M.Sc. Mathematics Part - II

Paper - II

FOURIER ANALYSIS

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CONTENTS

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1

FOURIER SERIES

Unit Structure

1.1 DEFINITION : PERIODIC FUNCTION :

A real or complex valued function f is said to be periodic with period $T > 0$, if $f(x+nt) = f(x)$, $\forall$ x and $\forall n \in \mathbb{Z}$.

Example : 1) $\sin(x+2n\pi) = \sin x$

2) $\cos(x+2n\pi) = \cos x$

hence $\sin x$ and $\cos x$ are periodic function with period 2π.

The Orthogonality Relations of Trigonometric functions:

1) $$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & m \neq n \quad m,n = 0,1,2,.... \\ \pi & m = n = 1,2,... \\ 2\pi & m = n = 0 \end{cases}$$

2) $$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & m \neq n \quad m,n = 1,2,... \\ \pi & m = n = 1,2,... \\ 0 & m = n = 0 \end{cases}$$

3) $$\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0, \ \forall m,n = 0,1,2,...$$

4) $$\int_{-\pi}^{\pi} e^{imx} \, e^{-inx} \, dx = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$$

Definition : Trigonometric Series : A series of the form

$$\frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + ...$$

$$................ + a_n \cos nx + b_n \sin nx + ...$$

where, $a_0, a_1, b_1, a_2, b_2, ..., a_n, b_n, ...$ are constants is called as trigonometric series.

1.2 DIRICHLET'S CONDITIONS :

If $f(x)$ is a periodic function of period 2π defined in the interval $C \le x \le C + 2\pi$ where C is any constant then following condition are known to be Dirichlet's conditions

i) Function $f(x)$ and its integrals are finite and single valued in the interval.
ii) Function $f(x)$ has at most finite number of finite discontinuities in the interval.
iii) Function $f(x)$ has at most finite number of maxima and minima in the interval.

1.3 FOURIER SERIES OF PERIODIC CONTINUOUS FUNCTIONS :

Definition : If $f(x)$ is a periodic function of period 2π defined in the interval $C \le x \le C + 2\pi$ and satisfies the Dirichlet's conditions then, function $f(x)$ can be represented by the trigonometric series as $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$. This representation of a function $f(x)$ as a trigonometric series is known as **Fourier series expansion** of function $f(x)$ and its co-efficients a_0, a_n, b_n are called **Fourier coefficients**.

Example :

1) $f(x) = \tan x$ cannot be expanded as a Fourier series in the interval $[0, 2\pi]$ since $\tan\frac{\pi}{2} = \infty$.

2) $f(x) = e^{ax}$ where a is constant can be expressed in terms of Fourier series in any interval.

Note : The Fourier series expansion of $f(x)$ converges to $\frac{1}{2}\left[f(x^+) + f(x^-)\right]$, i.e. $\frac{\textit{Right hand limit} + \textit{left hand limit}}{2}$ at the point of discontinuity.

Calculation of Fourier coefficients :

Let $f(x)$ be a periodic function of period 2π defined in the interval $C \le x \le C+2\pi$ satisfying Dirichlet's conditions then its Fourier series expansion is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \qquad (1)$$

1) To calculate Fourier coefficient a_0, integrate equation (1) from C to $C+2\pi$.

$$\int_C^{C+2\pi} f(x)\,dx = \frac{a_0}{2}\int_C^{C+2\pi} dx + \sum_{n=1}^{\infty}\left[a_n \int_C^{C+2\pi} \cos nx\,dx + b_n \int_C^{C+2\pi} \sin nx\,dx\right]$$

$$\int_C^{C+2\pi} f(x)\,dx = \frac{a_0}{2}[2\pi] + (0+0) = a_0\pi$$

$$\Rightarrow\ a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\,dx$$

2) To determine the Fourier coefficient a_n multiply equation (1) by $\cos nx$ and the integrate from C to $C+2\pi$.

$$f(x)\cos nx = \frac{a_0}{2}\cos nx + \sum_{n=1}^{\infty}\left(a_n \cos^2 nx + b_n \sin nx \cos nx\right)$$

$$\int_C^{C+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_C^{C+2\pi} \cos nx\,dx + \sum_{n=1}^{\infty}\left(a_n \int_C^{C+2\pi} \cos^2 nx dx + b_n \int_C^{C+2\pi} \sin nx \cos nx\ dx\right)$$

$$\Rightarrow\ a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\,dx$$

3) To determine the Fourier coefficient b_n multiply equation (1) by $\sin nx$ and integrate from C to $C+2\pi$.

