

$$\mathcal{L}[r(t)] = \left[t \cdot \frac{e^{-st}}{-s} \right]_0^{\infty} - \int_0^{\infty} 1 \cdot \frac{e^{-st}}{-s}\, dt$$

$$\frac{te^{-st}}{-s} = \frac{\infty \cdot e^{-s\times\infty} - 0}{-s} = 0 \text{ if } s > 0$$

$$\mathcal{L}[r(t)] = 0 - \int_0^{\infty} \frac{e^{-st}}{-s}\, dt = \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{1}{s} \cdot \frac{e^{-s\times\infty} - e^0}{-s} = \frac{1}{s^2} \text{ if } s > 0$$

$$r(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s^2}, \text{ROC} : s > 0 \text{ or } \sigma > 0$$

Similarly,

$$\frac{t^{n-1}}{(n-1)!} \xleftrightarrow{\mathcal{L}} \frac{1}{s^n}, \text{ ROC} : Re(s) > 0 \text{ or } \sigma > 0$$

iv) $x(t) = t\, e^{-at}\, u(t)$

Solution :

$$\underbrace{e^{-at} u(t)}_{x_1(t)} \xleftrightarrow{\mathcal{L}} \underbrace{\frac{1}{s+a}}_{X_1(s)}, \text{ROC} : s > -a \text{ or } Re(s) > -a$$

By differentiation in s-domain property,

$$-t\,x_1(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}, X_1(s)$$

$$-t \cdot e^{-at}\, u(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds} \cdot \frac{1}{s+a}$$

$$te^{-at}\, u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^2}, \textbf{ROC} : Re(s) \text{ or } \sigma > -a$$

Similarly,

$$-t[t\,e^{-at}\, u(t)] \xleftrightarrow{\mathcal{L}} \frac{d}{ds} \cdot \frac{1}{(s+a)^2}$$

$$\frac{t^2}{2} e^{-at}\, u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^3}$$

$$\frac{t^{n-1}}{(n-1)!}\, e^{-at}\, u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^n}$$

Example 3.4: Obtain the Laplace transform of following signals :

i) $x(t) = A \sin \omega_0 t$

Solution :

$$X(s) = \int_{-\infty}^{\infty} A \sin \omega_0 t\; u(t)\; e^{-st}\; dt$$

$$= A \int_{-\infty}^{\infty} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} u(t)\; e^{-st}\; dt, \text{ since } \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{A}{2j}\left[\int_{-\infty}^{\infty} e^{j\omega_0 t}\, u(t)\, e^{-st}\, dt - \int_{-\infty}^{\infty} e^{-j\omega_0 t}\, u(t)\, e^{-st}\, dt\right]$$

$$= \frac{A}{2j}\left[\mathcal{L}\,[e^{j\omega_0 t}\, u(t)] - \mathcal{L}[e^{-j\omega_0 t}\, u(t)]\right]$$

We know that ,

$$e^{at}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a}, \ \textbf{ROC} : s > a.$$

The above equation can be written as ,

$$X(s) = \frac{A}{2j}\left\{\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right\} \ \textbf{ROC} : \text{s} > j\omega \ \text{ and s} > -j\omega.$$

Here s > jω can be written as σ+ jω > 0 +jω , hence σ >0 .

Therefore ROC will be Re(s) or σ >0.

$$X(s) = \frac{A}{2j}\left\{\frac{2j\omega_0}{s^2+\omega^2}\right\} = \frac{A\omega_0}{s^2+\omega^2}$$

$$A\ \sin\omega_0\, t\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{A\omega_0}{s^2+\omega^2}, \ \textbf{ROC} : \sigma > 0$$

ii) x(t) = A cos ω_0 t u(t)

Solution:

$$X(s) = \int_{-\infty}^{\infty} A\ \cos\omega_0 t\ u(t)\ e^{-st}\,dt$$

$$= A\int_{-\infty}^{\infty} \frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\cdot u(t)\ e^{-st}\,dt, \ \cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$= \frac{A}{2}\left[\int_{-\infty}^{\infty} e^{j\omega_0 t}\,u(t)\,e^{-st}\,dt + \int_{-\infty}^{\infty} e^{-j\omega_0 t}\,u(t)\,e^{-st}\,dt\right]$$

$$= \frac{A}{2}\left\{\mathcal{L}\,[e^{j\omega_0 t}\,u(t)] + \mathcal{L}[e^{-j\omega_0 t}\,u(t)]\right\}$$

$$= \frac{A}{2}\left\{\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right\}, \ \textbf{ROC} : \text{s} > j\omega \ \text{ and s} > -j\omega \ \text{ i.e. } Re(s) \text{ or } \sigma > 0$$

$$= \frac{A\cdot s}{s^2+\omega_0^2}$$

Therefore,

$$A\cos\omega_0 t\, u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{A\cdot s}{s^2+\omega_0^2},\ \textbf{ROC}: \sigma > 0$$

3.7 Unilateral Laplace Transform

The unilateral Laplace transform is given as ,

$$X(s) = \int_{0-}^{\infty} x(t)\, e^{-st}\, dt$$

The lower limit is taken as 0^- to indicate that initial conditions at t=0 are also considered.

Note that unilateral Laplace transform will be always convergent since ROC will be always R.H.S. of S-plane.

3.7.1 Differentiation in Time Domain

Let x(t) ←→ X(S) Laplace transform pair

Then,

$$\frac{dx(t)}{dt} \leftrightarrow s\,X(s) - x(0^-)$$

Here $x(0^-)$ is value of x(t) at $t=0^-$. It is initial value of x(t).

Proof :

By definition of Laplace transform,

$$\mathcal{L}\left[\frac{d}{dt}x(t)\right] = \int_{0-}^{\infty} \frac{d}{dt}x(t)\, e^{-st}\, dt$$

Integrating above equation by parts,

$$\mathcal{L}\left[\frac{d}{dt}x(t)\right] = \left[e^{-st}x(t)\right]_{0-}^{\infty} - \int_{0-}^{\infty} x(t)(-s)\,e^{-st}\,dt = \left[e^{-st}x(t)\right]_{0-}^{\infty} + s\int_{0-}^{\infty} x(t)\,e^{-st}\,dt$$

The integration term in above equation is Laplace transform of x(t) .

Hence,

$$\mathcal{L}\left[\frac{d}{dt}x(t)\right] = \left[e^{-st}x(t)\right]_{0-}^{\infty} + sX(s) = \left[e^{-\infty}x(\infty) - e^{0}x(0^-)\right] + s\,X(s)$$

We know that $e^{-\infty} = 0$ and $e^0 = 1$. Hence the above equation becomes,

$$\mathcal{L}\left[\frac{d}{dt}x(t)\right] = 0 - x(0-) + s\,X(s) \quad \text{i.e.} \quad \mathcal{L}\left[\frac{d}{dt}x(t)\right] = s\,X(s) - x(0-)$$

This property can be further expanded for multiple differentiations as follows:

$$\frac{d^n}{dt^n} x(t) \overset{\mathcal{L}}{\longleftrightarrow} s^n X(s) - \frac{d^{n-1}}{dt^{n-1}} x(t)\Big|_{t=0-} - s \frac{d^{n-2}}{dt^{n-2}} x(t)\Big|_{t=0-} - \dots - s^{n-2} \frac{d}{dt} x(t)\Big|_{t=0-} - s^{n-1} x(0-)$$

3.7.2 Initial Value Theorem

Let x(t) ←→ X(S) Laplace transform pair

Then initial value of x(t) is given as

$$x(0+) = \lim_{t \to 0+} x(t) = \lim_{s \to \infty} [sX(s)]$$

Provided that the first derivatives of x(t) should be Laplace transformable.

Proof: From the differentiation property of Laplace transform we know that,

$$\mathcal{L}\left[\frac{d}{dt} x(t)\right] = sX(s) - x(0-)$$

Let us take limit of the above equation as s→∞ ,i.e.,

$$\lim_{s \to \infty} \mathcal{L}\left[\frac{d}{dt} x(t)\right] = \lim_{s \to \infty} \{sX(s) - x(0-)\}$$

Consider L.H.S of above equation i.e.,

$$\lim_{s \to \infty} \mathcal{L}\left[\frac{d}{dt} x(t)\right] = \lim_{s \to \infty} \int_{0-}^{\infty} \frac{d}{dt} x(t) e^{-st}\, dt = 0, \text{ since } \lim_{s \to \infty} e^{-st}\, dt = 0$$

Therefore above equation becomes

$$0 = \lim_{s \to \infty} \{sX(s) - x(0-)\} = \lim_{s \to \infty} [sX(s)] - x(0-)$$

$$x(0-) = \lim_{s \to \infty} [sX(s)]$$

$x(0^-)$ indicates the value of x(t) just before t=0 and x(0+) indicates value of x(t) just after t=0 .

If the function x(t) is continuous at t=0 , then its value just before and after t=0 will be same i.e.,

$$x(0+) = \lim_{s \to \infty} [sX(s)]$$

$$x(0+) = x(0-) \text{ for } x(t) \text{ continuous at } t = 0.$$

$$x(0+) = \lim_{s\to\infty} [sX(s)]$$

This equation is used to determine the initial value of x(t) and its derivative.

3.7.3 Final Value Theorem

Let x(t) ←→ X(S) Laplace transform pair

Then initial value of x(t) is given as

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} [sX(s)]$$

Proof : From differentiation property we know that,

$$\mathcal{L}\left[\frac{d}{dt}x(t)\right] = sX(s) - x(0-)$$

Let us take limit of above equation as s→0 , i.e.

$$\lim_{s\to 0}\mathcal{L}\left[\frac{d}{dt}x(t)\right] = \lim_{s\to 0}\{sX(s) - x(0-)\} = \lim_{s\to 0}\{sX(s)\} - x(0-)$$

Consider L.H.S of above equation,

$$\lim_{s\to 0}\mathcal{L}\left[\frac{d}{dt}x(t)\right] = \lim_{s\to 0}\int_{0-}^{\infty}\frac{d}{dt}x(t)\,e^{-st}\,dt = \int_{0-}^{\infty}\frac{d}{dt}x(t)\,dt, \quad \text{since } \lim_{s\to 0} e^{-st} = 1.$$

$$= [x(t)]_{0-}^{\infty} = \lim_{t\to\infty} x(t) - x(0-)$$

Hence equation can be written as

$$\lim_{t\to\infty} x(t) - x(0-) = \lim_{s\to 0}[sX(s)] - x(0-)$$

$$\therefore \quad \lim_{t\to\infty} x(t) = \lim_{s\to 0}[sX(s)]$$

Application of Initial and Final Value Theorem

The initial voltage on the capacitor or current through an indicator can be evaluated with the help of initial value theorem.

The final charging voltage on capacitor or saturating currents through an inductor can be evaluated with the help of final value theorem.

Example 3.5: Find f(∞) final value of function whose Laplace transform is given by

$$\mathbf{F(s)} = \frac{5}{s} - \frac{1}{s-4}$$

Solution:

Final value is given as,

$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,F(s) = \lim_{s \to 0} s\left[\frac{5}{s} - \frac{1}{s-4}\right] = \lim_{s \to 0}\left[5 - \frac{s}{s-4}\right]$$

$$= 5 - \lim_{s \to 0} \frac{s}{s-4} = 5.$$

Example 3.6: Use the s- domain shift property and Fourier transform pair

$$e^{-at}\, u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$$

To derive the unilateral Laplace transform of $x(t) = e^{-at}\, u(t) \cos \omega_1 t\, u(t)$

Solution:

$$x(t) = e^{-at} \cos(\omega_1 t)\; u(t) = e^{-at}\, \frac{e^{j\omega_1 t} + e^{-j\omega_1 t}}{2}\, u(t)$$

$$= \frac{1}{2}\left\{e^{j\omega_1 t}\, e^{-at}\, u(t) + e^{-j\omega_1 t}\, e^{-at}\, u(t)\right\}$$

$$e^{-at}\, u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$$

Here

$$e^{-at}\, u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$$

By shifting in s-domain property,

$$\mathcal{L}\,[e^{s_0 t}\; x(t)] = X(s - s_0)$$

$$X(s) = \left\{\frac{1}{(s - j\omega) + a} + \frac{1}{(s + j\omega) + a}\right\} = \frac{s+a}{(s+a)^2 + \omega^2}$$

Example 3.7 : Determine initial and final values of signal x(t) whose unilateral Laplace transform :

$$X(s) = \frac{7s+10}{s(s+10)}$$

Solution:

Initial value is given by,

$$x(0+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{7s+10}{s+10} = \lim_{s\to\infty} \frac{7+\frac{10}{s}}{1+\frac{10}{s}} = 7$$

Final value is given as,

$$\lim_{t\to\infty} x(t) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{7s+10}{s+10} = 1$$

SUMMARY

Laplace transform represents continuous time signals in terms of complex exponentials i.e. e^{-st}

Continuous time systems are also analyzed more effectively using Laplace transform.

Laplace transform can be applied to the analysis of unstable systems also.

$$X(s)= \int_{-\infty}^{\infty} x(t) \quad e^{-st} \ dt$$

Here the independent variable 's' is complex in nature and it is given as

$$s = \sigma + j\omega$$

Here σ is real part of 's' It is called attenuation constant.

jω is the imaginary part of 's' and it is called complex frequency.

Types of Laplace Transform i) Bilateral or two sided Laplace transform

ii) Unilateral or one sided Laplace transform

The Inverse Laplace transform is given as

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) \ e^{st} \ dt$$

If Fourier transform of x(t) $e^{-\sigma t}$ exists , then Laplace transform of x(t) exits.

For the Fourier transform to exists , x(t) $e^{-\sigma t}$ must be absolutely integrable.

$$\int_{-\infty}^{\infty} |\ x(t)\ e - \sigma t|\ |\ dt < \infty$$

The range of values of σ for which Laplace transform converges is called region of convergence or ROC.

No poles lie in ROC.

ROC of the causal signal is right hand sided. It is of the form Re(s) > a.

ROC of the noncausal signal is left hand sided. It is of the form Re(s) < a.

The system is stable if its ROC includes $j\omega$ axis of s-plane.

Properties of Laplace Transform are Linearity, Time Shifting, Shifting in s- Domain, Time Scaling, Differentiation in Time Domain, Differentiation in s- Domain, Convolution in Time Domain, Integration in Time domain, and Integration in s- Domain.

The unilateral Laplace transform is given as ,

$$X(s) = \int_{0-}^{\infty} x(t)\, e^{-st}\, dt$$

The lower limit is taken as 0^- to indicate that initial conditions at t=0 are also considered.

Note that unilateral Laplace transform will be always convergent since ROC will be always R.H.S. of S-plane.

Questions:

1) Calculate Laplace transform of $e^{-at}\, u(t)$.
   [Ans. $\frac{1}{s+a}$, ROC: s > -a]
2) Calculate Laplace transform of $-e^{-at}\, u(t)$.
   [Ans. $\frac{1}{s+a}$, ROC: s > -a]
3) Calculate Laplace transform of $-e^{-3t}\, u(-t)$.
   [Ans. $-\frac{1}{s+3}$, ROC: s > -3]
4) Obtain Laplace transform of the following signals.
   i. $x(t) = \sin(3t)\, u(t)$
   ii. $x(t) = e^{-2t}\, u(t + 1)$

5) Find the Laplace transform of the following with ROC:

   i. $x(t) = u(t - 5)$
   ii. $x(t) = e^{5t}\, u(-t + 3)$

6) State and prove initial value theorem of Laplace transforms.

7) State and prove final value theorem of Laplace transforms.

8) Explain properties of Laplace Transform.

9) State and prove Convolution in Time Domain property.

10) State and prove Integration in Time domain property.

11) State and prove Integration in s- Domain property.

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Unit 3 :

Chapter 4 : Z-transform

Unit Structure

4.0 OBJECTIVE

By the end of this chapter, student will be able to understand Z-transform as a tool for the solution of linear constant difference equations. Also one can analyse discrete time systems in the frequency domain.

4.1 INTRODUCTION

Z-transform simplies signal analysis by reducing the number of poles and zeros to a finite number in z-plane. Z-transform has real and imaginary parts, whose plot is called Z-plane.

Z-transform maps(transforms) any point $s = \pm\sigma \pm jw$ in s-plane to a corresponding point $z(r|\theta)$ in the z-plane using the relationship :

$$z = e^{sT}, \quad \text{where T is the sampling period}$$

The poles and zeros of discrete time system are plotted in the complex z-plane.

Figure 4.1 shows Mapping of s-plane into z-plane for $z = e^{jwT}$

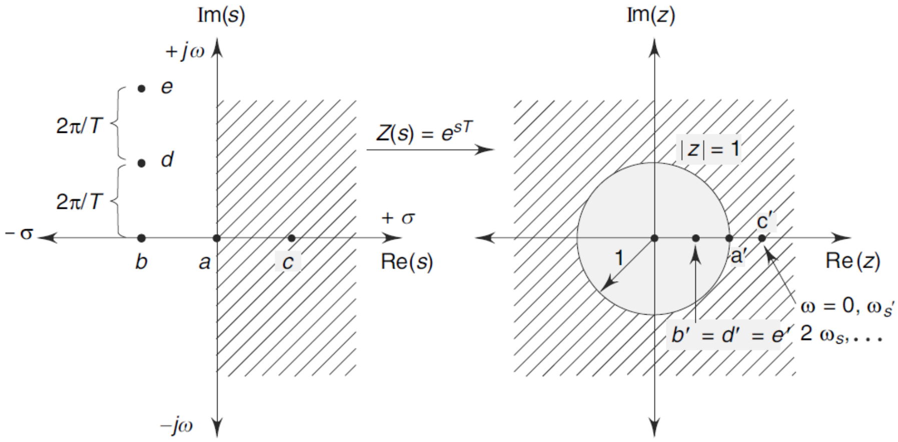


Fig 4.1 Mapping of s-plane into z-plane for $z = e^{jwT}$

The stability of the system can be checked using pole-zero plot. Also z-transform can be used to analyse discrete time systems for finding system transfer function and digital network realisation.

4.2 Definition of Z-transform

The z-transform of a discrete time signal $x(n)$ can be defined as :

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Where z is a complex variable. This equationis also called two sided z transform.

One sided z-transform is given as :

$$Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

Inverse Z-transform

Inverse z-transform is defined as :

$$x(n) = Z^{-1}[X(z)]$$

Inverse z-transform is applied to recover original time domain discrete signal from ite frequency domain signal.

Z-transform can be denoted as:

$$x(n) \overset{Z}{\leftrightarrow} X(z)$$

Or z-transform can also be denoted as:

$$X(z) = Z[x(n)]$$

4.2.1.1 Region of Convergence (ROC)

If the output signal magnitude of thedigital signal system, $x(n)$ is to be finite, then the magnitude of its z-transform must be finite. The Z values in the z-plane for which the magnitude of $X(z)$ is finite is called the **Region of Convergence (ROC).**

ROC for $X(z)$ is the area outside the unit circle in the z-plane.