$$\int_C^{C+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_C^{C+2\pi} \sin nx\,dx + \sum_{n=1}^{\infty}\left[a_n \int_C^{C+2\pi} \cos nx \sin nx\,dx + b_n \int_C^{C+2\pi} \sin^2 nx\right]$$

$$= \frac{a_0}{2}\left[\frac{-\cos nx}{n}\right]_C^{C+2\pi} + \sum_{n=1}^{\infty}\left[a_n \int_C^{C+2\pi} \frac{\sin(2nx)}{2}\,dx + b_n \int_C^{C+2\pi}\left(\frac{1-\cos nx}{n}\right)dx\right]$$

$$\Rightarrow\ b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\,dx$$

Thus we have complete set of formulation for Fourier series expansion of periodic function $f(x)$ of period 2π satisfying Dirichlet's conditions as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n \cos nx+b_n \sin nx\right)$$

where

$$a_0=\frac{1}{\pi}\int_{C}^{C+2\pi} f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_{C}^{C+2\pi} f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{C}^{C+2\pi} f(x)\sin nx\,dx \qquad for\ C\le x\le C+2\pi$$

Note :

(1) If $C=0$ then $0\le x\le 2\pi$ and

$$a_0=\frac{1}{\pi}\int_{0}^{2\pi} f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin nx\,dx \qquad for\ 0\le x\le 2\pi$$

2) If $C=-\pi$ then $-\pi\le x\le\pi$ then

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad for\ -\pi\le x\le\pi$$

1.4 FOURIER SERIES EXPANSION OF EVEN AND ODD FUNCTIONS :

Definition :

The function f is said to be even, if $f(-x)=f(x),\quad \forall x,\ -c\le x\le c$.

The function f is said to be odd, if $f(-x)=-f(x),\quad \forall x,\ -c\le x\le c$.

Example : $\cos\theta$ is even function since $\cos(-\theta) = +\cos\theta$.
$\sin\theta$ is odd function since $\sin(-\theta) = -\sin\theta$.

Property : $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f \text{ is even} \\ 0 & \text{if } f \text{ is odd} \end{cases}$

Hence Fourier series expansion of even function defined in the interval $-\pi \le x \le \pi$ is given by

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = 0$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad -\pi \le x \le \pi$$

This series is also called as **Fourier Cosine series**.

Fourier Series expansion of odd function defined in the interval $-\pi \le x \le \pi$ is given by

$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad -\pi \le x \le \pi$$

This series is also known as **Fourier Sine series**.

1.5 FOURIER SERIES EXPANSION OF A PERIODIC FUNCTION HAVING ARBITRARY PERIOD:

Let $f(x)$ be a periodic function of period $2L$ defined in the interval $C \le x \le C + 2L$ then substitute $z = \frac{\pi x}{L}$ or $x = \frac{zL}{\pi}$

when $x = C \Rightarrow \quad z = \frac{\pi C}{L} = d\ (say)$

when $x = C + 2L \Rightarrow \quad z = \frac{\pi}{L}(C + 2L) = d + 2\pi$

Thus $f(z)$ is a periodic function of period 2π defined in the interval $d \le z \le d + 2\pi$.

Hence Fourier series expansion of a periodic function $f(x)$ of a period 2L defined in the interval $C \le x \le C+2L$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

where the Fourier coefficients are given by

$$a_0=\frac{1}{L}\int_{C}^{C+2L} f(x)\,dx$$

$$a_n=\frac{1}{L}\int_{C}^{C+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n=\frac{1}{L}\int_{C}^{C+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Note :

If $C=-L$ then $-L \le x \le L$. In this case we can verify whether the given periodic function is given either even or odd.

Hence Fourier series expansion of even function defined in the interval $-L \le x \le L$ is given by

$$a_0=\frac{2}{L}\int_{0}^{L} f(x)\,dx$$

$$a_n=\frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n=0$$

$$\therefore\ f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right) \qquad -L \le x \le L$$

This series is also called as **Fourier Cosine series**.

Fourier series expansion of odd function defined in the interval $-\pi \le x \le \pi$ is given by

$$a_0=0,\quad a_n=0,\quad b_n=\frac{2}{\pi}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\therefore\ f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right) \qquad -L \le x \le L$$

This series is also known as **Fourier Sine series**.

Examples

Ex. 1. Find Fourier series expansion of $f(x)=|x| \quad -\pi \le x \le \pi$ and show that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.....=\frac{\pi^2}{8}$.

Solution : $\because f(-x)=|-x|=x=f(x)$

$\therefore f$ is even function.

$$\therefore a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi} x\,dx=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=\pi$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$=\frac{2}{\pi}\left[\left.\frac{x\sin nx}{n}\right|_0^{\pi}-\int_0^{\pi}\frac{\sin nx}{n}\,dx\right]$$

$$=\frac{2}{\pi}\left[0+\frac{1}{n}\left[\frac{\cos nx}{n}\right]_0^{\pi}\right]$$

$$=\frac{2}{\pi}\left[\frac{1}{n^2}\left[(-1)^n-1\right]\right]=\frac{2\left[(-1)^n-1\right]}{n^2\pi}$$

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$=0$$

$$\therefore f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$\therefore |x|=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2\left[(-1)^n-1\right]}{n^2\pi}\cos nx$$