Z-transform of the unit step $u(n)$ is $X(z) = \frac{z}{z-1}$ which has a zero at $z = 0$ and pole at $z = 1$ and the ROC is $|z| > 1$ extending to ∞ as shown in Fig 4.2

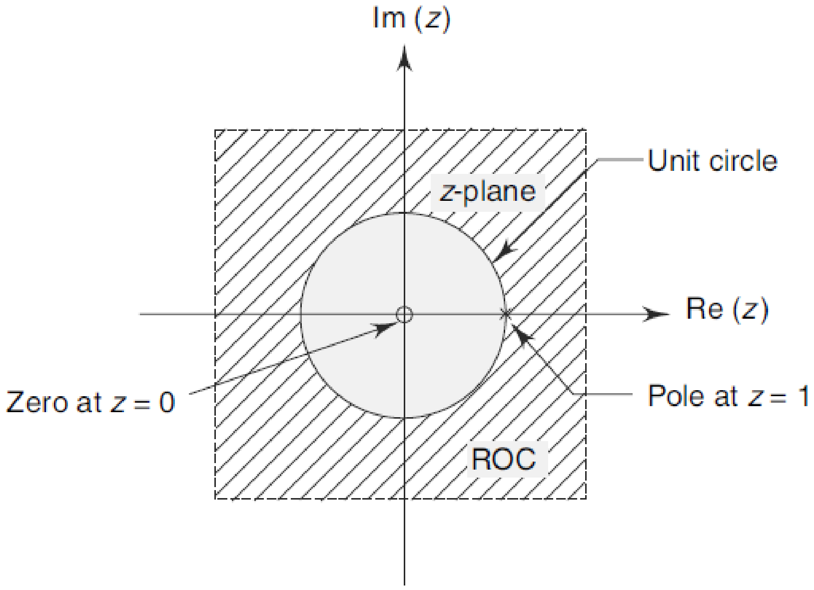


Fig 4.2 Pole-zero plot and ROC of the Unit-Step response $u(n)$

Properties of ROC :

1) ROC does not contain any poles
2) System stability can be checked with ROC
3) ROC also determines the types of sequence as
   a) Causal or Non-causal signal
   b) Finite or Infinite signal

Table 4.1) Finite Duration causal, anti-causal and two-sided signals with their ROCs.

Finite Duration Signals and their ROCs

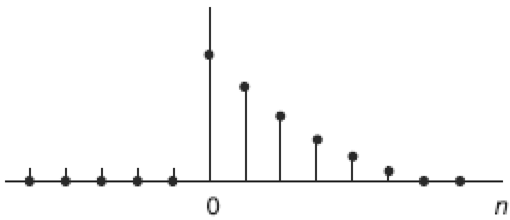
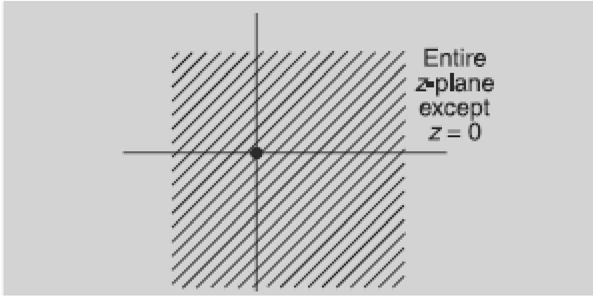
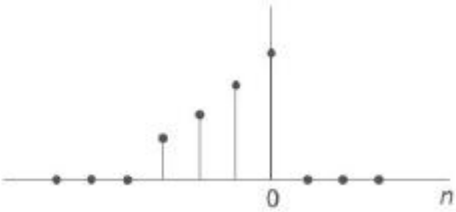
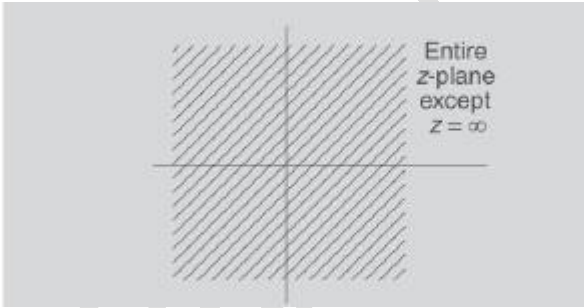
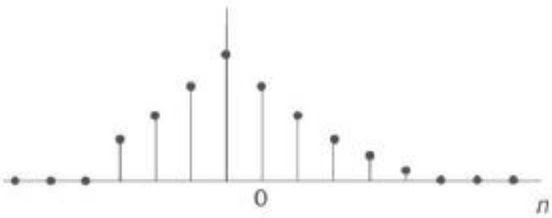
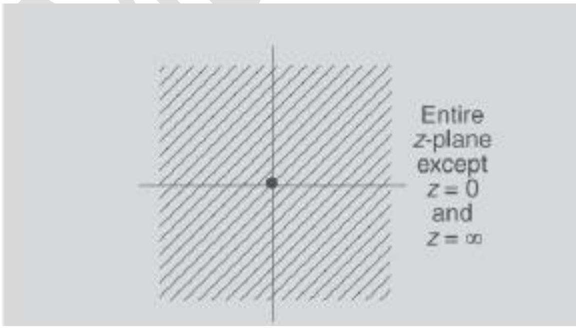
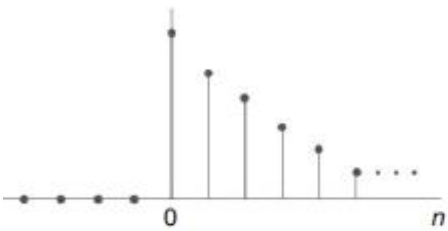
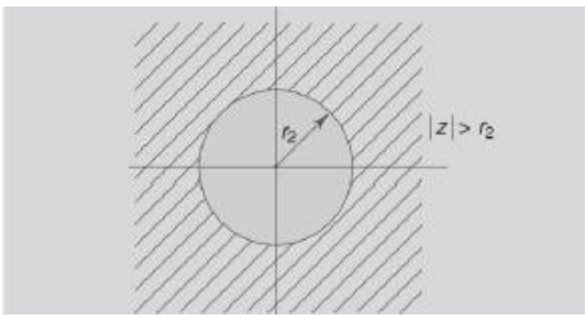
Causal 	
Anti-causal 	
Two-sided 	

Table 4.2) Infinte Duration causal, anticausal and two-sided signals with their ROCs

Infinite Duration Signals and their ROCs	
Causal 	

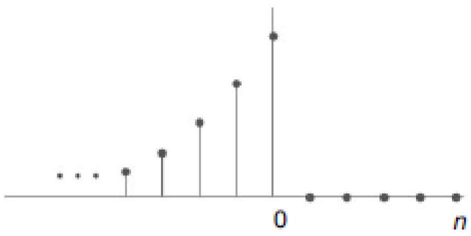
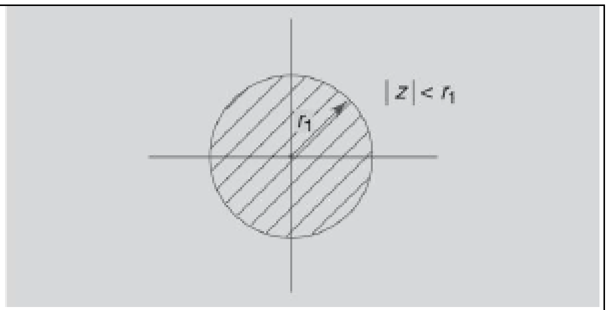
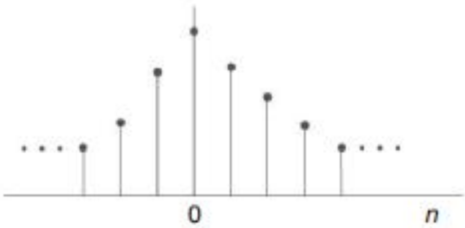
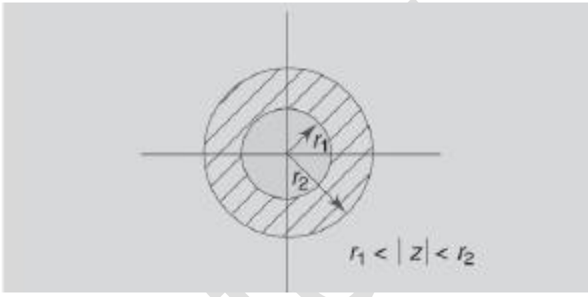
Anti-causal 	
Two-sided 	

Table 4.3) Some z-transform pairs

Sl	Signal $x(t)$	Sequence $x(n)$	Laplace Transform $X(s)$	z-transform $X(z)$	ROC
1.	$\delta(t)$	$\delta(n)$		1	All z-plane
2.	$\delta(t-k)$	$\delta(n-k)$	e^{-ks}	z^{-k}	$\lvert z\rvert > 0, \quad k > 0$ $\lvert z\rvert < \infty, \quad k < 0$
3.	$u(t)$	$u(n)$	$\frac{1}{s}$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\lvert z\rvert > 1$
4.		$-u(-n-1)$	$\frac{1}{s}$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\lvert z\rvert < 1$
5.	e^{-at}	e^{-an}	$\frac{1}{s+a}$	$\frac{1}{1-e^{-a}z^{-1}} = \frac{z}{z-e^{-a}}$	$\lvert z\rvert > \lvert e^{-a}\rvert$

Example 4.1) Determine z-transform of following finite duration signals.

(a) $x(n) = \{4, \underset{\uparrow}{2}, 3, 1, 3\}$

(b) $x(n) = \{4, 3, 1, 4, \underset{\uparrow}{4}, 0, 2\}$

(c) $x(n) = \delta(n)$

(d) $x(n) = \{4\,,\ 2\,,\ 3\,,\ 1\,,\ 3\}$

Solution :

(a) $x(n) = \{4\,,\ \underset{\uparrow}{2},\ 3\,,\ 1\,,\ 0\}$

Taking z-transform,

$$X(z) = 4z + 2 + 3z^{-1} + z^{-2} + 3z^{-3}$$

ROC entire plane except $z = 0$ and $z = \infty$

(b) $x(n) = \{4\,,\ 3\,,\ 1\,,\ 4\,,\ \underset{\uparrow}{4},\ 0\ \ 2\}$

Taking z-transform,

$$X(z) = 4z^4 + 3z^3 + z^2 + 4z^1 + 4 + 2z^{-2}$$

ROC entire plane except $z = 0$ and $z = \infty$

(c) $x(n) = \delta(n)$, hence $X(z) = 1,$ ROC : Entire z-plane

(d) $x(n) = \{4\,,\ 2\,,\ 3\,,\ 1\,,\ 3\}$,

since there is no bottom arrow, it is assumed to be below first element i.e. 4

$$X(z) = 4z^4 + 3z^3 + z^2 + 4z^1 + 4 + 2z^{-2}$$

ROC entire plane except $z = 0$ and $z = \infty$

4.3 Properties of Z-transform

Table 4.4 shows some of the important properites of z-transform

Table 4.4) Z-tranform properties

S. No.	Property or operation	Signal	z-transform
1.	Transformation	$x(n)$	$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$
2.	Inverse transformation	$\frac{1}{2\pi j} \int X(z) z^{n-1} dz$	$X(z)$
3.	Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$
4.	Time reversal	$x(-n)$	$X(z^{-1})$
5.	Time shifting	(i) $x(n-k)$ (ii) $x(n+k)$	(i) $z^{-k} X(z)$ (ii) $z^{k} X(z)$
6.	Convolution	$x_1(n) * x_2(n)$	$X_1(z) X_2(z)$
7.	Correlation	$r_{x_1 x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n) x_2(n-l)$	$R_{x_1 x_2}(z) = X_1(z) X_2(z^{-1})$
8.	Scaling	$a^n x(n)$	$X(a^{-1} z)$
9.	Differentiation	$nx(n)$	$z^{-1} \frac{dX(z)}{dz^{-1}}$ or $-z \frac{dX(z)}{dz}$
10.	Time differentiation	$x(n) - x(n-1)$	$X(z)(1 - z^{-1})$
11.	Time integration	$\sum_{k=0}^{\infty} X(k)$	$X(z) = \left(\frac{z}{z-1}\right)$
12.	Initial value theorem	$\lim_{n \to 0} x(n)$	$\lim_{\lvert z \rvert \to \infty} X(z)$
13.	Final value theorem	$\lim_{n \to \infty} x(n)$	$\lim_{\lvert z \rvert \to 1} \left(\frac{z-1}{z}\right) X(z)$

Example 4.2) Find z-tranform of the signal

$$x(n) = \delta(n+1) + 2\delta(n-1)$$

Solution : Taking z-transform of the given signal,

$$X(z) = Z[\delta(n+1) + 2\delta(n-1)]$$

Using linearity property,

$$X(z) = Z[\delta(n+1)] + 2Z[\delta(n-1)]$$

$$X(z) = z + 2z^{-1} = \{1, \underset{\uparrow}{0}, 1\}$$ **(Ans)**

Example 4.3) By applying time shifting property, determine the inverse z-transform of the signal

$$X(z) = \frac{z^{-1}}{1-2z^{-1}}$$

Solution : By applying time shifting propery, we have $k = 1$ and $x(n) = (2)^n u(n)$

Hence, $x(n) = (2)^{(n-1)}u(n-1)$ **(Ans)**

Example 4.4) Determine the convolution of two sequences

$$x(n) = \{2, 0, 1\} \;\; and \;\; h(n) = \{3, 2, 1, 2\}$$

Solution : Taking z-transorm of the two given signals $x(n)$ and $h(n)$,

$$X(z) = 2 + z^{-2} \quad \text{and} \quad \mathrm{H}(z) = 3 + 2z^{-1} + z^{-2} + 2z^{-3}$$

$$Y(z) = X(z)H(z) = (2 + z^{-2})(3 + 2z^{-1} + z^{-2} + 2z^{-3})$$

$$= (6 + 4z^{-1} + 2z^{-2} + 4z^{-3} + 3z^{-2} + 2z^{-3} + z^{-4} + 2z^{-5})$$

$$= (6 + 4z^{-1} + 5z^{-2} + 6z^{-3} + z^{-4} + 2z^{-5})$$

Taking Inverse z-transform,

$y(n) = \{\underset{\uparrow}{6}, \; 4, \; 5, \; 6, \; 1, \; 2\}$ **(Ans)**

Initial Value Theorem

If $x(n)$ is a causal sequence with z-transform $X(z)$, the initial value can be determined as :

$$x(0) = \lim_{n\to 0} x(n) = \lim_{|z|\to\infty} [X(z)]$$

Final Value Theorem

If $X(z) = Z[x(n)]$ and the poles of $X(z)$ are all inside the unit circle, then the final value of the sequence, $x(\infty)$ can be determined as :

$$\lim_{n\to\infty} x(\infty) = \lim_{|z|\to 1} x\,(1 - z^{-1})\,X(z)$$

if $x(\infty)$ exists

Example 4.5) Find Initial and final values of $x(n)$ for $X(z) = 1 + 2z^{-1} + 3z^{-2}$

Solution :

$$x(0) = \lim_{|z|\to\infty} [\,1 + 2z^{-1} + 3z^{-2}] = 1 + \frac{2}{\infty} + \frac{3}{\infty} = 1$$

$$x(\infty) = \lim_{|z|\to 1} [(1 - z^{-1})(\,1 + 2z^{-1} + 3z^{-2})]$$

$$= \lim_{|z|\to 1}[\,1 + 2z^{-1} + 3z^{-2} - z^{-1} - 2z^{-2} - 3z^{-3}]$$

$$= \lim_{|z|\to 1}[\,1 + z^{-1} + z^{-2} - 3z^{-3}]$$

$$= 1 + 1 + 1 - 3 = 0$$

(Ans)

4.4 Evaluation of Inverse Z-transform

Various methods are available to take Inverse Z-transform. We will use **Partial Fraction method** for taking Inverse z-transform.

Example 4.6) Find Inverse z of the following :

$$X(z) = \frac{-12}{1 + 4z^{-1}} - \frac{6}{1 + 3z^{-1}}$$

Solution : Taking Inverse z-transform,

$$x(n) = [-12(-4)^n - 6(-3)^n]u(n)$$ **(Ans)**

Example 4.7) Find the signal $x(n)$, whose z-transform is given as

$$X(z) = \frac{1}{(1 + z^{-1})(1 - z^{-1})}$$

Solution :

$$X(z) = \frac{1}{(1 + z^{-1})(1 - z^{-1})}$$

$$X(z) = \frac{a}{(1 + z^{-1})} + \frac{b}{(1 - z^{-1})}$$

Equating numerators,

$$1 = a(1 - z^{-1}) + b(1 + z^{-1})$$

$$1 = (a - az^{-1} + b + bz^{-1})$$

$$1 = a + b + bz^{-1} - az^{-1}$$

$$1 = (a + b) + (b - a)z^{-1}$$

Equating like terms,

$$(a+b)=1, \qquad (b-a)=0$$

Solving simultaneously,

$$a=\frac{1}{2}, \qquad b=\frac{1}{2}$$

$$X(z)=\frac{(\frac{1}{2})}{(1+z^{-1})}+\frac{(\frac{1}{2})}{(1-z^{-1})}$$

Taking Inverse z-transform,

$$x(n)=\left[\frac{1}{2}(-1)^n+\frac{1}{2}(1)^n\right]u(n)$$

Example 4.8) Find $x(n)$ for given $X(Z)$

$$X(z)=\frac{2+3z^{-1}}{1+\frac{5}{4}z^{-1}+\frac{1}{8}z^{-2}-\frac{1}{8}z^{-3}}$$

Solution : Writing denominator in terms of its factors,

$$X(z)=\frac{2+3z^{-1}}{(1+z^{-1})(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

$$X(z)=\frac{a}{(1+z^{-1})}+\frac{b}{(1+\frac{1}{2}z^{-1})}+\frac{c}{(1-\frac{1}{4}z^{-1})}$$

$$a=\left.\frac{2+3z^{-1}}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}\right|_{z^{-1}=-1}=-\frac{8}{5}$$

$$b=\left.\frac{2+3z^{-1}}{(1+z^{-1})(1-\frac{1}{4}z^{-1})}\right|_{z^{-1}=-2}=\frac{8}{3}$$

$$c=\left.\frac{2+3z^{-1}}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})}\right|_{z^{-1}=4}=\frac{14}{15}$$

$$X(z) = \frac{(-\frac{8}{5})}{(1+z^{-1})} + \frac{(\frac{8}{3})}{(1+\frac{1}{2}z^{-1})} + \frac{(\frac{14}{15})}{(1-\frac{1}{4}z^{-1})}$$

Taking Inverse z-transform,

$$x(n) = Z^{-1}[X(z)]$$

$$x(n) = \left[-\frac{8}{5}(-1)^n + \frac{8}{3}\left(-\frac{1}{2}\right)^n + \frac{14}{15}\left(\frac{1}{4}\right)^n\right] u(n)$$

(Ans)

Example 4.9) Determine inverse z-transform of $X(z) = \frac{z^2}{(z-a)^2}$ for ROC $|z| > |a|$

Solution : We will use Residue method.

$$X(z) = \frac{z^2}{(z-a)^2}$$

$$X(z)z^{n-1} = \frac{z^2}{(z-a)^2}\, z^{n-1} = \frac{z^{n+1}}{(z-a)^2}$$

We note that, the pole is at $z = a$ and order $m = 2$.

The Residue of $X(z)z^{n-1}$ at $z = a$ is calculated as :

$$\underset{z=a}{Res}[X(z)z^{n-1}] = \frac{1}{(2-1)!}\lim_{z\to p_i}\{\frac{d^{2-1}}{dz^{2-1}}(z-p_i)^2 X(z)z^{n-1}\}_{z=a}$$

$$= \{\frac{d}{dz}(z-a)^2\,\frac{z^{n+1}}{(z-a)^2}\}_{z=a}$$

$$= \{\frac{d}{dz}z^{n+1}\}_{z=a}$$

$$= (n+1)a^n$$

$$x(n) = \sum_{i=1} \underset{z=a}{Res} [X(z)z^{n-1}]$$

$$x(n) = (n+1)a^n \, u(n) \qquad \text{since ROC } |z| > |a|$$

(Ans)

4.5 SUMMARY

1) Z-transform is a powerful tool in digital signal analysis

2) Z-transform is used to analyze discrete-time systems for finding the transfer function, stability and network realization of system

3) $Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$, where $z = e^{sT}$, T being smapling period

4) The Z values in the z-plane for which the magnitude of $X(z)$ is finite is called the Region of Convergence (ROC).

5) The stability of the system can be determined by using the location of poles of $H(z)$

6) Some of the important properties of x-transform are linearity, time reversal, time shifting, time scling, differentiation, concoulution and correlation.

7) Inverse z-transform can be obtained by either of the methods : long division method, partial fraction method or residue method

4.6 EXERCISE

1) Define z-transform

2) Explain shift property of z-transform

3) Explain what is discrete convolution

4) Explain inverse z-transform

5) Determine z-trasnform of $u(n-5)$

6) Determine the inverse z-transform of $X(z) = \frac{1}{(z+2)^2}$, for $|z| < \frac{1}{2}$

7) Determine the causal signal $x(n)$ having z-transform $X(z) = \frac{z^2+z}{\left(z-\frac{1}{2}\right)^3\left(z-\frac{1}{4}\right)}$ for $|z| > \frac{1}{2}$

4.7 LIST OF REFERENCES

1) Digital Signal Processing, S Salivahanan, TMH

2) Digital Signal Processing, Sanjit Mitra, TMH

3) Signals and Systems, A Anand Kumar, PHI

4) Digital Signal Processing, Apte, Wiley India

Unit 4 :

Chapter 5 : Linear Time Invariant Systems

Unit Structure

5.0 OBJECTIVE

This chapter deals with Linear time invariant systems analysis using tool like z-transform. Important methods like convolution, response to discrete systems by using standrd digital inputs like impulse and unit step signals are discussed in detail. Concept of difference equation is introduced for signal analysis.

5.1 INTRODUCTION

A typical Digital System has input $x(n)$ and has an output $y(n)$ as a response to input. The sytem output depends on the input as well as on the system parameters. Some of the inputs used to study digital systems are **Impulse** and **Unit Step Signals**, among others. Similarly the Systems can also be classified as **Liner Time Variant, Time Invariant** and others.

5.2 PROPERTIES OF DSP SYSTEMS

Various system properties as discussed in this chapter are linearity, time invariance, causality and stability.

Linearity

$$F[a_1x_1(n) \pm a_2x_2(n)] = a_1F[x_1(n)] \pm a_2F[x_2(n)] = a_1y_1(n) \pm a_2y_2(n)$$

Where F is an operator

Example 5.1) Check if the system $\boldsymbol{F[x(n)] = 3\,n\,x(n) + 4}$ is linear :

Solution :

$$F[x_1(n) + x_2(n)] = 3\,n\,[x_1(n) + x_2(n)] + 4$$

$$F[x_1(n)] + F[x_2(n)] = [3\,n\,x_1(n) + 4] + [3\,n\,x_2(n) + 4]$$

Since, $F[x_1(n) + x_2(n)] \neq F[x_1(n)] + F[x_2(n)]$, the systemin non-linear.

Time Invariance

A system is said to be time-invariant if the relationship between the input and output does not change with time.

If $y(n) = F[x(n)]$, then $y(n-k) = F[x(n-k)] = z^{-k}F[x(n)]$

z^{-k} represents a single delay of k samples.

Example 5.2) Check if system $y(n) = a\,n\,x(n)$ is time invariant or not

Solution :

$$F[x(n-k)] = a\,n\,x(n-k)$$

The delayed repsonse is :

$$y(n-k) = a\,(n-k)\,[x(n-k)]$$

Since, $F[x(n-k)] \neq y(n-k)$, the system is not time invariant. **(Ans)**

Causality

The systems in which changes in the output are only dependent on the changes in the present and past values of the input and/or previous output values, and are not dependent on future input values are called Causal Sytems.

The Causality condition for linear time invariant systems is given as :

$$h(n) = 0 \quad for\; n < 0$$

Example 5.3) Check if system $y(n) = x(n-2) + x(n-3)$ is causal or not.

Solution :

In this system, the output is computed only on past sample values i.e. $x(n-2)$ and $x(n-3)$, the system is causal.

Example 5.4) Check if system $y(n) = x(n-2) + x(n+3)$ is causal or not.

Solution :

In this system, the output is computed on past sample values i.e. $x(n-2)$ and also future values, i.e. $x(n+3)$, the system is non-causal.

<u>Stability</u>

A DSP is said to be stable, if system poles are given as :

$$|p_i| < 1 \text{ i.e. } \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

The pole-zero plot in Fig. 5.1 shows pole position of stable and unstable systems.

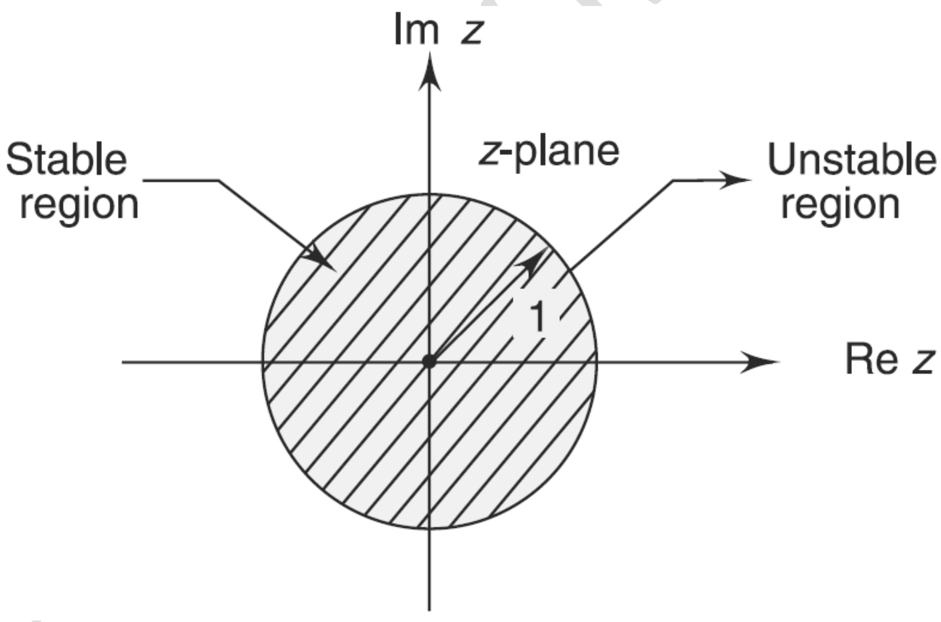


Fig 5.1 Z-plane regions for stable and unstable systems

Example 5.5) Check stability of the system $H(z) = \frac{z^2 - z + 1}{z^2 - z + \frac{1}{2}}$

Solution :

$$H(z) = \frac{z^2 - z + 1}{z^2 - z + \frac{1}{2}} = \frac{z^2 - z + 1}{(z + \frac{1}{2} + j\frac{1}{2})(z + \frac{1}{2} - j\frac{1}{2})}$$

Since, $\left|\frac{1}{2} \pm j\frac{1}{2}\right| < 1$, the given system is stable. **(Ans)**

Bounded Input – Bunded Output (BIBO) stability :

A system is said to BIBO stable, if and only if every bounded input gives bounded output.

The impulse response of the system decides the BIBO stability of the linear time invariant system.

The necessary and sufficient condition for the BIBO stability is :

$$\sum_{k=0}^{\infty} |h(k)| < \infty$$

Example 5.6) Check BIBO stability of system $y(n) = 3x(n) + 5$

Solution :

$$y(n) = 3x(n) + 5$$

If $x(n) = \delta(n)$, then $y(n) = h(n)$

Hence, impulse response is $h(n) = 3\delta(n) + 5$

When, $n = 0,\ h(0) = 3\delta(0) + 5 = 3 + 5 = 8$

When, $n = 1,\ h(1) = 3\delta(1) + 5 = 0 + 5 = 5$

so, $h(1) = h(2) = \ldots = h(k) = 5$

therefore,

$$h(n) = 8, \quad when\, n = 0$$

$$h(n) = 5, \quad when\, n \neq 0$$

The necessary and sufficient condition for BIBO stability is :

$$\sum_{k=0}^{\infty} |h(k)| < \infty$$

therefore, $\sum_{k=0}^{\infty} |h(k)| = |h(0)| + |h(a)| + \ldots + |h(k)|$

$= 8 + 5 + 5 + \ldots + 5 + \ldots$

This is diverging series, hence the given system is BIBO unstable. **(Ans)**

5.3 DISCRETE CONVOLUTION

Convoluting two signals in time domain is same as muliplying two signals in frequency domain. Convolution is useful in studying analysing input signal response to the given system. The concolution of the two signals is given as :

$$y(n) = x(n) * h(n)$$

$$y(n) = \sum_{k=0}^{\infty} x(k)\, h(n-k)$$

<u>Properties of Convolution :</u>

a) Commutative law :

$$x(n) * h(n) = h(n) * x(n)$$

b) Associative law :

$$\text{For} \quad y_1(n) = x(n) * h_1(n)$$

$$\text{and} \quad y(n) = y_1(n) * h_2(n)$$

$$y(n) = [x(n) * h_1(n)] * h_2(n)$$

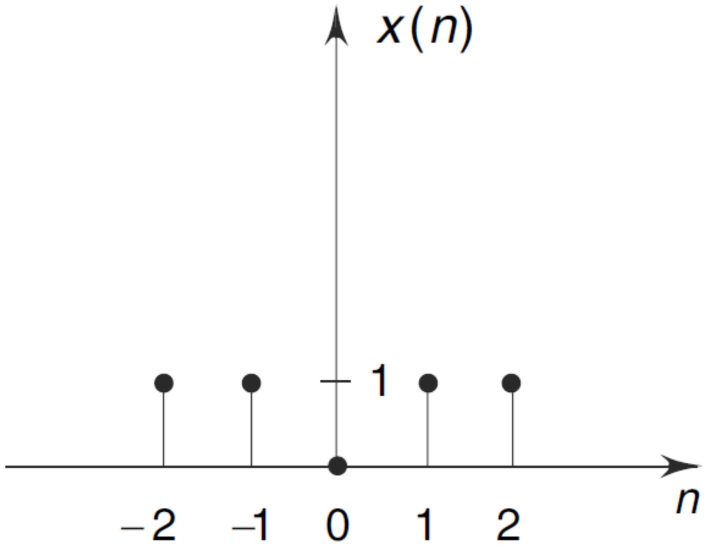
$$y(n) = x(n) * [h_1(n) * h_2(n)]$$

c) Distributive law :

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

Example 5.7) Plot the signal given by sequence $\{\frac{1}{},\ \frac{1}{},\ \underset{\uparrow}{\frac{0}{}},\ \frac{1}{},\ \frac{1}{}\}$

Solution :



5.4 SOLUTION OF LINEAR CONSTANT COEFFICIENT DIFFERENCE EQUATION

A discrete time system transforms an input sequence into an output sequence according to the recursion formula that represents the solution of a difference equation.

The general formof the difference equation is :

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k)$$

Where N is the order of the differenc equation.

The solution of the dfference equation has two parts :

$$y(n) = y_h(n) + y_p(n)$$

Where, $y_h(n)$ is the solution to the homogeneous difference equation and $y_p(n)$ represents the particular solution to the difference equation.

Table 5.1) Particular solution of several types of inputs

Input Signal $x(n)$	Particular Solution $y_p(n)$
A (Step input)	K
AM^n	KM^n
An^M	$K_0 n^M + K_1 n^{M-1} \cdots + K_M$
$A^n n^M$	$A^n(K_0 n^M + K_1 n^{M-1} \cdots + K_M)$
$\left.\begin{matrix} A\cos\omega_0 n \\ A\sin\omega_0 n \end{matrix}\right\}$	$K_1 \cos \omega_0 n + K_2 \sin \omega_0 n$

5.5 FREQUENCY DOMAIN REPRESENTATION OF DISCRETE TIME SIGNALS AND SYSTEMS

In a discrete time invariant system, if the input is of the form $e^{i\omega t}$, the output is $H(w)(\,e^{i\omega t})$. $H(\,e^{i\omega t})$ is a function of w, which denotes the frequency response of the system.

$$H\left(\,e^{i\omega t}\right) = H_r\left(\,e^{i\omega t}\right) + jH_i\left(\,e^{i\omega t}\right) \text{ or}$$

$$H\left(\,e^{i\omega t}\right) = |H_r\left(\,e^{i\omega t}\right)|e^{j\emptyset}, \text{ where } \emptyset = tan^{-1}\frac{H_i(\,e^{i\omega t})}{H_r(\,e^{i\omega t})}$$

The input-output relation is :

$$y(n) = H\left(\,e^{i\omega}\right)e^{i\omega n}$$

Example 5.8) Find transfer function of the system :

$$y(n) = -2y(n-1) - 3y(n-2) + x(n) + x(n-1)$$

Solution :

The given difference equation can be written as :

$$y(n) + 2y(n-1) + 3y(n-2) = x(n) + x(n-1)$$

The system transfer function can be written as :

$$H\left(e^{j\omega}\right) = \frac{1 + e^{-j\omega}}{1 + 2e^{-j\omega} + 3e^{-j2\omega}}$$

(Ans)

5.6 DIFFERENCE EQUATION AND ITS RELATIONCHIP WITH SYSTEM FUNCTION, IMPULSE RESPONSE AND FREQUENCY RESPONSE

A causal LIT system is defined by a linear constant coefficient difference equation :

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)$$

The system function $H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$

<u>IIR Systems – Infinite Impulse Response Systems</u>

An LTI system is said to be an Infinite Impulse Response (IIR) system if its unit sample response $h(n)$ is of infinte duration. Recursive filter having feedback has an impulse response that is theoretically continues for ever.

<u>FIR Systems – Finite Impulse Response Systems</u>

An LTI system is said to be a finite impulse response (FIR) system if its unit sample response $h(n)$ is of finite duration. Non-recursive filters can be FIR systems.

Example 5.9) A DSP is given as a difference equation :

$$y(n) = 0.2\, x(n) - 0.5\, x(n-2) + 0.4\, x(n-3)$$

Digital input sequence $\{-1, 1, 0, -1\}$ is applied to this DSP. Find output response.

Solution :

$$y(n) = 0.2\, x(n) - 0.5\, x(n-2) + 0.4\, x(n-3)$$

Taking z-transform of the given difference equation,

$$Y(z) = 0.2\, X(z) - 0.5\, z^{-2}X(z) + 0.4\, z^{-3}X(z)$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)} = 0.2 - 0.5\, z^{-2} + 0.4\, z^{-3}$$

Input sequence is, $x(n) = \{-1, 1, 0, -1\}$, and its z-transform is

$$X(z) = -1 + z^{-1} - z^{-3}$$

Since, $Y(Z) = H(z).X(z)$

$$Y(z) = \{-0.2 + 0.2\, z^{-1} + 0.5\, z^{-2} - 1.1\, z^{-3} + 0.4\, z^{-4} + 0.5\, z^{-5} - 0.4\, z^{-6}$$

Taking Inverse transform,

$$y(n) = \{-0.2, 0.2, 0.5, -1.1, 0.4, 0.5, -0.4\}$$

(Ans)

Example 5.9) Determine the impulse response of the systems described by the difference equation :

$$y(n) = 0.7y(n-1) - 0.1y(n-2) + 2x(n) - x(n-2)$$

Solution :

The given difference equation is :

$$y(n) = 0.7y(n-1) - 0.1y(n-2) + 2x(n) - x(n-2)$$

This equation canbe written as :

$$y(n) - 0.7y(n-1) + 0.1y(n-2) = 2x(n) - x(n-2)$$

Input impulse is give as $x(n) = \delta(n)$.

Hence, $y(n) = u(n)$

Taking z-transform,

$$H(z)[1 - 0.7z^{-1} + 0.1z^{-2} = (2 - z^{-2})$$

$$\frac{H(z)}{z} = \frac{2z^2 - 1}{z(z-0.5)(z-0.2)}$$

$$\frac{H(z)}{z} = \frac{A_1}{z} + \frac{A_2}{(z-0.5)} + \frac{A_3}{(z-0.2)}$$

$$\frac{H(z)}{z} = \frac{10}{z} - \frac{10}{3}\frac{1}{(z-0.5)} + \frac{46}{3}\frac{1}{(z-0.2)}$$

$$H(z) = 10 - \frac{10}{3}\frac{z}{(z-0.5)} + \frac{46}{3}\frac{z}{(z-0.2)}$$

Taking inverse z-transform,

$$h(n) = -10\,\delta(n) - \frac{10}{3}(0.5)^n\,u(n) + \frac{46}{3}(0.2)^n\,u(n)$$

(Ans)

Example 5.10) Determine the unit step response of the systems described by the difference equation :

$$y(n) = 0.6y(n-1) - 0.08y(n-2) + x(n)$$

Solution :

The given difference quation of the system is :

$$y(n) = 0.6y(n-1) - 0.08y(n-2) + x(n)$$

This can be written as :

$$y(n) - 0.6y(n-1) + 0.08y(n-2) = x(n)$$

Taking z-transform,

$$Y(z)[1 - 0.6z^{-1} + 0.08z^{-2}] = \frac{1}{1 - z^{-1}}$$

$$Y(z) = \frac{1}{(1 - z^{-1})[1 - 0.6z^{-1} + 0.08z^{-2}]}$$

$$Y(z) = \frac{1}{(1 - z^{-1})(1 - 0.4z^{-1})(1 - 0.2z^{-1})}$$

$$Y(z) = \frac{A_1}{(1 - z^{-1})} + \frac{A_2}{(1 - 0.4z^{-1})} + \frac{A_3}{(1 - 0.2z^{-1})}$$

$$Y(z) = \frac{25}{12}\frac{1}{(1 - z^{-1})} - \frac{4}{3}\frac{1}{(1 - 0.4z^{-1})} + \frac{1}{4}\frac{1}{(1 - 0.2z^{-1})}$$

Taking inverse z-transform,

$$s(n) = y(n) = \frac{25}{12}\,u(n) - \frac{4}{3}(0.4)^n\,u(n) + \frac{1}{4}(0.2)^n\,u(n)$$

(Ans)

5.7 FREQUENCY RESPONSE

Frquency response describes the magnitude and phase shift over a range of frequencies.