Note that $(-1)^n-1=\begin{cases}0 & if\ \ n\ is\ even\\ -2 & if\ \ n\ is\ \text{odd}\end{cases}$

hence replace n by 2n – 1, we have

$$\therefore |x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2(-2)}{(2n-1)^2 \pi} \cos\left[(2n-1)x\right]$$

$$= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{-2\cos((2n-1)x)}{(2n-1)^2}$$

Put $x = 0$

$$0 = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{-2\cos 0}{(2n-1)^2}$$

$$= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{-2}{(2n-1)^2}$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right]$$

$$\therefore \frac{-4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right] = \frac{-\pi}{2}$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$$

Ex 3. Find Fourier series expansion of $f(x) = x^2 \quad -\pi \le x \le \pi$. Evaluate series at $x = \pi$ and find $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Solution :

$$f(x) = x^2$$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$\therefore f(x)$ is even function

$$\therefore a_0 = \frac{2}{\pi}\int_o^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_o^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi} \times \frac{\pi^3}{3}$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_o^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} \Big|_o^{\pi} - \int_o^{\pi} \frac{\sin nx}{n}(2x)\,dx\right]$$

$$= \frac{2}{\pi}\left[0 - \frac{2}{n}\int_o^{\pi} x \sin nx\,dx\right]$$

$$=\frac{2}{\pi}\left[\frac{2}{n}\left[x\left(\frac{-\cos nx}{n}\right)\Big|_{o}^{\pi}-\int_{o}^{\pi}\frac{\cos nx}{n}1\,dx\right]\right]$$

$$=\frac{-2}{\pi}\left[\frac{2}{n}\left(-\pi\frac{(-1)^n}{n}+\frac{1}{n}\frac{\sin nx}{n}\Big|_{o}^{\pi}\right)\right]$$

$$=\frac{-2}{\pi}\left[\frac{2}{n}\left(\frac{-\pi(-1)^n}{n}-\frac{1}{n}(0)\right)\right]$$

$$=\frac{4}{\pi n}\left(\pi\frac{(-1)^n}{n}\right)$$

$$=\frac{4(-1)^n}{n^2}$$

$$b_n=0$$

$\therefore$ The Fourier Cosine series is given by

$$f(x)=\frac{a_o}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

$$\therefore x^2=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

$at\ x=\pi$

$$\pi^2=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}(-1)^n=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^{2n}}{n^2}$$

$$\therefore\sum_{n=1}^{\infty}\frac{4}{n^2}=\pi^2-\frac{\pi^2}{3}=\frac{2\pi^2}{3}$$

$$\therefore\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Ex. 4. Compute Fourier series of $f(x)=e^{ax}$ where a is +ve and hence prove that

$$1=\frac{\sinh\pi a}{\pi a}\left[1+2\sum_{n=1}^{\infty}(-1)^n\frac{a^2}{n^2+a^2}\right]$$

Solution : Let $f(x) = e^{ax}$

$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_{-\pi}^{\pi} = \frac{2}{\pi a}\left[\frac{e^{\pi a} - e^{-\pi a}}{2}\right]$$

$$= \frac{2}{\pi a}\sinh \pi a$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

Let $a_n = I = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx\,dx$

$$= \frac{1}{\pi}\left[\cos nx\,\frac{e^{ax}}{a}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{e^{ax}}{a}(-\sin nx)\,n\,dx\right]$$

(by LIATE)

$$= \frac{1}{\pi}\left[\cos n\pi\,\frac{e^{a\pi}}{a} - \cos n\pi\,\frac{e^{-a\pi}}{a} + \frac{n}{a}\int_{-\pi}^{\pi}\sin nx\,e^{ax}\,dx\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\frac{e^{a\pi}}{a} - (-1)^n\frac{e^{-a\pi}}{a} + \frac{n}{a}\left(\sin nxa\frac{e^{ax}}{a}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{e^{ax}}{a}n\cos nx\right)\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\frac{e^{a\pi}}{a} - (-1)^n\frac{e^{-a\pi}}{a} + \frac{n}{a}\left(\sin nx\frac{e^{ax}}{a}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{e^{ax}}{a}n\cos nx\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{a}\left(e^{a\pi} - e^{-a\pi}\right) + \frac{n}{a}\times\left(-\frac{n}{a}\int_{-\pi}^{\pi}e^{ax}\cos nx\,dx\right)\right]$$

$$= \frac{(-1)^n}{\pi a}\left(e^{a\pi} - e^{-a\pi}\right) - \frac{n^2}{a^2}\frac{1}{\pi}\int_{-\pi}^{\pi}e^{ax}\cos nx\,dx$$

$$I = \frac{(-1)^n}{\pi a}\left(e^{a\pi} - e^{-a\pi}\right) - \frac{n^2}{a^2}I$$

$$\therefore I + \frac{n^2}{a^2}I = \frac{(-1)^n}{\pi a}\left(e^{a\pi} - e^{-a\pi}\right)$$

$$\left(\frac{a^2 + n^2}{a^2}\right)I = \frac{(-1)^n}{\pi a}\left(e^{a\pi} - e^{-a\pi}\right)$$

$$\therefore a_n = \frac{a^{\not 2}}{\left(a^2 + n^2\right)}\times\frac{(-1)^n}{\pi \not a}\left(e^{a\pi} - e^{-a\pi}\right)$$

$$= \frac{a}{\left(a^2 + n^2\right)}\frac{(-1)^n}{\pi}\left(e^{a\pi} - e^{-a\pi}\right) = \frac{2a(-1)^n}{\left(a^2 + n^2\right)\pi}\sinh \pi a$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

(by LIATE)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax} \sin nx\, dx$$

$$\therefore b_n = \frac{1}{\pi}\left[\sin nx \frac{e^{ax}}{a}\right]_{\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{ax}}{a} \cos nx.ndx$$

$$= \frac{1}{\pi}\left[0 - \frac{n}{a}\int_{-\pi}^{\pi} e^{ax} \cos nx.ndx\right]$$

$$= \frac{-n}{a} \times \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax} \cos nx\, dx$$

$$= \frac{-n}{\cancel{a}} \times \frac{\cancel{a}\,(-1)^n}{\left(a^2+n^2\right)\pi}\left(e^{a\pi} - e^{-a\pi}\right)$$

$$= \frac{n(-1)^{n+1}}{\left(a^2+n^2\right)\pi}\left(e^{a\pi} - e^{-a\pi}\right) = \frac{2n(-1)^{n+1}}{\left(a^2+n^2\right)\pi}\sinh \pi a$$

Thus the Fourier series expansion of f is given by

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$e^{ax} = \frac{\sinh \pi a}{\pi a} + \sum_{n=1}^{\infty}\left(\frac{2a(-1)^n \sinh \pi a}{\left(a^2+n^2\right)\pi}\right)\cos nx + \left(\frac{2n(-1)^{n+1}\sinh \pi a}{\left(a^2+n^2\right)\pi}\right)\sin nx$$

$at\ x = 0$

$$1 = \frac{\sinh \pi a}{\pi a} + \sum_{n=1}^{\infty}\frac{2a(-1)^n}{\left(a^2+n^2\right)\pi}\sinh \pi a$$

$$\therefore 1 = \frac{\sinh \pi a}{\pi a}\left[1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n a^2}{\left(a^2+n^2\right)}\right]$$

$$\therefore 1 = \frac{\sinh \pi a}{\pi a}\left[1 + 2\sum_{n=1}^{\infty}(-1)^n \frac{a^2}{\left(a^2+n^2\right)}\right]$$

Hence proved

Ex. 6. Show that $\frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(x)\right]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n{}^2 + b_n{}^2\right)$

where a_n & b_n are Fourier coefficients of Fourier series expansion of periodic function f defined in $[-\pi, \pi]$

(This is known as **Parseval's Identity**)

Solution: The Fourier series expansion of a periodic function $f(x)$ of period 2π defined in the interval $-\pi \le x \le \pi$ satisfying Dirichlet's conditions is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

On squaring both sides we get

$$f^2(x) = \frac{a_0^2}{4} + \sum_{n=1}^{\infty} a_n^2 \cos^2 nx + \sum_{n=1}^{\infty} b_n^2 \sin^2 nx$$
$$+\sum_{n=1}^{\infty} a_0 a_n \cos nx + \sum_{n=1}^{\infty} a_0 b_n \sin nx + 2\sum_{n=1}^{\infty} a_n b_n \sin nx\, conx$$

Assuming term by term integration on R.H.S. of above equation is permissible.

Integrating both side of above equation with the limit $-\pi$ to π.

$$\int_{-\pi}^{\pi}\left[f(x)\right]^2 dx = \int_{-\pi}^{\pi}\frac{a_0^2}{4}\,dx + \sum_{n=1}^{\infty} a_n^2 \int_{-\pi}^{\pi}\cos^2 nx\, dx$$
$$+\sum_{n=1}^{\infty} b_n^2 \int_{-\pi}^{\pi}\sin^2 nx\, dx + + \sum_{n=1}^{\infty} a_0 a_n \int_{-\pi}^{\pi}\cos nx\, dx$$
$$+\sum_{n=1}^{\infty} a_0 b_n \int_{-\pi}^{\pi}\sin nx dx + 2\sum_{n=1}^{\infty} a_n b_n \int_{-\pi}^{\pi}\sin nx \cos nx\, dx$$

Using orthogonality relations we get

$$\therefore \int_{-\pi}^{\pi}\left[f(x)\right]^2 dx = \frac{a_0^2\pi}{2} + \pi\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(x)\right]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left({a_n}^2 + {b_n}^2\right)$$

This relation is known as **Parseval's Identity**.