Properties of Frequency Response

Properties of frequency response of a real sequence $h(n)$ are given as :

a) $H(e^{j\omega})$ takes on values for all ω

b) $H(e^{j\omega})$ is periodic in ω with period 2π

c) The magnitude response $|H(e^{j\omega})|$ is an even function of ω and symmetric about π

d) The magnitude response $|H(e^{j\omega})|$ is an odd function of ω and antisymmetric about π

Frequency Response of an inter connection of Systems

Parallel connection:

When there are L number of linear time invariant systems in time domain connected in parallel, the impilse response $h(n)$ of the resultant sytem is given as :

$$h(n) = \sum_{k=1}^{L} h_k(n)$$

Using, linearity property of z-transform, the frequency response of the complete system is :

$$H(z) = \sum_{k=1}^{L} H_k(z) = H_1(z) + H_2(z) + \ldots + H_L(z)$$

Where $z = e^{j\omega}$

Parallel interconnection of linear discrete time signal is shown in figure 5.2

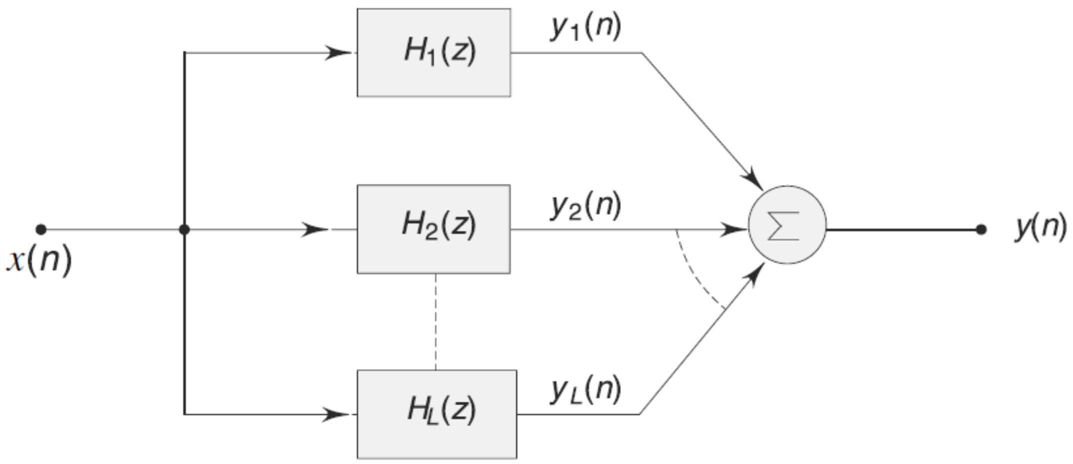


Figure 5.2 Parallel interconnection of linear discrete time systems

Cascade connection :

The impulse respopnse of L linear time invriant systems connected in cascade is given as :

$$h(n) = h_1(n) * h_2(n) * \ldots * h_L(n)$$

Using Convolution property, z-transform is obtained as :

$$H(z) = H_1(z)H_2(z) \ldots H_L(z)$$

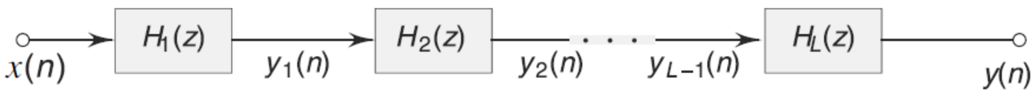


Figure 5.3 Cascade interconnection of linear discrete time systems

5.8 SUMMARY

1) Linear time invariant systems have properties like linearity, time-invariance and causality which can be used in system analysis

2) A DSP is said to be stable, if system poles are given as :

$$|p_i| < 1 \text{ i.e. } \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

3) A system is said to BIBO stable, if and only if every bounded input gives bounded output.

4) Convolution is useful in studying analysing input signal response to the given system.

5) A discrete time system transforms an input sequence into an output sequence according to the recursion formula that represents the solution of a difference equation.

6) Frquency response describes the magnitude and phase shift over a range of frequencies.

5.9 EXERCISE

1) Explain what do you understand by discrete tim invariant systems

2) Explain conditions of causality and stability of a linear time invariant systems

3) What is BIBO stability?

4) Write a note on system transfer function

5) Find the stability region of the causal system

$$H(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}$$

6) Write a note on discrete convolution

7) Check if the system $y(n) = x(-n)$ is causal or not

8) Explain how can the linearity of a discrete system be found out?

5.10 LIST OF REFERENCES

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UNIT V

Discrete and Fast Fourier Transforms

5.0 Introduction

The DFT is important because it is the mathematical relation that is implemented by the various Fast Fourier Transform (FFT) algorithms. In this section we will discuss

- Relationships between periodic and finite-duration time functions
- The discrete Fourier series (DFS) for periodic time functions
- The discrete Fourier transform (DFT) for finite-duration time functions

The discrete Fourier series (DFS) is used to represent periodic time functions and the DFT is used to represent finite-duration time functions. The two representations are virtually identical mathematically, and they are closely related because a finite-duration time function can be thought of as a single period of a periodic time function. Conversely, a periodic time function can be easily constructed from a finite-duration time function simply by repeating the finite-duration sequence over and over again, *ad infinitum.*

Let's adopt the notation used by OSYP to distinguish these functions: let $\tilde{x}[n]$ represent a periodic discrete-time sequence with period N, and let $x[n]$(without the tilde) represent a finite-duration sequence that is nonzero for $0 <= n <= N-1$. We than note (trivially) that

$$x[n] = \tilde{x}[n], 0 \le n \le N-1 \text{ and}$$
$$\tilde{x}[n] = x(n \bmod N) \equiv x[((n))_N]$$

5.1 Discrete Fourier series

- Definitions of the DFS

  If a time function $\tilde{x}[n]$ is periodic has period N, we can write

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{jk\frac{2\pi}{N}n} \quad \text{and}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-jk\frac{2\pi}{N}n}$$

There are only N unique frequency components because for integer k, n, and r.

$$e^{j2\pi kn/N} = e^{j2\pi(k+rN)n/N} = e^{j2\pi kn/N} e^{j2\pi rnN/N} = e^{j2\pi kn/N} \quad (1)$$

Comments:

1. Both the time function $\tilde{x}[n]$ and the Fourier series coefficients $\tilde{X}[k]$ are periodic with period N, so they are represented by only N distinct (possibly complex) numbers.

2. The series can be evaluated over any consecutive of values of n or k of length N.

3. The frequency components of the DFS are the complex exponentials $e^{j\frac{2\pi}{N}kn}$, where the frequency $\frac{2\pi}{N}$ can be considered to be the "fundamental frequency" of the periodic waveform and all other frequencies are integer multiples of it.

4. The frequency components are N equally-spaced samples of the frequencies of the DTFT, or alternatively they represent N equally-spaced locations around the unit circle of the z-plane.

We commonly use the notational shorthand $W_N = e^{-j2\pi/N}$, so the DFS equations can be rewritten as

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-nk} \text{ and}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{nk}$$

Properties of the DFS

In general, the properties of the DFS are very similar to what we would expect from the DTFT, except that the functions considered are periodic.

1. **Linearity**: $a\tilde{x}_1[n] + b\tilde{x}_2[n] \Leftrightarrow a\tilde{X}_1[n] + b\tilde{X}_2[n]$ provided that the period of the two time functions is the same.

2. **Time shift:** $\tilde{x}[n-m] \Leftrightarrow W_N^{km}\tilde{X}[k]$

3. **Multiplication by a complex exponential:** $\tilde{x}[n]W_N^{-rn} \Leftrightarrow \tilde{X}[k-r]$

Note that in the previous two properties, shift in one domain corresponds to multiplication in the other by a complex exponential, both of which may be easier to evaluate when you apply the definition for W_N.

4. **Multiplication-convolution:** $\tilde{x}_1[n]\tilde{x}_2[n] \Leftrightarrow \frac{1}{N}\sum_{r=0}^{N-1} \tilde{X}_1[r]\tilde{X}_2[k-r]$

The sum on the right hand side is very important for our work and is referred to as *periodic convolution* or *circular convolution.* The operator for circular convolution is normally written as an asterisk with a circle around it, possibly with an accompanying number that indicates the size of the convolution (which matters).

5.2 Discrete time Fourier transform (DTFT)

The discrete-time Fourier transform has essentially the same properties as the continuous-time Fourier transform, and these properties play parallel roles in continuous time and discrete time. As with the continuous-time Fourier transform, the discrete-time Fourier transform is a complex-valued function whether or not the sequence is real-valued. Furthermore, as we stressed in Lecture 10, the discrete-time Fourier transform is always a periodic function of fl. If x(n) is real, then the Fourier transform is conjugate symmetric, which implies that the real part and the magnitude are both even functions and the imaginary part and phase are both odd functions. Thus for real-valued signals the Fourier transform need only be specified for positive frequencies because of the conjugate symmetry. Whether or not a sequence is real, specification of the Fourier transform over a frequency range of 2π specifies it entirely. For a real-valued sequence, specification over the frequency range from, for example, 0 to π is sufficient because of conjugate symmetry.

The time-shifting property together with the linearity property plays a key role in using the Fourier transform to determine the response of systems characterized by linear constant-coefficient difference equations. As with continuous time, the convolution property and the modulation property are of particular significance. As a consequence of the convolution property, which states that the Fourier transform of the convolution of two sequences is the product of their Fourier transforms, a linear, time-it variant system is represented in the frequency domain by its frequency response. This representation corresponds to the scale factors applied at each frequency to the Fourier transform of the input. Once again, the convolution property can be thought of as a direct consequence of the fact that the Fourier transform decomposes a signal into a linear combination of complex exponentials each of which is an Eigen function of a linear, time-invariant system. The frequency response then corresponds to the eigenvalues. The concept of filtering for discrete-time signals is a direct consequence of the convolution property. The modulation property in discrete time is also very similar to that in continuous time, the principal analytical difference being that in discrete time the Fourier transform of a product of sequences is the periodic convolution.

DISCRETE-TIME FOURIER TRANSFORM

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega \qquad \text{synthesis}$$

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\Omega n} \qquad \text{analysis}$$

$$x[n] \overset{\mathfrak{F}}{\longleftrightarrow} X(\Omega)$$

$$X(\Omega) = Re\{X(\Omega)\} + j\,Im\{X(\Omega)\}$$

$$= |X(\Omega)|\, e^{j\measuredangle X(\Omega)}$$

Figure 5.1: **DTFT**

5.3 Fast Fourier transforms (FFT)

The time taken to evaluate a DFT on a digital computer depends principally on the number of multiplications involved, since these are the slowest operations. With the DFT, this number is directly related to V (matrix multiplication of a vector), where is the length of the transform. For most problems, is chosen to be at least 256 in order to get a reasonable approximation for the spectrum of the sequence under consideration – hence computational speed becomes a major consideration. Highly efficient computer algorithms for estimating Discrete Fourier Transforms have been developed since the mid-60's. These are known as Fast Fourier Transform (FFT) algorithms and they rely on the fact that the standard DFT involves a lot of redundant calculations.

Re-writing $F[n] = \sum_{k=0}^{N-1} f[k] e^{-j\frac{2\pi}{N}nk}$ as $F[n] = \sum_{k=0}^{N-1} f[k] W_N^{nk}$

It is easy to realise that the same values of W_N^{nk} : are calculated many times as the computation proceeds. Firstly, the integer product *nk* repeats for different combinations of n and k; secondly, W_N^{nk} is a periodic function with N only distinct values.

5.4 Inverse DFT

The inverse transform of

$$F[n] = \sum_{k=0}^{N-1} f[k] e^{-j\frac{2\pi}{N}nk}$$

Is given as:

$$f[k] = \frac{1}{N} \sum_{n=0}^{N-1} F[n] e^{+j\frac{2\pi}{N}nk}$$

i.e. the inverse matrix is 1/N : times the complex conjugate of the original (symmetric) matrix.

From the inverse transform formula, the contribution to $f[k]$ of $F[n]$ and $F[N-n]$ is:

$$f_n[k] = \frac{1}{N}\{F[n]e^{j\frac{2\pi}{N}nk} + F[N-n]e^{j\frac{2\pi}{N}(N-n)k}\} \qquad (7.2)$$

For all $f[k]$ real, $\quad F[N-n] = \sum_{k=0}^{N-1} f[k] e^{-j\frac{2\pi}{N}(N-n)k}$

But $\quad e^{-j\frac{2\pi}{N}(N-n)k} = \underbrace{e^{-j2\pi k}}_{1 \text{ for all } k} \; e^{+j\frac{2\pi n}{N}k} = e^{+j\frac{2\pi}{N}nk}$

i.e. $F[N-n] = F^*(n) \qquad$ (i.e. the complex conjugate)

5.5 Composite radix FFT

It is not always possible to work with sequences whose length is a power of 2. However, efficient computation of the DFT is still possible if the sequence length may be written as a product of factors. For example, suppose that N may be factored as follows: N=N1.N2.

We then decompose x(n) into N2 sequences of length NI and arrange these sequences in an array as follows:

$$\mathbf{x} = \begin{bmatrix} x(0) & x(N_2) & \cdots & x(N_2(N_1-1)) \\ x(1) & x(N_2+1) & \cdots & x(N_2(N_1-1)+1) \\ \vdots & \vdots & & \vdots \\ x(N_2-1) & x(2N_2-1) & \cdots & x(N_1N_2-1) \end{bmatrix}$$

5.6 Fast (Sectioned) convolution

5.6.1 Fast Circular Convolution

Since

$$\sum_{m=0}^{N-1} (x(m)(h(n-m))\, modN) = y(n) \text{ is equivalent to} Y(k) = X(k)H(k)$$

$$y(n) \text{ can be computed as } y(n) = \text{IDFT}[\text{DFT}[x(n)]\,\text{DFT}[h(n)]]$$

Cost

- Direct
  - N2 complex multiplies.
  - N (N - 1) complex adds.
- **Via FFTs**
  - 3 FFTs $+\ N$ multipies.
  - $N + \frac{3N}{2}\log_2 N$ complex multiplies.
  - $3(N\log_2 N)$ complex adds.

If $H(k)$ can be precomputed, cost is only 2 FFts $+\ N$ multiplies.

5.6.2 Fast Linear Convolution

For linear convolution, we must zero-pad sequences so that circular wrap-around always wraps over zeros.

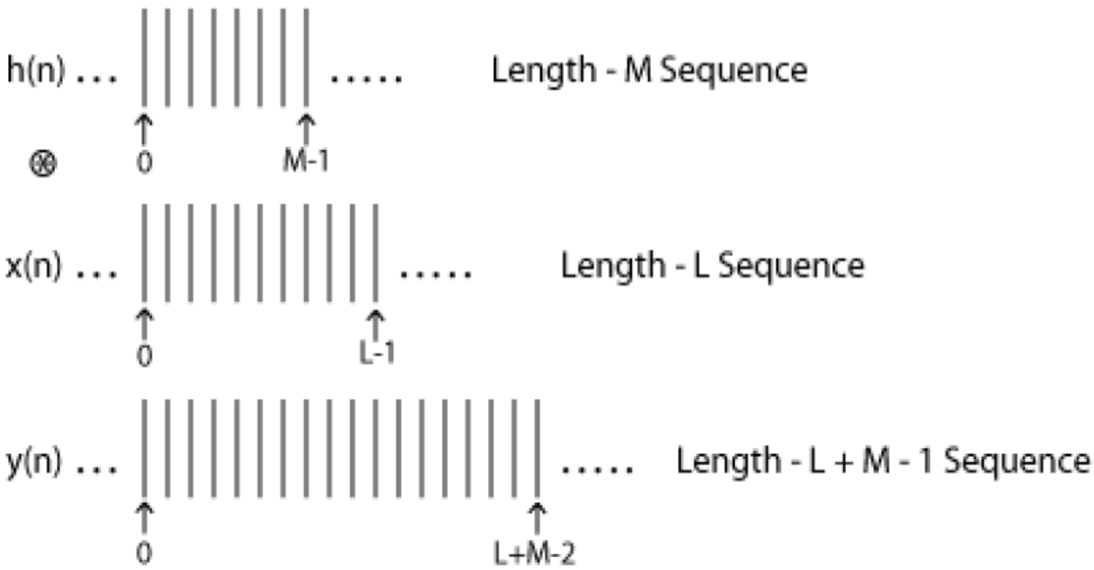

Figure 5.2: Fast Linear Convolution

To achieve linear convolution using fast circular convolution, we must use zero-padded DFTs of length $N \geq L + M - 1$.

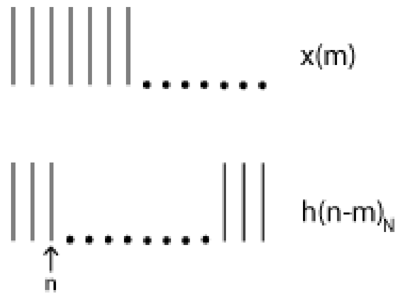

Figure 5.3

Choose shortest convenient N (usually smallest power-of-two greater than or equal to L +M - 1).

$$y(n) = \mathrm{IDFT}_N\left[\mathrm{DFT}_N[x(n)]\,\mathrm{DFT}_N[h(n)]\right]$$

5.6.3 Running Convolution

Suppose L = ∞, as in a real time filter application, or (L >> M). There are efficient block methods for computing fast convolution.

5.6.3.1 Overlap-Save (OLS) Method

Note that if a length-M filter h (n) is circularly convolved with a length-N segment of a signal x (n),

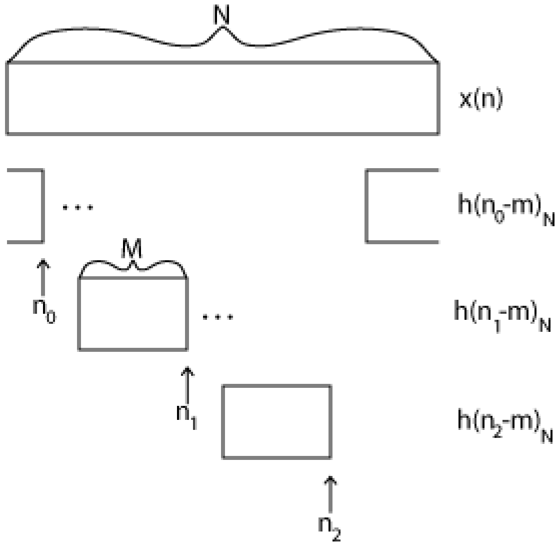

Figure 5.4: Overlap –Save method

The Overlap-Save Method: Break long signal into successive blocks of N samples, each block overlapping the previous block by M-1 samples. Perform circular convolution of each block with filter h (m). Discard first M - 1 point in each output block, and concatenate the remaining points to create y (n).