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BASIC PROPERTIES OF FOURIER SERIES

Unit Structure

2.1 COMPLEX FORM OF FOURIER SERIES :

Let $f(x)$ be a periodic function of period 2π defined in the interval $C \le x \le C + 2\pi$ then its Fourier series expansion is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

We have $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + b_n \left(\frac{e^{inx} - e^{-inx}}{2i} \right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - ib_n}{2} \right) e^{inx} + \left(\frac{a_n + ib_n}{2} \right) e^{-inx} \right]$$

Setting $\dfrac{a_0}{2} = C_0$

$$\frac{a_n - ib_n}{2} = C_n \quad ; \frac{a_n + ib_n}{2} = C_{-n}$$

$$\therefore f(x) = C_0 + \sum_{n=1}^{\infty} \left(C_n e^{inx} + C_{-n} e^{-inx} \right)$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

This is **Complex form of Fourier series** where, C_n is Fourier coefficient which is given by, $C_n = \frac{a_n - ib_n}{2}$.

Using value of Fourier coefficient a_n & b_n we can simplify for C_n as.

$$C_n = \frac{1}{2}\left[\frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\,dx - \frac{i}{\pi}\int_C^{C+2\pi} f(x)\sin nx\,dx\right]$$

$$C_n = \frac{1}{2\pi}\left[\int_C^{C+2\pi} f(x)\ (\cos nx - i\sin nx)\,dx\right]$$

$$C_n = \frac{1}{2\pi}\int_C^{C+2\pi} f(x)\,e^{-inx}\,dx$$

This is general formula for **Fourier coefficient in the complex form**.

Note :

1) The Fourier series coefficients C_n in complex form is also denoted by $\hat{f}(n)$.

i.e. $\hat{f}(n) = C_n = \frac{1}{2\pi}\int_C^{C+2\pi} f(x)\,e^{-inx}dx$

2) If $f(x)$ is a periodic function of period 2π defined in the interval $-\pi \le x \le \pi$ then $\hat{f}(n) = C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}dx$.

3) We have $C_n = \frac{a_n - ib_n}{2}$ and $C_{-n} = \frac{a_n + ib_n}{2}$

$\Rightarrow C_n + C_{-n} = a_n$

$C_n - C_{-n} = ib_n$

$b_n = -i(C_n - C_{-n})$

$C_o = \frac{a_o}{2}$

4) Similarly, we can find the Fourier series expansion of a periodic function $f(x)$ of arbitrary period 2L defined in the interval $C \le x \le C + 2L$ in complex form of as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{\frac{in\pi x}{L}}$$

where,

$$\hat{f}(n) = C_n = \frac{1}{2L}\int_{C}^{C+2L} f(x)\, e^{\frac{-in\pi x}{L}}\, dx$$

Ex. 1. Find complex form of Fourier series of
$g(\theta) = \theta \quad -\pi \le \theta \le \pi$

Solution: We have Fourier coefficient in Complex Fourier series expansion as

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, e^{-in\theta} d\theta$$

$$= \frac{1}{2\pi}\int_{\pi}^{\pi} \theta\, e^{-in\theta} d\theta$$

$$= \frac{1}{2\pi}\left[\theta\, \frac{e^{-in\theta}}{-in}\Bigg|_{-\pi}^{\pi} - \frac{e^{-in\theta}}{-in} d\theta\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\pi\, \frac{e^{-in\pi}}{-in} + \pi\, \frac{e^{-in\pi}}{-in} + \frac{1}{in}\left[e^{-in\theta}\right]_{-\pi}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\pi\, \frac{e^{-in\pi}}{-in} + \frac{-\pi e^{-in\pi}}{-in} + \frac{1}{n^2}\left(e^{in\pi} - e^{in\pi}\right) + \frac{1}{n^2}\left(\cos n\pi - i\sin n\pi - \cos n\pi - i\sin n\pi\right)\right]$$

$$= \frac{1}{2\pi}\left[\pi\, \frac{(-1)^n}{-in} + \frac{\pi}{-in}(-1)^n\right]$$

$$= \frac{1}{2\pi}\left[2\pi\, \frac{(-1)^{n+1}}{in}\right]$$

$$= \frac{(-1)^{n+1}}{in}$$

$$g(\theta) = \sum_{n=-\infty}^{\infty} C_n e^{in\theta}$$

$$\theta = \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} e^{in\theta}}{in}$$

$At\ n = 0$

To find the value C_0 consider Fourier coefficient in complex form

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, e^{-in\theta} d\theta$$

$Put\ n = 0$

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, d\theta = 0 \text{ …….. \{since g is odd function\}.}$$

Thus complex form of Fourier series of a given function is given by

$$\theta = \sum_{\substack{n=-\infty \\ n\neq o}}^{\infty} \frac{(-1)^{n+1} e^{in\theta}}{in}$$

$$g(\theta) = 0 \quad at\ n = 0$$

We have

$$g(\theta) = \theta = \sum_{\substack{n=-\infty \\ n\neq o}}^{\infty} \frac{(-1)^{n+1} e^{in\theta}}{in}$$

$$g(\theta) = 0 \qquad at\ \ n = 0$$

Since $(-1)^n = (-1)^{-n}$ as n varies from -ve to +ve integer.
Hence we can combine n^th term & (-n^th) term as.