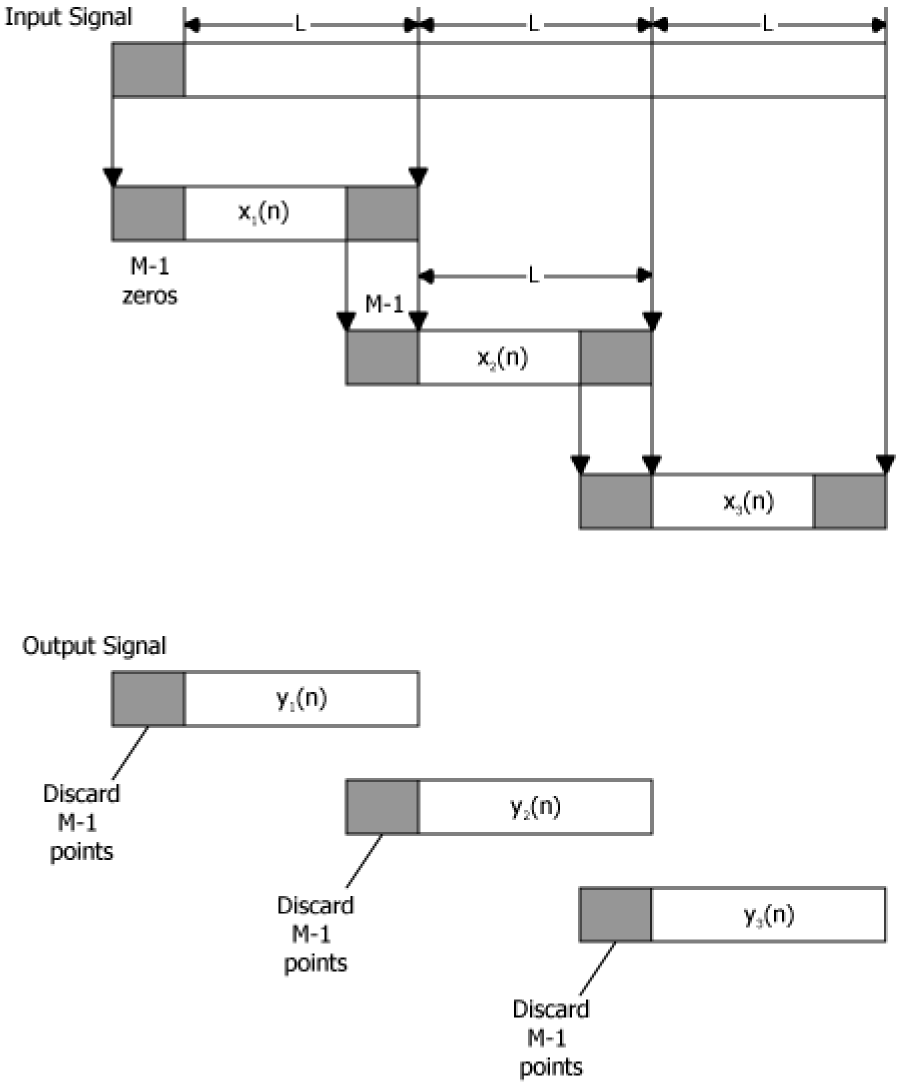

Figure 5.5

5.7 Correlation

The concept of correlation can best be presented with an example. Figure 5.6 shows the key elements of a radar system. A specially designed antenna transmits a short burst of radio wave energy in a selected direction. If the propagating wave strikes an object, such as the helicopter in this illustration, a small fraction of the energy is reflected back toward a radio receiver located near the transmitter. The transmitted pulse is a specific shape that we have selected, such as the triangle shown in this example. The received signal will consist of two parts: (1) a shifted and scaled version of the transmitted pulse, and (2) random noise, resulting from interfering radio waves, thermal noise in the electronics, etc. Since radio signals travel at a known rate, the speed of light, the shift between the transmitted and received pulse is a direct measure of the distance to the object being detected. This is the problem: given a signal of some known shape, what is the best way to determine where (or if) the signal occurs in another signal. Correlation is the answer. Correlation is a mathematical operation that is very similar to convolution. Just as with convolution, correlation uses two signals to produce a third signal. This third signal is called the cross-correlation of the two input signals. If a signal is correlated with itself, the resulting signal is instead called the autocorrelation. The convolution machine was presented in the last chapter to show how convolution is performed. Figure 5.6 is a similar illustration of a correlation machine. The received signal, x[n], and the cross-correlation signal, y[n], are fixed on the page. The waveform we are looking for, t[n], commonly called the target signal, is contained within the correlation machine. Each sample in y[n] is calculated by moving the correlation machine left or right until it points to the sample being worked on. Next, the indicated samples from the received signal fall into the correlation machine, and are multiplied by the corresponding points in the target signal. The sum of these products then moves into the proper sample in the cross correlation signal.

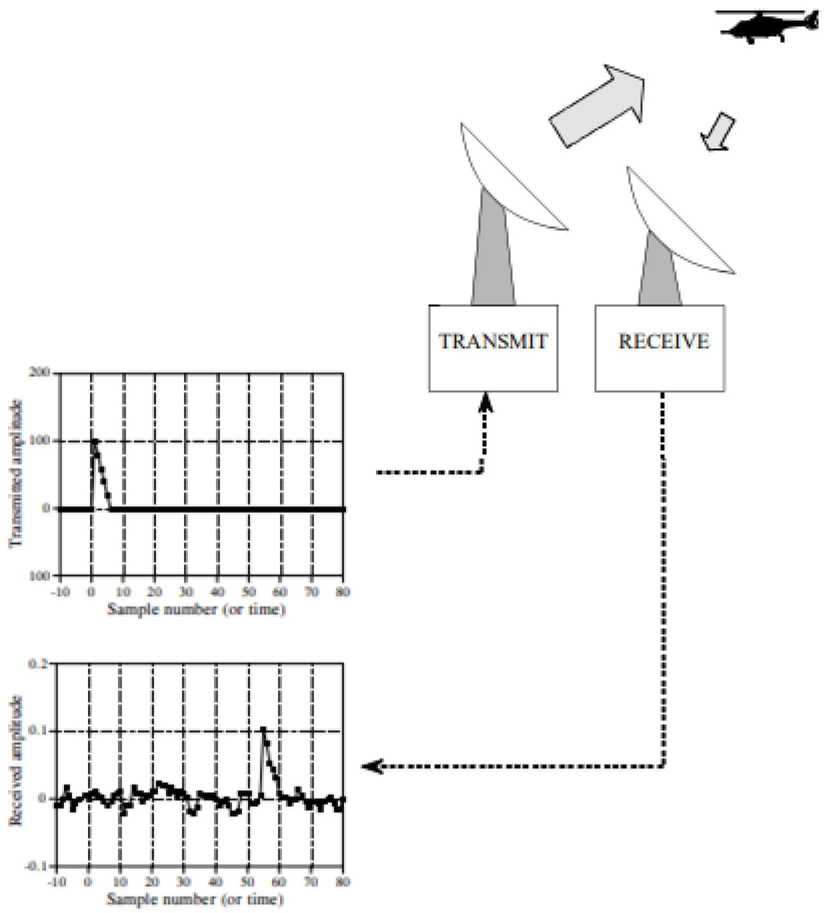

Figure 5.6

The amplitude of each sample in the cross-correlation signal is a measure of how much the received signal resembles the target signal, at that location. This means that a peak will occur in the cross-correlation signal for every target signal that is present in the received signal. In other words, the value of the cross-correlation is maximized when the target signal is aligned with the same features in the received signal. What if the target signal contains samples with a negative value? Nothing changes. Imagine that the correlation machine is positioned such that the target signal is perfectly aligned with the matching waveform in the received signal. As samples from the received signal fall into the correlation machine, they are multiplied by their matching samples in the target signal. Neglecting noise, a positive sample will be multiplied by itself, resulting in a positive number. Likewise, a negative sample will be multiplied by itself, also resulting in a positive number. Even if the target signal is completely negative, the peak in the cross-correlation will still be positive. If there is noise on the received signal, there will also be noise on the cross correlation signal. It is an unavoidable fact that random noise looks a certain amount like any target signal you can choose. The noise on the cross-correlation signal is simply measuring this similarity. Except for this noise, the peak generated in the cross-correlation signal is symmetrical between its left and right. This is true even if the target signal isn't symmetrical. In addition, the width of the peak is twice the width of the target signal. Remember, the cross-correlation is trying to detect the target signal, not recreate it. There is no reason to expect that the peak will even look like the target signal. Correlation is the optimal technique for detecting a known waveform in random noise. That is, the peak is higher above the noise using correlation than can be produced by any other linear system. (To be perfectly correct, it is only optimal for random white noise). Using correlation to detect a known waveform is frequently called matched filtering.

The correlation machine and convolution machine are identical, except for one small difference. As discussed in the last chapter, the signal inside of the convolution machine is flipped left-for-right. This means that samples numbers: 1, 2, 3¿ run from the right to the left. In the correlation machine this flip doesn't take place, and the samples run in the normal direction.

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Unit 6

6.0 OBJECTIVES

After going through this unit you will be able to :

- Define finite impulse response as well as infinite impulse response filters
- Describe design techniques of FIR filters
- Explain IIR filter design by approximation of derivatives,impulse invariant method,bilinear transformation
- Understand butterworth and chebyshev filters.
- Explain frequency transformation
- Describe frequency response of linear phase FIR filters.

6.1 Finite Impulse Response (FIR) Filters

6.1.1 Introduction

A filter is a frequency selective system.Digital filters are classified as finite duration unit impulse response filters or infinite duration unit impulse response filters.In signal processing,a finite impulse response filter is a filter whose impulse response is of finite duration. I.e it has a finite number of non zero terms.FIR filter is a filter with no feedback in its equation.the response of FIR filter depends only on past and present samples.FIR filters are usually implemented using non-recursive structure,however they can be realized in both recursive as well as non recursive structures.

Following are the main advantages of FIR filters:

1. FIR filters are always stable.
2. FIR filters are free of limit cycle oscillations , when implemented on a finite word length digital system.
3. Excellent design methods are available for various kinds of FIR filters.

Disadvantages of FIR filters are as follows:

1. Memory requirement for FIR filter is very high
2. The implementation of FIR filters is very costly, since it requires more arithmetic operations and hardware components such as multipliers,adders and delay elements.

The basis FIR filter is characterized by two equations :

$$y(n) = h(k)\,(x(n-k) \qquad \text{(6.1a)}$$

$$H(z) = h(k) \qquad \text{(6.1b)}$$

Where $h(k), k = 0,1, \ldots N-1$, are the impulse response coefficient of the filter, $H(z)$is the transfer function of the filter and N is the filter length,i.e the number of filter coefficients.equation 6.1a is the FIR difference equation.

y(n) is a function only of past and present values of input x(n).FIR filters are always stable ,if they are implemented by direct evaluation as shown in equation 6.1a.Equation 6.1b represents the transfer function of filter,which provides a mean of analyzing the filter.

All DSP processors available have architecture suited to FIR filtering.FIR filters are very simple to implement.

6.1.2 Magnitude response and phase response of digital filters

The magnitude response of the filter can be characterized in terms of frequency bands the filter will pass or reject.The transfer function of a FIR causal filter is given by

$$H(z) = \sum_{n=0}^{N-1} \quad h(n)\, z^{-n}$$

where h(n) is the impulse response of the filter. The frequency response [Fourier transform

of h(n)] is given by

$$H(\omega) = \sum_{n=0}^{N-1} \quad h(n)\, e^{-n\omega j}$$

which is periodic in frequency with period 2 p , i.e.,

$$H(\omega) = H(\omega + 2\pi k), \text{ k = 0,1,2..}$$

Since $H(\omega)$ is complex it can be expressed as

$$H(\omega) = \pm|H(\omega)|\, e^{-n\omega j}$$

Where $|H(\omega)|$ is the magnitude response and $\theta(\omega)$is the phase response .

We define the phase delay $\tau_p$and group delay $\tau_g$of a filter as:

$\tau_p = -d\theta(\omega) \,/\, d(\omega)$ = $-d\theta(\omega) \,/\, d(\omega)$ =

6.1.3 Frequency response of linear phase FIR filters

The frequency response of the filter is the fourier transform of its impulse response.if h(n) is the impulse response of the system, then the frequency system is denoted by $H(\omega)$ or $H(e^{j\omega})$.$H(\omega)$is a complex function of frequency ω and so it can be expressed as magnitude function

$|H(\omega)|$ and phase function $\angle H(\omega)$

Linear phase filters have 4 possible type of impulse response, depending on N and the type of symmetry :

1. Symmetrical impulse response when *N* is odd.
2. Symmetrical impulse response when *N* is even.
3. Asymmetrical impulse response when *N* is odd.
4. Asymmetrical impulse response when *N* is even.

6.1.3.1 Frequency response of linear phase FIR filter when impulse response is symmetrical and *N* is odd.

The equation for frequency response of linear filter when impulse response is symmetrical and N is odd is given by :

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n\right\}$$

The magnitude function is given by

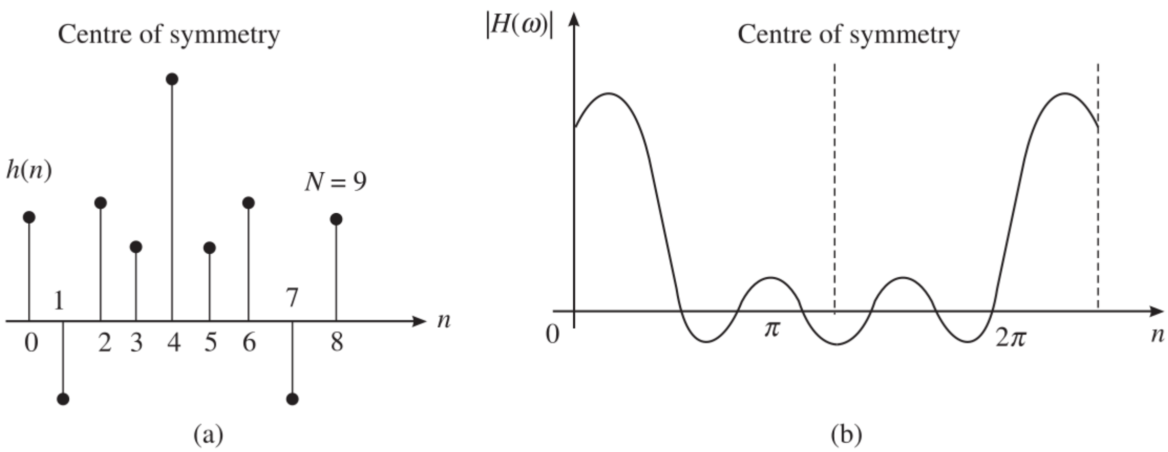
$$|H(\omega)| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n$$

The phase function is given by

$$\angle H(\omega) = -\omega\left(\frac{N-1}{2}\right) = -\omega\alpha \qquad \text{where } \alpha = \frac{N-1}{2}$$

Figure a shows symmetrical impulse response when N is 9,wheres figure b shows

the corresponding magnitude function of frequency response.



(a) Symmetrical impulse response, N = 9 (b) Magnitude function of $H(\omega)$.

From the figure it can be observed that the magnitude function of h is symmetric with $\omega = \Pi$, when the impulse response is symmetric and N is an odd number.

6.1.3.2 Frequency response of linear phase FIR filter when impulse response is symmetrical and *N* is even.

The expression for frequency response of linear phase FIR filter when impulse response is symmetrical and N is even is given by

$$H(\omega) = e^{-j\omega\frac{N-1}{2}}\left\{\sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right)\cos\omega\left(n-\frac{1}{2}\right)\right\}$$

The magnitude function of H(w) is given by

$$|H(\omega)| = \left[\sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right)\cos\omega\left(n-\frac{1}{2}\right)\right]$$

The phase function of H(w) is given by

$$\angle H(\omega) = -\omega\left(\frac{N-1}{2}\right) = -\omega\alpha \qquad \text{where } \alpha = \frac{N-1}{2}$$

Following figure (a) shows a symmetrical impulse response when N = 8, and figure (b) shows the corresponding magnitude function of frequency response. From these figures it can be observed that the magnitude function of $H(\omega)$is antisymmetric with$\omega = \Pi$, when impulse response is symmetric and N is even number.

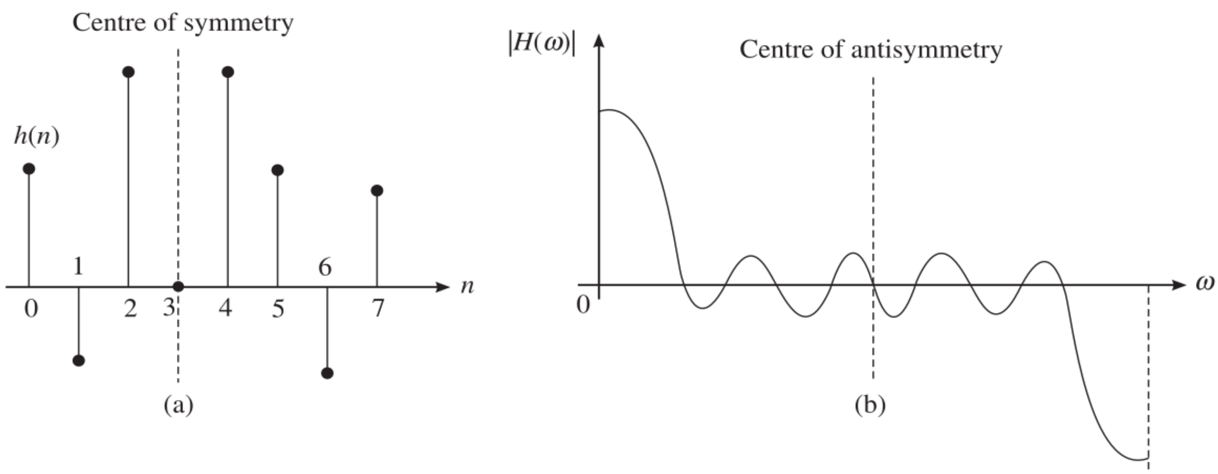


Fig (a) symmetrical impulse response(N=8) Fig (b) magnitude function of frequency response

6.1.3.3 Frequency response of linear phase FIR filter when impulse response is anti symmetric and *N* is odd.

This is the equation for frequency response of linear phase FIR filter when impulse response is antisymmetric and *N* odd.

$$H(\omega) = e^{j\left(\frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right)\right)} \left[\sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n\right) \sin \omega n \right]$$

The magnitude function is given by

$$|H(\omega)| = \sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n\right) \sin \omega n$$

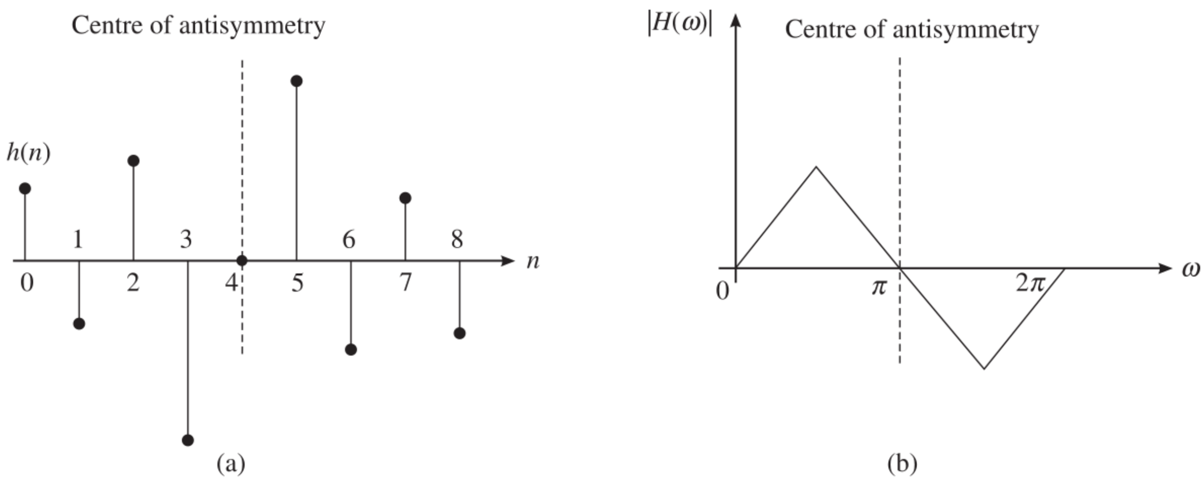
The phase function is given by

$$\angle H(\omega) = \frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right) = \beta - \alpha\omega$$

Where

$$\beta = \frac{\pi}{2} \text{ and } \alpha = \frac{N-1}{2}$$

Figure a) shows an antisymmetric impulse response when N = 9, and Figure (b) shows the corresponding magnitude function of frequency response. From these figures, it can be observed that the magnitude function is antisymmetric with $\omega = \Pi$, when the impulse response is antisymmetric and *N* is odd.