$$\frac{(-1)^{n+1} e^{in\theta}}{in} = (-1)^{n+1}\left(\frac{e^{in\theta}}{in} + \frac{e^{-in\theta}}{-in}\right)$$

$$= \frac{(-1)^{n+1}}{n}\left(-ie^{in\theta} + ie^{-in\theta}\right)$$

$$= \frac{(-1)^{n+1}}{n} i\left[-e^{in\theta} + e^{-in\theta}\right]$$

$$= \frac{(-1)^{n+1}}{n} i\left[-\left(\cancel{\cos n\theta} + i\sin n\theta\right) + \left(\cancel{\cos n\theta} - i\sin n\theta\right)\right]$$

$$= \frac{(-1)^{n+1}}{n} i\left(-2i\sin n\theta\right)$$

$$= \frac{2(-1)^{n+1}}{n}\sin n\theta$$

$$\therefore g(\theta) = \theta = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin \mathrm{n}\theta$$

Note : The function $g(\theta) = \theta \quad -\pi \le \theta \le \pi$ is odd function. Hence we can expand this function in terms of Fourier Sine series.

Ex. 2. Show that $\sum_{n=-\infty}^{\infty} |C_n|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$

where C_n is complex Fourier coefficient of Fourier series expansion of periodic function f defined in $[-\pi, \pi]$

(This relation is known as **Bessel's Inequality**.)

Solution : The complex form of Fourier series expansion of periodic function $f(x)$ is given by $f(x)=\sum_{n=-\infty}^{\infty} C_n e^{inx}$.

We have property of the complex number z, $|z|^2 = z\bar{z}$

Consider,

$$\left| f(x) - \sum_{-N}^{N} C_n e^{inx} \right|^2 = \left(f(x) - \sum_{-N}^{N} C_n e^{inx} \right)\left(\overline{f(x)} - \sum_{-N}^{N} \overline{C}_n\, e^{-inx} \right)$$

$$= f(x)\,\overline{f(x)} - \sum_{-N}^{N}\left[C_n\, \overline{f(x)}\, e^{inx} + \overline{C}_n\; f(x) e^{-inx} \right] + \sum_{m=n-N}^{N} C_m \overline{C}_n\; e^{i(m-n)x}$$

Divide both side of above equation by 2π and integrate within limit $-\pi$ to π also using

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = C_n \;\&\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)}\, e^{-inx}\, dx = \overline{C}_n \text{ and}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x} dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

We obtains

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| f(x) - \sum_{-N}^{N} C_n e^{inx} \right|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx$$

$$- \sum_{-N}^{N}\left(C_n \overline{C}_n + \overline{C}_n C_n \right) + \sum_{-N}^{N} C_n \overline{C}_n$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx - \sum_{-N}^{N} 2|C_n|^2 + \sum_{-N}^{N} |C_n|^2$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx - \sum_{-N}^{N} |C_n|^2$$

$$\geq 0$$

$$\Rightarrow \sum_{-N}^{N} |C_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx$$

Letting $N \to \infty$ *we get*

$$\sum_{n=-\infty}^{\infty} |C_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx$$

where C_n is complex Fourier coefficient.

This relation is known as **Bessel's Inequality**.

Note :

1) $\frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right)$

$$= \sum_{n=-\infty}^{\infty} |C_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx .$$

2) From above Bessel's Inequality the series $\sum|a_n|^2$, $\sum|b_n|^2$ $\sum|C_n|^2$ are convergent.

2.2 PROPERTIES OF FOURIER COEFFICIENT

The following statements are equivalent

1) 2π Periodic function on R like exponential function.
2) Function defined on the interval of length 2π .
3) Function defined on the unit circle.

Since a point on the unit circle takes the form $e^{i\theta}$, θ is real and unique up to integer multiple of 2π . If F is a function on the circle then we may define for each real number θ

$$f(\theta) = F(e^{i\theta})$$

Observe that $f(\theta + 2\pi) = f(\theta)$ for all θ .

Thus f is periodic of period 2π . The integrability, continuity and other smoothness properties of F are determined by those of f.

Definition : The Fourier coefficient of an integrable periodic function f are the complex number $\hat{f}(n)$ defined by the integral.

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} \, dx, \qquad n \in z$$

The L^1 norm of an integrable periodic function f is given by

$$\|f\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)| \, dx .$$

The L^2 norm of square integrable periodic function f is given by

$$\|f\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx\right)^{\frac{1}{2}} .$$

Properties of Fourier Coefficient:

Theorem 1: Suppose that f is an integrable periodic function then $\left|\hat{f}(n)\right| \leq \|f\|_1, \qquad \forall\, n \in Z$.

Proof:

We have,

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

Taking mod on both sides

$$\left|\hat{f}(n)\right| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx\right|$$

$$\left|\hat{f}(n)\right| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(x)\right|\left|e^{-inx}\right| dx \qquad \left\{\because \left|\int f\right| \le \int |f|\right\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(x)\right|\left|e^{-inx}\right| dx$$

since

$$\left|e^{-inx}\right| = \left|\cos nx - i\sin nx\right| = \sqrt{\cos^2 nx + \sin^2 nx} = 1$$

$$\therefore \left|\hat{f}(n)\right| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(x)\right| dx = \|f\|_1$$

$$\therefore \left|\hat{f}(n)\right| \le \|f\|_1 \qquad \forall n \in Z$$

Theorem 2: Translation Property : Suppose that f is an integrable periodic function. Given a in R. Let f_a translate function f as $f_a(x) = f(x-a)$ then $\hat{f}_a(n) = e^{-ina}\,\hat{f}(n) \;\; \forall \;\; n \in Z.$

Proof : We have,

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$\Rightarrow \hat{f}_a(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_a(x)\, e^{-inx}\, dx$$

$$\therefore \hat{f}_a(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-a)\, e^{-inx}\, dx \qquad \left\{\because f_a(x) = f(x-a)\right\}$$

Put $x - a = y \Rightarrow x = a + y$

$dx = dy$

when $x = -\pi, \quad y = -\pi - a$

when $x = \pi, \quad y = \pi - a$

$$\therefore \hat{f}_a(n) = \frac{1}{2\pi}\int_{-\pi-a}^{\pi-a} f(y)\, e^{-in(a+y)}\, dy$$

$$= \frac{e^{-ina}}{2\pi}\int_{-\pi-a}^{\pi-a} f(y)\, e^{-iny}\, dy$$

Since f is periodic function of period 2π .

$$\therefore \hat{f}_a(n) = e^{-ina}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, e^{-iny} dy\right)$$

$$\hat{f}_a(n) = e^{-ina}\ \hat{f}_a(n)$$

Theorem 3: Suppose that f is continuous function with continuous derivative f' then $\hat{f}'(n) = in\ \hat{f}(n)$, $\forall\, n \in \mathbb{Z}$.

Proof : We have,

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

On integrating by parts

$$\hat{f}(n) = \frac{1}{2\pi}\left[\left(\frac{f(x)\, e^{-inx}}{-in}\right)_{-\pi}^{\pi} + \int_{-\pi}^{\pi} f'(x)\frac{e^{-inx}}{in}\, dx\right]$$

Since f is periodic function of period 2π , we have

$$f(-\pi) = f(-\pi + 2\pi) = f(\pi)$$

The 1st term in above equation vanishes

$$\therefore \hat{f}(n) = \frac{1}{2\pi}\left[\frac{1}{in}\int_{-\pi}^{\pi} f'(x)\, e^{-inx}\, dx\right]$$

$$= \frac{1}{in}\times\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\, e^{-inx}\, dx$$

$$= \frac{1}{in}\hat{f}'(n) \qquad \left\{\therefore \hat{f}'(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\, e^{-inx}\, dx\right\}$$

$$\therefore \hat{f}'(n) = in\,\hat{f}(n)\ , \qquad \forall\ n \in \mathbb{Z}$$

Notation :

$\hat{f}(n) = O\left(\dfrac{1}{|n|^2}\right)$ as $|n| \to \infty$ means L.H.S. is bounded by constant multiple of R.H.S. i.e. there exist constant $C > 0$ such that

$$\left|\hat{f}(n)\right| \le \frac{C}{|n|^2} \quad \forall \text{ large } |n|.$$

In general, $f(x) = O\left[g(x)\right]$ as $x \to a$ means for some +ve constant C, $\left|f(x)\right| \le C\left|g(x)\right|$ as $x \to a$.

Note : $f(x) = O(1)$ means f is bounded function.

Theorem 4: Suppose that function f is twice continuously differentiable function defined on the circle then $\left|\hat{f}(n)\right| = O\left(\frac{1}{|n|^2}\right)$ as $|n| \to \infty$. So that Fourier series of f converges absolutely & uniformly to f.

Proof : We have

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

Integrating R.H.S. by part

$$\hat{f}(n) = \frac{1}{2\pi}\left[\left(f(x)\frac{e^{-inx}}{-in}\right)_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{e^{-inx}}{-in} f'(x)\, dx\right]$$

$$2\pi\ \hat{f}(n) = \left(f(x)\frac{e^{-inx}}{-in}\right)_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{e^{-inx}}{-in} f'(x)\, dx$$

Since f is periodic function with period 2π. 1st term of R.H.S. Vanishes

$$\therefore 2\pi\,\hat{f}(n) = \frac{+1}{in}\int_{-\pi}^{\pi} e^{-inx} f'(x)\, dx$$