Fig(a) antisymmetric impulse response(N=9) Fig(b) magnitude function of frequency response

6.1.3.4 Frequency response of linear phase FIR filter when impulse response is anti symmetric and *N* is even.

This is the equation for the frequency response of linear phase FIR filter when impulse response is antisymmetric and N is even.

$$H(\omega) = e^{j\left(\frac{\pi}{2} - \omega\frac{N-1}{2}\right)}\left[\sum_{n=1}^{N/2} 2h\left(\frac{N}{2} - n\right)\sin\left(\omega\left(n - \frac{1}{2}\right)\right)\right]$$

The magnitude function is given by

$$|H(\omega)| = \sum_{n=1}^{N/2} 2h\left(\frac{N}{2} - n\right)\sin\left(\omega\left(n - \frac{1}{2}\right)\right)$$

The phase function is given by

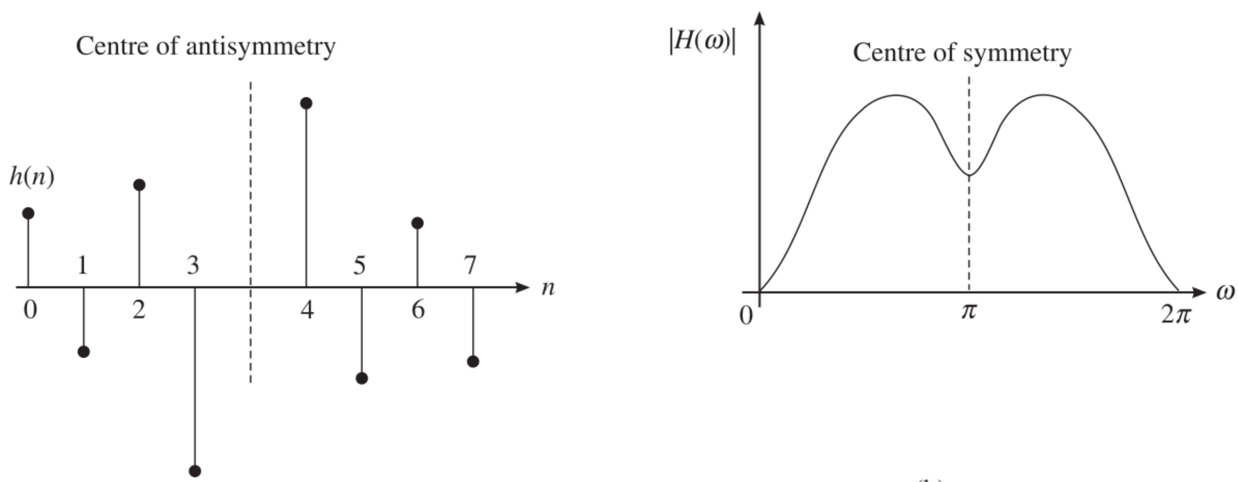
$$\angle H(\omega) = \frac{\pi}{2} - \omega\frac{N-1}{2} = \beta - \alpha\omega$$

Where

$$\beta = \frac{\pi}{2} \text{ and } \alpha = \frac{N-1}{2}$$

Figure (a) shows an antisymmetric impulse response when N = 8, and Figure (b)

shows its corresponding magnitude function of frequency response. it can be observed that the magnitude function of $H(\omega)$ is symmetric with $\omega = \Pi$ when the impulse response is antisymmetric and *N* is an even number.



(a) (b)

Fig (a) Antisymmetric impulse response for N = 8 Fig (b) Magnitude function of $H(\omega)$.

6.1.4 Design techniques of FIR filters

Design of digital filter involves :

1. Filter specification : This may include stating the type of filter, for example lowpass filter, the desired amplitude and/or phase responses and the tolerance, the sampling frequency, and the word length of input data.
2. Coefficient calculation to find transfer function: At this step,we determine the coefficient of transfer function.
3. Realization function : This involves converting the transfer function into a suitable filter network or structure.
4. Analysis of finite wordlength effects : Here, we analyze the effect of quantizing the filter coefficients and the input data as well as the effect carrying out the filtering operations
5. Implementation : This involves producing the software and/or hardware and performing the actual filtering.

To design FIR filters following methods are followed :

1. Fourier series method
2. Window method
3. Frequency sampling method
4. Optimum filter design

6.1.4.1 Fourier series method of design

The procedure for designing FIR filters by Fourier series method is as follows:

Step 1: Choose the desired frequency response $H_d(\omega)$ of the filter.

Step 2: Evaluate the Fourier series coefficients of $H_d(\omega T)$which gives the desired

impulse response$h_d(n)$.

Step 3: Truncate the infinite sequence $h_d(n)$ to a finite sequence$h \quad (n)$.

Step 4: Take Z-transform of $h \quad (n)$ to get a non-causal filter transfer function $H(z)$.

Step 5: Multiply $H(z)$ by$z^{-(N-1)/2}$ to convert the non-causal transfer function to a

realizable causal FIR filter transfer function.

We know that any periodic function can be expressed as a linear combination of complex exponentials. The frequency response of a digital filter is periodic with period equal to sampling frequency. Therefor, the desired frequency response of an FIR filter can be represented by fourier series as :

$$H_d(\omega)|_{\omega=\omega T} = H_d(\omega T) = \sum_{n=-\infty}^{\infty} h_d(n)\, e^{-j\omega nT}$$

Where the fourier coefficients hd(n) are the desired impulse response sequence of the filter.the samples of hd(n) can be determined using the equation :

$$h_d(n) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H_d(\omega T)\, e^{j\omega nT}\, d\omega$$

Where ws is a sampling frequency in rad/sec. Fs is sampling frequency in Hz. T = 1/Fs is a sampling period in sec.

The impulse response from the above equation is an infinite duration sequence.

For FIR filters , we truncate this infinite impulse response to a finite duration sequence of length N , where N is odd.Therefore ,

$$h(n) = \begin{cases} h_d(n), & \text{for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right) \\ 0, & \text{otherwise} \end{cases}$$

Taking Z - transform of the above equation for $h\ \ (n)$, we get

$$H(z) = \sum_{n=-(N-1)/2}^{(N-1)/2} h(n)\, z^{-n}$$

This transfer function of the filter $H(z)$ represents a non-causal filter. Hence the transfer function represented by the above equation for $H(z)$ is multiplied by $z^{-(N-1)/2}$. Therefore

$$= z^{-\left(\frac{N-1}{2}\right)} \left[\sum_{n=1}^{(N-1)/2} h(-n)\, z^{n} + h(0) + \sum_{n=1}^{(N-1)/2} h(n)\, z^{-n} \right]$$

Since $h\ \ (n)$=h(-n) , we express $H(z)$as :

$$H(z) = z^{-(N-1)/2} \left[h(0) + \sum_{n=1}^{(N-1)/2} h(n) \left[z^{n} + z^{-n} \right] \right]$$

Hence we see that casualty is brought by multiplying the transfer function by the delay factor

$\alpha = (N-1)/2$. This modification does not affect the amplitude response of the filter, however the abrupt truncation of the fourier series results in oscillations in the pass band and stop band.these oscillations are due to slow convergence of the fourier series.This effect is known as gibbs phenomenon.

6.1.4.2 Design of FIR filter using windows.

A finite weighing sequence w(n) with which the infinite impulse response is multiplied to obtain a finite impulse response is called a window.A finite weighing sequence w(n) with which the infinite impulse response is multiplied to obtain a finite impulse response is called a window. This is necessary because abrupt truncation of the infinite impulse response will lead to oscillations in the pass band and stop band, and these oscillations can be reduced through the

use of less abrupt truncation of the Fourier series.

The desirable characteristics of the window :

1. The central lobe of the frequency response of the window should contain most of the energy and should be narrow.
2. The highest side lobe level of the frequency response should be small.
3. The side lobes of the frequency response should decrease in energy rapidly as w tends to pi.

The procedure for designing FIR filters using windows is:

Step 1: For the desired frequency response$H(\omega)$, find the impulse response$h_d(n)$

using the equation:

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega$$

Step 2: Multiply the infinite impulse response with a chosen window sequence

$\omega(n)$ of length *N* to obtain filter coefficients $h \quad (n)$, i.e.

$$h(n) = \begin{cases} h_d(n)\,w(n), & \text{for } |n| \le \dfrac{N-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Step 3: Find the transfer function of the realizable filter

$$H(z) = z^{-(N-1)/2}\left[h(0) + \sum_{n=0}^{(N-1)/2} h(n)\,[z^{n} + z^{-n}]\right]$$

Some common window functions are :

1. Rectangular
2. Bartlett
3. Hanning
4. Hamming
5. Blackmann

Rectangular window:

The weighting function (window function) for an N-point rectangular window is given by

$$w_R(n) = \begin{cases} 1, & -\dfrac{(N-1)}{2} \le n \le \left(\dfrac{N-1}{2}\right) \\ 0, & \text{elsewhere} \end{cases} \quad \text{or} \quad w_R(n) = \begin{cases} 1, & 0 \le n \le (N-1) \\ 0, & \text{elsewhere} \end{cases}$$

The spectrum (frequency response) of rectangular window $W_R(\omega)$ is given by the Fourier transform of $w_R(n)$

The characteristic features of rectangular window are

(i) The main lobe width is equal to $4\Pi/N$

(ii) The maximum side lobe magnitude is –13 dB.

(iii) The side lobe magnitude does not decrease significantly with increasing ω.

In a rectangular window , the width of the transition region is related to the width of the main lobe of window spectrum.Gibbs oscillations are noticed in the pass band and stop band.The attenuation in the stop band is constant and cannot be varied.

Bartlett Window :

Bartlett window is also called a triangular window. This window has been chosen such that it has tapered sequences from the middle on either side. The window function w T (n) is defined as

$$w_T(n) = \begin{cases} 1 - \dfrac{2|n|}{N-1}, & \text{for } -\left(\dfrac{N-1}{2}\right) \le n \le \left(\dfrac{N-1}{2}\right) \\ 0 \quad , & \text{otherwise} \end{cases}$$

$$w_T(n) = \begin{cases} 1 - \dfrac{2|n-(N-1)/2|}{N-1}, & 0 \le n \le N-1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

In magnitude response of triangular window, the side lobe level is smaller than that of the rectangular window being reduced from –13 dB to –25 dB. However, the main lobe width is now $8\Pi/N$ or twice that of the rectangular window.

The triangular window produces a smooth magnitude response in both pass band and stop band, but it has the following disadvantages when compared to magnitude response obtained by using rectangular window:

1. The transition region is more.

2. The attenuation in the stop band is less.

Because of these characteristics, the triangular window is not usually a good choice

Hanning window :

The Hanning window function is given by

$$w_H(n) = \begin{cases} \alpha + (1-\alpha)\cos\left(\dfrac{2\pi n}{N-1}\right), & \text{for } -\left(\dfrac{N-1}{2}\right) \le n \le \left(\dfrac{N-1}{2}\right) \\ 0, & \text{elsewhere} \end{cases}$$

$$w_{Hn}(n) = \begin{cases} 0.5 - 0.5 \cos \dfrac{2n\pi}{N-1}, & \text{for } 0 \le n \le N-1 \\ 0 & , \text{otherwise} \end{cases}$$

The width of main lobe is $8\Pi/N$, i.e twice that of a rectangular window which results in doubling of the transition region of the filter. The peak of the first side lobe is –32 dB relative to the maximum value. This results in smaller ripples in both the pass band and stop band of the low-pass filter designed using the Hanning window. The minimum stop band attenuation of the filter is 44 dB. At higher frequencies the stop band attenuation is even greater. When compared to a triangular window, the main lobe width is the same, but the magnitude of the side lobe is reduced, hence the Hanning window is preferable to the triangular Window.

Hamming window :

The Hamming window function is given by

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5 \cos \left(\dfrac{2\pi n}{N-1} \right), & \text{for } -\left(\dfrac{N-1}{2} \right) \le n \le \left(\dfrac{N-1}{2} \right) \\ 0 & , \text{otherwise} \end{cases}$$

$$w_H(n) = \begin{cases} 0.54 - 0.46 \cos \left(\dfrac{2n\pi}{N-1} \right), & 0 \le n \le N-1 \\ 0 & , \text{otherwise} \end{cases}$$

In the magnitude response for N = 31, the magnitude of the first side lobe is down about 41

dB from the main lobe peak, an improvement of 10 dB relative to the Hanning window. But this improvement is achieved at the expense of the side lobe magnitudes at higher frequencies, which are almost constant with frequency. The width of the main lobe is 8 p /N. In the magnitude response of a low-pass filter designed using the Hamming window, the first side lobe peak is –51 dB, which is –7 dB lesser with respect to the Hanning window filter. However, at higher frequencies, the stop band attenuation is low when compared to that of Hanning window. Because the Hamming window generates lesser oscillations in the side lobes than the Hanning window for the same main lobe width, the Hamming window is generally preferred.

Blackman window :

The Blackman window function is another type of cosine window and given by the equation

$$w_B(n) = \begin{cases} 0.42 + 0.5\cos\dfrac{2\pi n}{N-1} + 0.08\cos\dfrac{4\pi n}{N-1}, & \text{for } -\left(\dfrac{N-1}{2}\right) \le n \le \left(\dfrac{N-1}{2}\right) \\ 0 & \text{, otherwise} \end{cases}$$

Or

$$w_B(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2n\pi}{N-1} + 0.08\cos\dfrac{4n\pi}{N-1}, & 0 \le n \le N-1 \\ 0 & \text{, otherwise} \end{cases}$$

In the magnitude response, the width of the main lobe is $12\Pi/N$, which is highest among windows. The peak of the first side lobe is at –58 dB and the side lobe magnitude decreases with frequency. This desirable feature is achieved at the expense of increased main lobe width. However, the main lobe width can be reduced by increasing the value of N. The side lobe attenuation of a low-pass filter using Blackman window is –78 dB.

Window's name	Mainlobe	Mainlobe/sidelobe	Peak $20\log_{10}\delta$
Rectangular	$4\pi/\ M$	-13dB	-21dB
Hanning	$8\pi/M$	-32dB	-44dB
Hamming	$8\pi/M$	-43dB	-53dB
Blackman	$12\pi/M$	-58dB	-74dB

Summary of window characteristics

6.1.4.3 Frequency sampling method

The ideal frequency response is sampled at sufficient number of points (i.e. N-points). These samples are the DFT coefficients of the impulse response of the filter. Hence the impulse response of the filter is determined by taking IDFT.

Let $H_d(\omega)$ = Idea frequency response

$H(k)$ = DFT sequence obtained by sampling $H_d(\omega)$

$h(n)$ = Impulse response of FIR filter

The impulse response $h(n)$ is obtained by taking IDFT of $H(k)$ .The samples of impulse response should be real. The terms $H(k)e^{j(2\Pi nk/N)}$ should be matched by the $e^{-j(2\Pi nk/N)}$.

Frequency sampling methods include two design techniques i.e,

1. type-I design
2. type-II design.

In the type-I design, the set of frequency samples includes the sample at frequency $\omega = 0$

When other set of samples are used instead of $\omega = 0$,such a design procedure is referred to as the type-II design

Procedure for type-I design

1. Choose the ideal (desired) frequency response $H_d(\omega)$.

2 Sample $H_d(\omega)$ at N-points by taking $\omega = \omega_k = \frac{2\Pi k}{N}$, where $k = 0, 1, 2, 3, \ldots, (N - 1)$ to generate the sequence $H(k)$.

$$\tilde{H}(k) = H_d(\omega)\,|_{\omega = (2\pi k)/N}; \quad \text{for } \; k = 0,\ 1,\ 2, \ldots, (N-1)$$

3. Compute the N samples of h(n) using the following equations:

When N is odd ,

$$h(n) = \frac{1}{N}\left[\tilde{H}(0) + 2\sum_{k=1}^{(N-1)/2} \mathrm{Re}\left(\tilde{H}(k)\, e^{j\frac{2\pi nk}{N}}\right)\right]$$

When n is even ,

$$h(n) = \frac{1}{N}\left[\tilde{H}(0) + 2\sum_{k=1}^{\left(\frac{N}{2}-1\right)} \left(\tilde{H}(k)\, e^{j\frac{2\pi nk}{N}}\right)\right]$$

4. Take Z-transform of the impulse response h(n) to get the transfer function $H(z)$

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

Procedure for type-II design

1. Choose the ideal frequency response $H_d(\omega)$.

2. Sample $H_d(\omega)$ at N-points by taking $\omega = \omega_k = \frac{2\Pi(2k+1)}{2N}$

where $k = 0, 1, 2, \ldots, (N - 1)$ to generate the sequence $H(k)$.

$$\tilde{H}(k) = H_d(\omega)|_{\omega = \frac{2\pi(2k+1)}{2N}}; \quad \text{for } k = 0, 1, 2, \ldots, (N-1)$$

3. Compute the N samples of$h(n)$ using the following equations:

When N is odd,

$$h(n) = \frac{2}{N} \sum_{k=0}^{(N-3)/2} \text{Re}\left[\tilde{H}(k)\, e^{jn\pi(2k+1)/N}\right]$$

When N is even,

$$h(n) = \frac{2}{N} \sum_{k=0}^{\left(\frac{N}{2}-1\right)} \text{Re}\left[\tilde{H}(k)\, e^{jn\pi(2k+1)/N}\right]$$

N

4. Take Z-transform of the impulse response $h(n)$ to get the transfer function $H(z)$.

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

6.1.4.4 Optimum filter design

In optimum filter design method, the weighted approximation error between the desired frequency response and the actual frequency response is spread evenly across the pass band and evenly across the stop band of the filter. This results in the reduction of maximum error. The resulting filter has ripples in both the pass band and the stop band. This concept of design is called optimum equiripple design criterion.

6.1.5 Design of optimal linear phase FIR filters

The optimal method is based on the concept of equiripple passband and stopband. In the passband , the practical response oscillates between 1-δ_p and 1+δ_p . in th stopband the filter response lies between 0 andδ_s.The difference between the ideal filter and the practical response can be viewed as error function

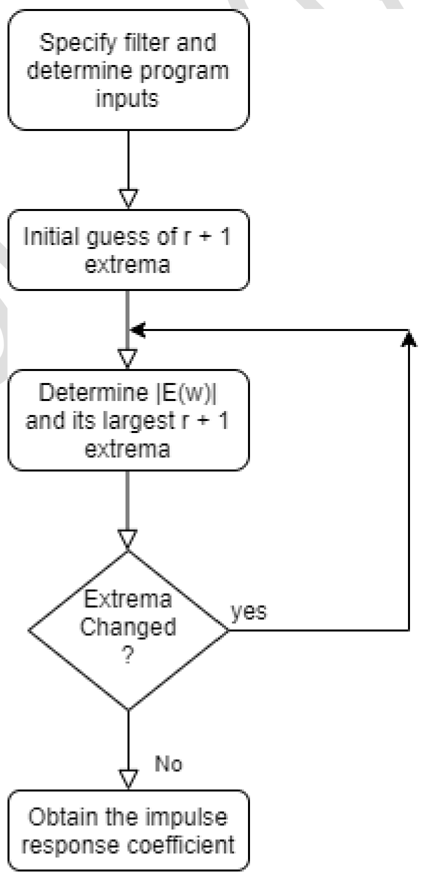
$$E(\omega) \ = \ W(\omega)[H_D(\omega) - H(\omega)$$

Where $H_D(\omega$ is the ideal or desired response and $W(\omega)$ is a weighing function that allows the relative error of approximation between different bands to be defined in the optimal method.

The main problem in the optimal method is to find the location of the external frequencies. A powerful technique which employs remez exchange algorithm to find the external frequencies has been developed.For a given set of specifications(that is passband edge frequencies N and the ratio between the passband and stopband ripples the optimal method involves the following key steps :

1. Use the remez exchange algorithm to find the optimum set of external frequencies.
2. Determine the frequency response using external frequencies
3. Obtain the impulse response coefficients.

Flow chart of the optimal method.:



Flowchart for optimal method

The heart of the optimal method is the first step where an iterative process is used to determine the external frequencies of a filter whose amplitude-frequency response satisfies the optimality condition.This step relies on the alternation theorem which specifies the number of external frequencies that can exist for a given value of N.

6.2 Infinite Impulse Response (IIR) Filters

6.2.1 Introduction

The type of filters which make use of feedback connection to get the desired filter implementation are known as recursive filters.Their impulse response is of infinite duration. So ,they are called IIR filters. IIR filters are designed by considering all the infinite samples of the impulse response. The impulse response is obtained by taking the inverse Fourier transform of ideal frequency response. There are several techniques available for the design of digital filters having an infinite duration unit impulse response. The popular methods for such filter design uses the technique of first designing the digital filter in analog domain and then transforming the analog filter into an equivalent digital filter because the analog filter design techniques are well developed.

IIR filters normally require fewer coefficients than FIR filters.These filters are mainly used when throughput and sharp cutoff is the important requirement.The physically realizable and stable IIR filter cannot have a linear phase. For a filter to have a linear phase, the condition to be satisfied is $h(n) = h(N – 1 – n)$ where N is the length of the filter and the filter would have a mirror image pole outside the unit circle for every pole inside the unit circle. This results in an unstable filter. As a result, a causal and stable IIR filter cannot have linear phase. In the design of IIR filters, only the desired magnitude

Important features of IIR filters:

1. The physically realizable IIR filters do not have linear phase.

2. The IIR filter specifications include the desired characteristics for the magnitude

response only.

6.2.2 IIR filter design by approximation of derivatives

The approximation of derivative method is also known as backward difference method.The analog filter having the rational system function H(s) can also be described by the linear

constant coefficient differential equation.

$$\frac{dy(t)}{dt} = \frac{dx(t)}{dt}$$

In this method of IIR filter design by approximation of derivatives, an analog filter is converted into a digital filter by approximating the above differential equation into an equivalent difference equation.

The backward difference formula is substituted for the derivative $\frac{dy(t)}{dt}$ at time $t = nT$

Thus,

$$\frac{dy(t)}{dt} = \frac{y(nT) - y(n-1)T}{T}$$

Or $$\frac{dy(t)}{dt} = \frac{y(n) - y(n-1)}{T}$$

where T is the sampling interval and $y(n) = y(nT)$

The system function of an analog differentiator with an output $\frac{dy(t)}{dt}$ is $H(s) = s$ and

the digital system which produces the output $[y(n) - y(n-1)]/T$ has the system function $H(z) = [1 - z^{-1}]/T$.

Comparing these two, we can say that the frequency domain equivalent

for the relationship $\frac{dy(t)}{dt} = \frac{y(n) - y(n-1)}{T}$ is:

$$s = \frac{1 - z^{-1}}{T}$$

Thus, this is the analog domain to digital domain transformation.

Mapping of the z-plane from the s-plane

We have $s = \frac{1-z^{-1}}{T}$ i.e $z = \frac{1}{1-sT}$

Substituting $s = j\Omega$ in the expression for z, we have

$$z = \frac{1}{1-j\Omega T}$$

$$= \frac{1}{1+j\Omega^2 T^2} + j\frac{\Omega T}{1+j\Omega^2 T^2}$$

It can be observed that the mapping of the equations$s = (1 - z^{-1)})/T$ takes the left half plane of s-domain into the corresponding points inside the circle of radius 0.5 and centre at z = 0.5. Also the right half of the s-plane is mapped outside the unit circle. Because of this,mapping results in a stable analog filter transformed into a stable digital filter. However, since the location of poles in the z-domain are confined to smaller frequencies, this design method can be used only for transforming analog low-pass filters and band pass filters which are having smaller resonant frequencies.

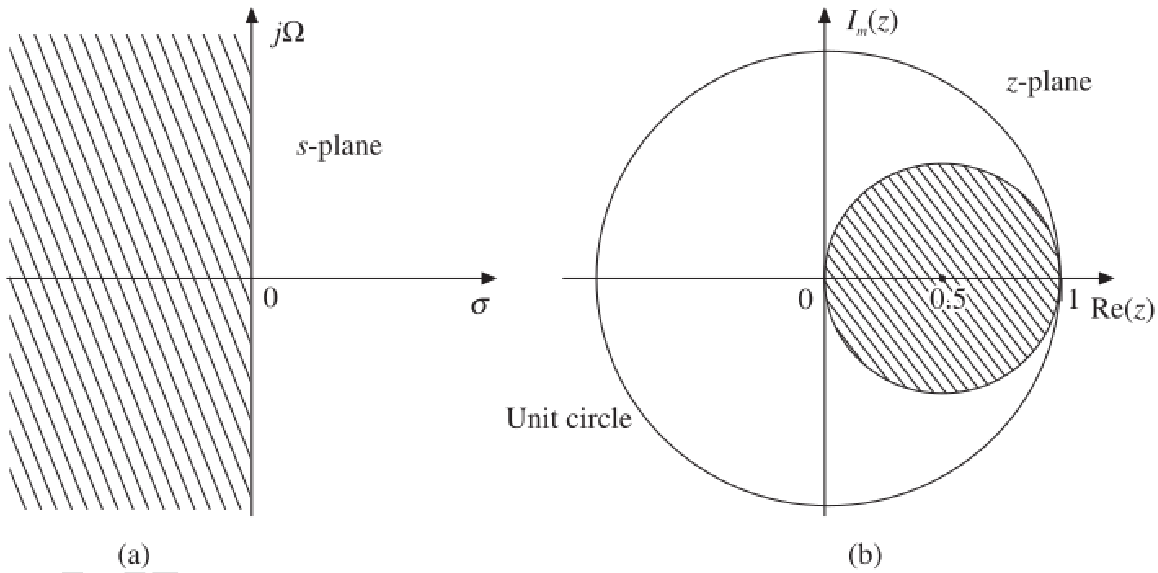


Fig : Mapping of s-plane into z-plane by the backward difference method.

6.2.3 IIR filter design by impulse invariant method

The desired impulse response of the digital filter is obtained by uniformly sampling the impulse response of the equivalent analog filter. The main idea behind this is to preserve the frequency response characteristics of the analog filter. For the digital filter to possess the frequency response characteristics of the corresponding analog filter, the sampling period T should be sufficiently small (or the sampling frequency should be sufficiently high) to minimize (or completely avoid) the effects of aliasing.

Let $h_a(t)$ = Impulse response of analog filter

T = Sampling period

$h(n)$ = Impulse response of digital filter

For impulse invariant transformation,

$h(n) = h_a(t) = h_a(nT)$

Analog filter's system function is given by

$$H_a(s) = \sum_{i=1}^{N} \frac{A_i}{s - p_i}$$

The relationship between the transfer function f the digital filter and analog filter is given by

$$H(z) = \sum_{i=1}^{N} \frac{A_i}{1 - e^{p_i T} z^{-1}}$$

Comparing the above expressions for $H_a(s)$and $H(z)$, we can say that the impulse invariant transformation is accomplished by the mapping.

$$\frac{1}{s - p_i} = \frac{1}{1 - e^{p_i T} z^{-1}}$$

The above mapping shows that the analog pole at $s = p_i$ is mapped into a digital pole at $z = e^{pT}$. Therefore, the analog poles and the digital poles are related by the relation.

$$z = e^{sT}$$

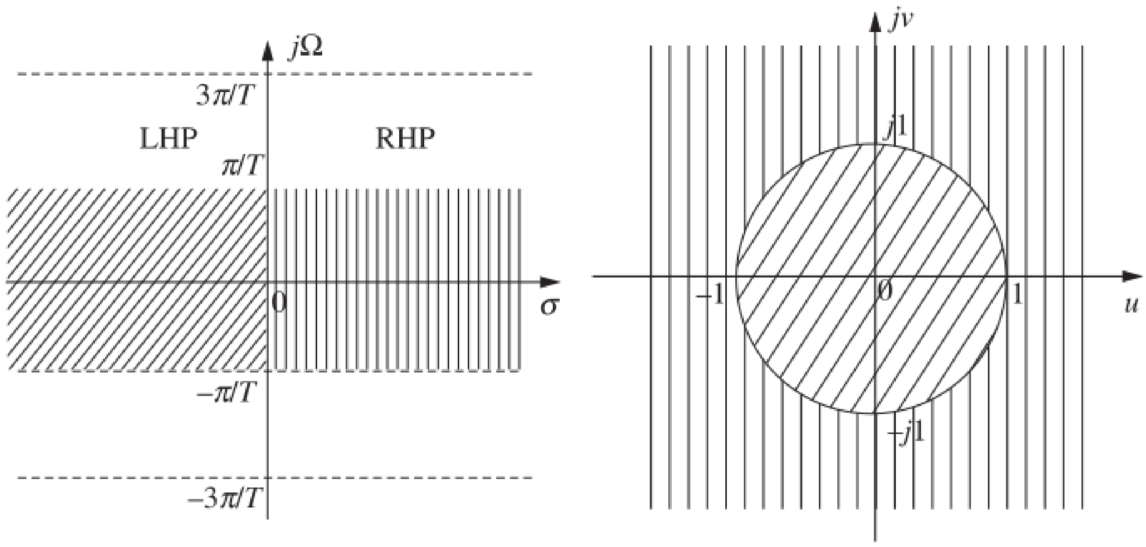


Fig : Mapping of (a) s-plane into (b) z-plane by impulse invariant transformation.

The mapping from the analog frequency Ω to the digital frequency ω by impulse invariant transformation is many-to-one which simply reflects the effects of aliasing due to sampling of the impulse response.

The stability of a filter (or system) is related to the location of the poles. For a stable analog filter the poles should lie on the left half of the s-plane. That means for a stable digital filter the poles should lie inside the unit circle in the z-plane.

6.2.4 IIR filter design by the bilinear transformation

The IIR filter design using impulse invariant as well as approximation of derivatives methods is appropriate only for the design of low-pass filters and band pass filters whose resonant frequencies are small. These techniques are not suitable for high-pass or band reject filters. The limitation is overcome in the mapping technique called the bilinear transformation. This transformation is a one-to-one mapping from the s-domain to the z-domain. That is, the bilinear transformation is a conformal mapping that transforms the imaginary axis of s-plane into the unit circle in the z-plane only once, thus avoiding aliasing

of frequency components. In this mapping, all points in the left half of s-plane are mapped inside the unit circle in the z-plane, and all points in the right half of s-plane are mapped outside the unit circle in the z-plane. So the transformation of a stable analog filter results in a stable digital filter. The bilinear transformation can be obtained by using the trapezoidal formula for the numerical integration.

Let the system function of analog filter be $H_a(s) = \frac{b}{s+a}$

The differential equation describing the above analog filter can be obtained as:

$$H_a(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

Or $$sY(s) + aY(s) = b\,X(s)$$

Taking inverse Laplace transform on both sides, we get

$$\frac{dy(t)}{dt} + a\,y(t) = bx(t)$$

Integrating the above equation between the limits (nT – T) and nT, we have

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt}\,dt + a\int_{nT-T}^{nT} y(t)\,dt = b\int_{nT-T}^{T} x(t)\,dt$$

The trapezoidal rule for numeric integration is expressed as:

$$\int_{nT-T}^{nT} a(t)\,dt = \frac{T}{2}[a(nT) + a(nT-T)]$$

Therefore, we get

$$y(nT) - y(nT-T) + a\frac{T}{2}y(nT) + a\frac{T}{2}y(nT-T) = b\frac{T}{2}x(nT) + b\frac{T}{2}x(nT-T)$$

After taking z-transform,the system function of a digital filter is

$$\frac{Y(z)}{X(z)} = H(z) = \frac{b}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + a}$$

Comparing this with the analog filter system function $H_a(s)$ we get

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

On rearranging,

$$z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}$$

This is the relation between analog and digital poles in bilinear transformation.

So to convert an analog filter function into an equivalent digital filter function, we need to put

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \text{ in } H_a(s)$$

The general characteristic of the **mapping** $z = e^{sT}$ may be obtained by putting $s = \sigma + j\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$ in the above equation for s.

Thus,

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{re^{j\omega} - 1}{re^{j\omega} + 1}\right)$$

Which is equal to

$$\frac{2}{T}\left[\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right]$$

On the imaginary axis of s-plane $\sigma = 0$ and correspondingly in the z-plane r = 1.

Therefore, The relation between analog and digital frequencies is:

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

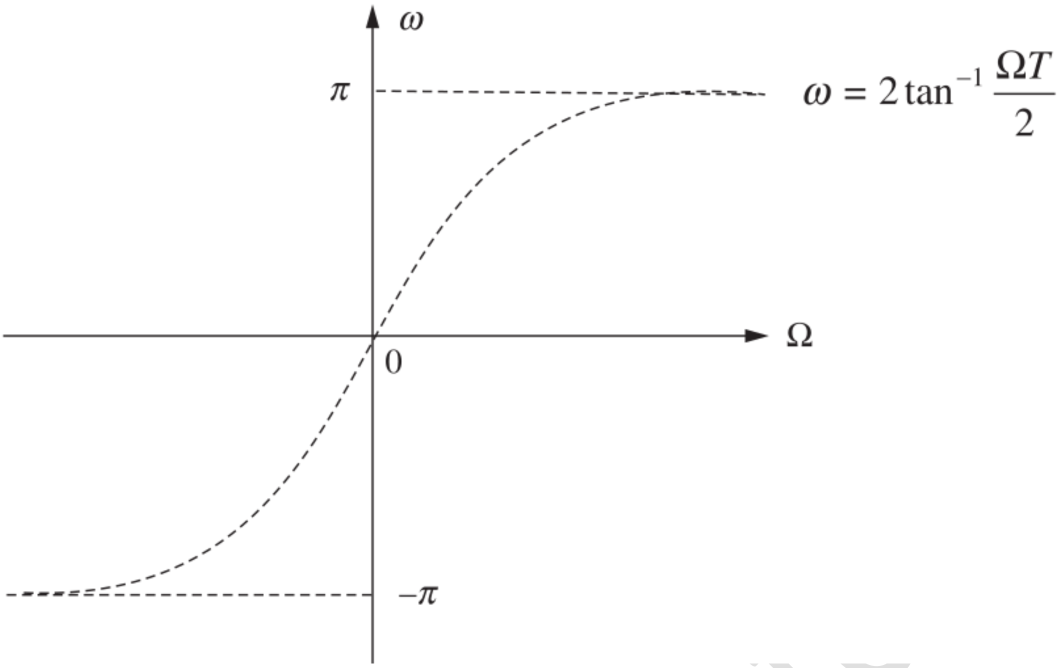


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Fig : Mapping between Ω and ω in bilinear transformation.

The mapping is non-linear and the lower frequencies in analog domain are expanded in the digital domain, whereas the higher frequencies are compressed. This is due to the nonlinearity of the arctangent function and is usually known as frequency warping.

6.2.5 Butterworth filters

To design a Butterworth IIR digital filter, first an analog Butterworth filter transfer function is determined using the given specifications. Then the analog filter transfer function is converted to a digital filter transfer function using either impulse invariant transformation or bilinear transformation.Infinite-duration Impulse Response (IIR) Filters

The analog Butterworth filter is designed by approximating the ideal frequency response using an error function. The error function is selected such that the magnitude is maximally flat in the passband and monotonically decreasing in the stopband. (Strictly speaking the magnitude is maximally flat at the origin, i.e., at W = 0, and monotonically decreasing with increasing W).

The magnitude response of low-pass filter obtained by this approximation is given by

$$|H_a(\omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

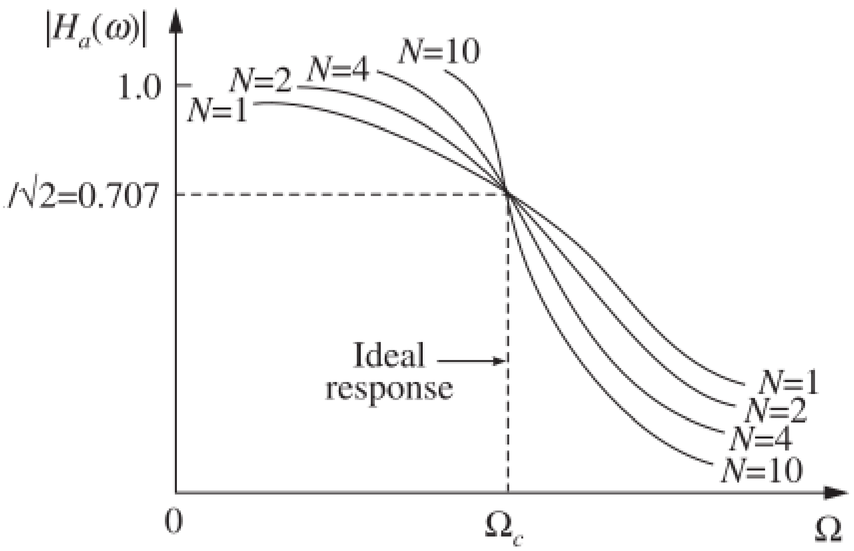
where W c is the 3 dB cutoff frequency and N is the order of the filter.

6.2.5.1 Frequency response of the Butterworth filter

The frequency response of Butterworth filter depends on the order N. The magnitude

response for different values of N are shown in Figure. From Figure 8.8, It can be observed that the approximated magnitude response approaches the ideal response as the value of N increases.

However, the phase response of the Butterworth filter becomes more nonlinear with increasing N.



Magnitude response of Butterworth low-pass filter for various values of N.

Design procedure for low-pass digital Butterworth IIR filter:

The low-pass digital Butterworth filter is designed as per the following steps:

Step 1 : Choose the type of transformation, i.e., either bilinear or impulse invariant transformation.

Step 2 : Calculate the ratio of analog edge frequencies depending upon the transformation chosen such as bilinear or impulse

Step 3 : Decide the order N of the filter. Choose N such that it is an integer just greater than or equal to the value obtained.

Step 4 : Calculate the analog cutoff frequency for both transformation

Step 5 : Determine the transfer function of the analog filter.

Step 6: Using the chosen transformation, transform the analog filter transfer function H a (s)

to digital filter transfer function H(z).

Step 7 : Realize the digital filter transfer function H(z) by a suitable structure.

Properties of Butterworth filters

1. The Butterworth filters are all pole designs (i.e. the zeros of the filters exist at ¥).
2. The filter order N completely specifies the filter.

The magnitude response approaches the ideal response as the value of N increases.

The magnitude is maximally flat at the origin.

The magnitude is a monotonically decreasing function of W.

At the cutoff frequency W c , the magnitude of normalized Butterworth filter is 1/
2 . Hence the dB magnitude at the cutoff frequency will be 3 dB less than the
maximum value.

6.2.6 Chebyshev filters

or designing a Chebyshev IIR digital filter, first an analog filter is designed using the given specifications. Then the analog filter transfer function is transformed to digital filter transfer function by using either impulse invariant transformation or bilinear transformation.

The analog Chebyshev filter is designed by approximating the ideal frequency response using an error function. There are two types of Chebyshev approximations. In type-1 approximation, the error function is selected such that the magnitude response is equiripple in the passband and monotonic in the stopband. In type-2 approximation, the error function is selected such that the magnitude function is monotonic in the passband and equiripple in the stopband. The type-2 magnitude response is also called inverse Chebyshev response. The type-1 design is presented in this book

The magnitude response of type-1 Chebyshev low-pass filter is given by:

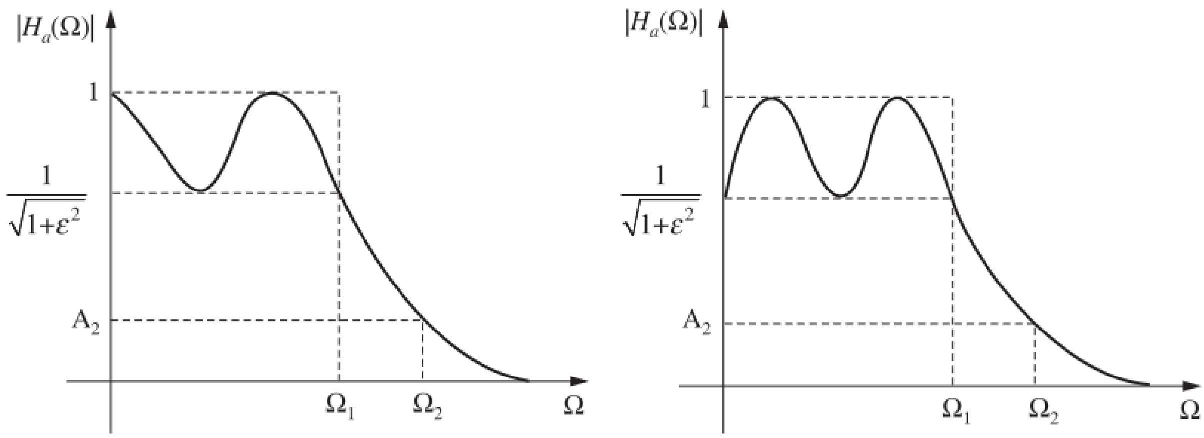
$$\left|H_a(\Omega)\right|^2 = \frac{1}{1 + \varepsilon^2 c_N^2\left(\dfrac{\Omega}{\Omega_c}\right)}$$

where e is attenuation constant given by

$$\varepsilon = \left[\frac{1}{A_1^2} - 1\right]^{\frac{1}{2}}$$

Frequency response of the Chebyshev filter:

The frequency response of Chebyshev filters depends on order N. The approximated response approaches the ideal response as the order N increases. The phase response of the Chebyshev filter is more nonlinear than that of the Butterworth filter for a given filter length N.



(a) (b)

Design procedure for low-pass digital Chebyshev IIR filter:

The low-pass Chebyshev IIR digital filter is designed following the steps given below.

Step 1 : Choose the type of transformation.

(Bilinear or impulse invariant transformation)

Step 2 : Calculate the attenuation constant e .

Step 3 : Calculate the ratio of analog edge frequencies W 2 /W 1 .

Step 4 : Decide the order of the filter N

Step 5 : Calculate the analog cutoff frequency W c . for both transformation.

Step 6 : Determine the analog transfer function H a (s) of the filter, wehn the order of N is odd or even

Step 7 : Using the chosen transformation, transform H a (s) to H(z), where H(z) is the transfer function of the digital filter.

Properties of Chebyshev filters (Type 1):

1. The magnitude response is equiripple in the passband and monotonic in the Stopband.
2. The chebyshev type-1 filters are all pole designs.
3. The normalized magnitude function has a value of 1/ 1 + F 2 at the cutoff frequency W c
4. The magnitude response approaches the ideal response as the value of N increases.

Inverse Chebyshev filters

Inverse Chebyshev filters are also called type-2 Chebyshev filters. A low-pass inverse Chebyshev filter has a magnitude response given by

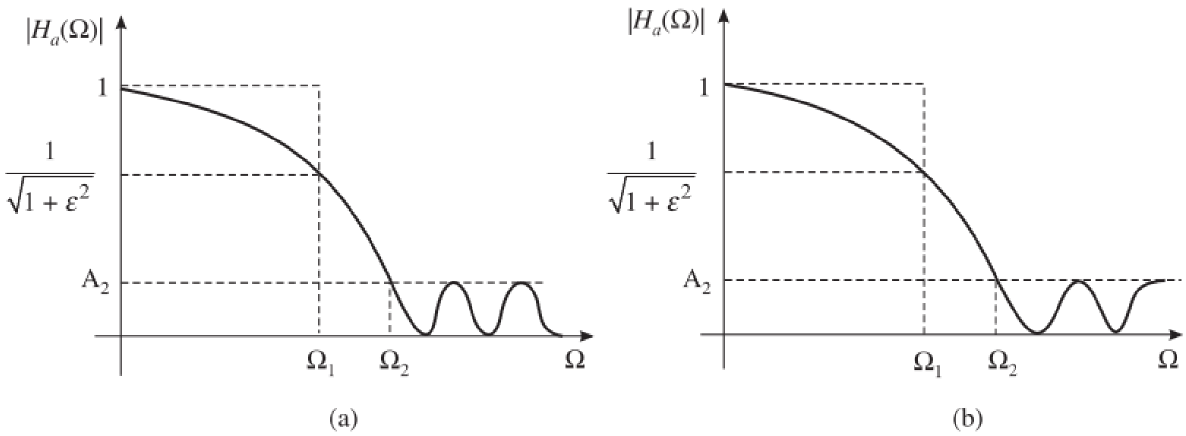
$$\left|H(\Omega)\right| = \frac{\varepsilon\, c_N(\Omega_2/\Omega)}{[1+\varepsilon^2 c_N^2(\Omega_2/\Omega)]^{\frac{1}{2}}}$$

where e is a constant and W c is the 3 dB cutoff frequency. The Chebyshev polynomial c N (x) is given by

$$c_N(x) = \cos(N\cos^{-1} x), \qquad \text{for } |x| \le 1$$
$$= \cosh(N\cosh^{-1} x), \quad \text{for } |x| > 1$$

The

magnitude response has maximally flat passband and equiripple stopband, just the opposite of the Chebyshev filters response. That is why type-2 Chebyshev filters are called the inverse Chebyshev filters.



6.2.7 Elliptic filters

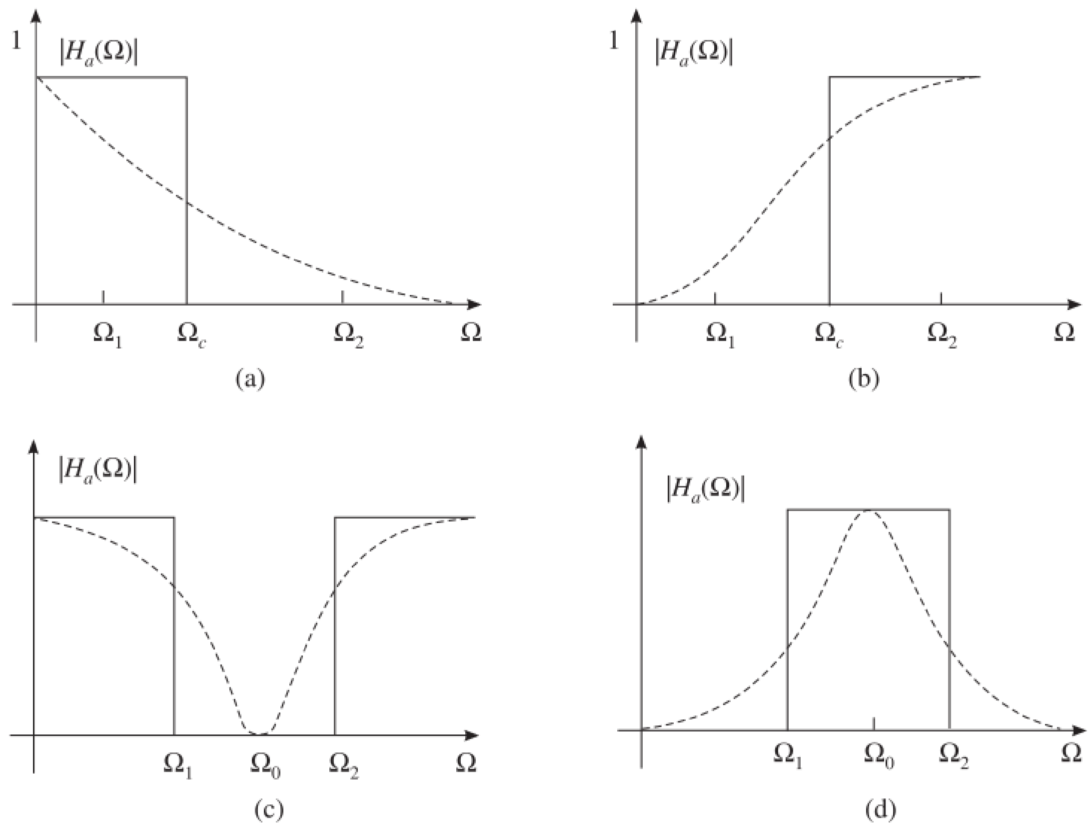
The elliptic filter is sometimes called the Cauer filter. This filter has equiripple passband and stopband. Among the filters discussed so far, for a given filter order, pass band and stop band deviations, elliptic filters have the minimum transition bandwidth. The magnitude response of an elliptic filter is given by

$$\left|H(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 U_N(\Omega/\Omega_c)}$$

where $U_N(x)$is the Jacobian elliptic function of order *N* and ε is a constant related to the passband ripple.

6.2.8 Frequency transformation

In the design techniques discussed so far, we have considered only low-pass filters.
This low-pass filter can be considered as a prototype filter and its system function H p (s) can be determined. The high-pass or band pass or band stop filters are designed by designing a low-pass filter and then transforming that low-pass transfer function into the required filter function by frequency transformation. Frequency transformation can be accomplished in two ways.
Basically there are four types of frequency selective filters, viz. low-pass, high-pass, band pass and the band stopped. In Figure 8.11, the frequency response of the ideal case is shown in solid lines and practical case in dotted lines



Frequency response of (a) Low-pass filter, (b) High-pass filter, (c) Band pass filter and (d) Band stop filter.

The high-pass or band pass or band stop filters are designed by designing a low-pass filter and then transforming that low-pass transfer function into the required filter function by frequency transformation. Frequency transformation can be accomplished in two ways:

1. Analog frequency transformation
2. Digital frequency transformation

Analog frequency transformation:

In the analog frequency transformation, the analog system function $H_p(s)$ of the prototype filter is converted into another analog system function $H(s)$ of the desired filter (a low-pass filter with another cutoff frequency or a high-pass filter or a band pass filter or a band stop filter). Then using any of the mapping techniques (impulse invariant transformation or bilinear transformation) this analog filter is converted into the digital filter with a system function $H(z)$.

The frequency transformation formulae used to convert a prototype low-pass filter into a low-pass (with a different cutoff frequency), high-pass, band pass or band stop are given in Table. Ω_c is the cutoff frequency of the low-pass prototype filter.Ω_c * cutoff frequency of new low-pass filter or high-pass filter and Ω_1 and Ω_2 are the cutoff frequencies of band pass or band stop filters.

Type	*Transformation*
Low-pass	$s \rightarrow \Omega_c \dfrac{s}{\Omega_c^*}$
High-pass	$s \rightarrow \Omega_c \dfrac{\Omega_c^*}{s}$
Band pass	$s \rightarrow \Omega_c \dfrac{s^2 + \Omega_1 \Omega_2}{s(\Omega_2 - \Omega_1)}$
Band stop	$s \rightarrow \Omega_c \dfrac{s(\Omega_2 - \Omega_1)}{s^2 + \Omega_1 \Omega_2}$

Digital Frequency Transformation

As in the analog domain, frequency transformation is possible in the digital domain also. The frequency transformation is done in the digital domain by replacing the variable z^{-1} by a function of z^{-1}, i.e.,$f(z^{-1})$. This mapping must take into account the stability criterion. All the poles lying within the unit circle must map onto itself and the unit circle must also map onto Itself.

Following table gives the formulae for the transformation of the prototype low pass digital filter into a digital low-pass, high-pass, band pass or band stop filters.

Type	*Transformation*	*Design parameter*
Low-pass	$z^{-1} \rightarrow \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \dfrac{\sin\left[(\omega_c - \omega_c^*)/2\right]}{\sin\left[(\omega_c + \omega_c^*)/2\right]}$
High-pass	$z^{-1} \rightarrow -\dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\dfrac{\cos\left[(\omega_c - \omega_c^*)/2\right]}{\cos\left[(\omega_c + \omega_c^*)/2\right]}$
Band pass	$z^{-1} \rightarrow -\dfrac{z^{-2} - \alpha_1 z^{-1} + \alpha_2}{\alpha_2 z^{-2} - \alpha_1 z^{-1} + 1}$	$\alpha_1 = \dfrac{-2\alpha k}{(k+1)}$
		$\alpha_2 = \dfrac{(k-1)}{(k+1)}$
		$\alpha = \dfrac{\cos[(\omega_2 + \omega_1)/2]}{\cos[(\omega_2 - \omega_1)/2]}$
		$k = \cot\left(\dfrac{\omega_2 - \omega_1}{2}\right)\tan\left(\dfrac{\omega_c}{2}\right)$
Band stop	$z^{-1} \rightarrow \dfrac{z^{-2} - \alpha_1 z^{-1} + \alpha_2}{\alpha_2 z^{-2} - \alpha_1 z^{-1} + 1}$	$\alpha_1 = \dfrac{-2\alpha}{(k+1)}$
		$\alpha_2 = \dfrac{(1-k)}{(1+k)}$
		$\alpha = \dfrac{\cos\left[(\omega_2 + \omega_1)/2\right]}{\cos\left[(\omega_2 - \omega_1)/2\right]}$
		$k = \tan\left(\dfrac{\omega_2 - \omega_1}{2}\right)\tan\left(\dfrac{\omega_c}{2}\right)$

The frequency transformation may be accomplished in any of the available two techniques, however, caution must be taken to which technique to use. For example, the impulse invariant transformation is not suitable for high-pass or bandpass filters whose resonant frequencies are higher. In such a case, suppose a low-pass prototype filter is converted into a high-pass filter using the analog frequency transformation and transformed later to a digital filter using the impulse invariant technique. This will result in aliasing problems. However, if the same prototype low-pass filter is first transformed into a digital filter using the impulse invariant technique and later converted into a high-pass filter using the digital frequency transformation, then it will not have any aliasing problem. Whenever the bilinear transformation is used, it is of no significance whether analog frequency transformation is used or digital frequency transformation. In this case, both analog and digital frequency transformation techniques will give the same result

6.3 Conclusion

- Based on impulse response, filters are of two types: (i) IIR filters and (ii) FIR filters.The IIR filters are designed using an infinite number of samples of impulse response.They are of recursive type, whereby the present output depends on the present input, past input and past output samples. The FIR filters are designed using only a finite number of samples of impulse response. They are non-recursive types whereby the present output depends on the present input and past input samples.
- The necessary and sufficient condition for the linear phase characteristic of FIR filter is that the phase function should be a linear function of ω, which in turn requires constant phase delay or constant phase and group delay.
- The transformation of analog filter to digital filter without modifying the impulse response of the filter is called impulse invariant transformation (i.e. in this transformation, the impulse response of the digital filter will be the sampled version of the impulse response of the analog filter).
- FIR filter is always stable because all its poles are at the origin.
- The two concepts that lead to the design of FIR filter by Fourier series are: (i) The frequency response of a digital filter is periodic with period equal to sampling frequency.(ii) Any periodic function can be expressed as a linear combination of complex exponentials.
- A finite weighing sequence w(n) with which the infinite impulse response is multiplied to obtain a finite impulse response is called a window. This is necessary because abrupt truncation of the infinite impulse response will lead to oscillations in the pass band and stop band, and these oscillations can be reduced through the use of less abrupt truncation of the Fourier series.
- Chebyshev approximation is one in which the approximation function is selected such that the error is minimized over a prescribed band of frequencies.
- Type-1 Chebyshev approximation is one in which the error function is selected such that the magnitude response is equiripple in the passband and monotonic in the stopband.
- Type-2 Chebyshev approximation is one in which the error function is selected such that the magnitude response is monotonic in the passband and equiripple in the stopband. The type-2 Chebyshev response is called inverse Chebyshev response.

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6.5 Bibliography

6.6 Unit End Exercises

1. What is an FIR filter? Compare an FIR filter with an IIR filter.

2. Write the steps in the design of FIR filters.

3. Explain FIR filter design using windowing method.

4. Find the frequency response of a rectangular window.

5.Design an FIR digital filter to approximate an ideal low-pass filter with pass band gain of unity, cutoff frequency of 1 kHz and working at a sampling frequency of f_s = 4 kHz. The length of the impulse response should be 11. Use the Fourier series method.

6. Compare analog and digital filters. State the advantages of digital filters over

analog filters.

7. Define infinite impulse response and finite impulse response filters and compare.

8. Justify the statement IIR filter is less stable and give reason for it.

9. Describe digital IIR filter characterization in time domain.

10. Describe digital IIR filter characterization in z-domain.

11. Discuss the impulse invariant method.

12. What are the limitations of impulse invariant method?

13 Compare impulse invariant and bilinear transformation methods.

14. Discuss the magnitude and phase responses of digital filters.