Once again integrating by parts,

$$2\pi\ in\ \hat{f}(n) = \left(f'(x)\frac{e^{-inx}}{-in}\right)_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{e^{-inx}}{-in} f''(x)\, dx$$

$$2\pi\ \hat{f}(n) = \left(f'(x)\frac{e^{-inx}}{-in}\right)_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{e^{-inx}}{-in} f''(x)\, dx$$

Since f' is periodic and

$$\because \left(e^{-inx}\right)_{-\pi}^{\pi} = e^{-in\pi} - e^{in\pi} = (\cos n\pi - i\sin n\pi) - (\cos n\pi + i\sin n\pi) = 0$$

$$\therefore 2\pi\, n^2\ \hat{f}(n) = -\int_{-\pi}^{\pi} f''(x) e^{-inx}\, dx$$

$$\left|2\pi\, n^2\ \hat{f}(n)\right| = \left|-\int_{-\pi}^{\pi} f''(x) e^{-inx}\, dx\right|$$

$$\therefore 2\pi\, n^2 \left|\hat{f}(n)\right| \le \int_{-\pi}^{\pi}\left|f''(x) e^{-inx}\right| dx \qquad \left(\because \left|e^{-inx}\right| = 1\right)$$

$$\therefore 2\pi\, n^2 \left|\hat{f}(n)\right| \le \int_{-\pi}^{\pi}\left|f''(x)\right| dx. \le C.$$

where C is a constant and independent of n. and since f is twice continuously differentiable, f'' is bounded function.

Setting $C = 2\pi B$ where, B is bound of f''

$$\therefore 2\pi\, n^2 \left|\hat{f}(n)\right| \le 2\pi B$$

$$\left|\hat{f}(n)\right| \le \frac{B}{|n|^2}$$

$$\therefore \left|\hat{f}(n)\right| = o\left(\frac{1}{|n|^2}\right) \text{ as } |n| \to \infty$$

2.3 THE RIEMANN - LEBESGUE LEMMA :

Statement : If f is integrable function defined on a circle then $\hat{f}(n) \to 0$ as $|n| \to \infty$.

OR

If f is integrable periodic function of period 2π then $\lim_{|n|\to\infty} \hat{f}(n) = 0$.

Proof : Since for any $\in > 0$, we can choose a continuous periodic function g with $\|f - g\| < \in$.

Since $\left|\hat{f}(n)\right| \le \|f\|, \quad \forall\ n \in Z$

$$\Rightarrow \left|\hat{f}(n) - \hat{g}(n)\right| \le \|f - g\| < \in \qquad (1)$$

i.e. the Fourier coefficient of function f and g differ by less than $\in$. So that $\hat{f}(n)$ are eventually less than $\in$ in modulus if $\hat{g}(n) \to 0$ as $|n| \to \infty$.

If g is continuous periodic function and $a \in \mathbb{R}$ then we have

$$g_a(x) = g(x - a)$$

$$\Rightarrow \hat{g}_a(n) = e^{-ina}\, \hat{g}(n),\ \forall\, n \in Z \qquad (2)$$

Choose $a = \dfrac{\pi}{n}$

$$\hat{g}_a(n) = e^{-in\frac{\pi}{n}}\, \hat{g}(n)$$

$$\therefore \hat{g}_a(n) = (-1)\, \hat{g}(n) \qquad (3)$$

We have,

$$\left|\hat{g}(n)\right| \le \|g\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)|\, dx \qquad (4)$$

Now consider,

$$|2\hat{g}(n)| = |\hat{g}(n) + \hat{g}(n)|$$

$$= |\hat{g}(n) - \hat{g}_a(n)| \qquad \text{(by equation (3))}$$

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x) - g_a(x)|\, dx \qquad \text{(by equation (4))}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x) - g(x-a)|\, dx$$

Put $a = \frac{\pi}{n}$

$$|2\hat{g}(n)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|g(x) - g\left(x - \frac{\pi}{n}\right)\right| dx$$

As $|n| \to \infty \quad a = \frac{\pi}{n} \to 0$

hence, $\left|g(x) - g\left(x - \frac{\pi}{n}\right)\right| \to 0$

$\Rightarrow |2\hat{g}(n)| \to 0$ as $|n| \to \infty$

$\therefore |\hat{g}(n)| \to 0$ as $|n| \to \infty$

$\therefore$ By (1), $\left|\hat{f}(n)\right| \to 0$ as $|n| \to \infty$

$$\therefore \lim_{|n| \to \infty} \hat{f}(n) = 0$$

hence proof.

2.4 GOOD KERNELS :

Definition : A family of Kernels $\{K_n(x)\}_{n=1}^{\infty}$ defined on the circle is said to be family of good Kernel if it satisfies the following property

1) for all $n \ge 1$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(X)\, dx = 1$

2) There exist M > 0 Such that for $n \ge 1$

$$\int_{-\pi}^{\pi} |K_n(x)|\, dx \le M$$

3) for every $\delta > 0$, $\int\limits_{\delta \le |x| \le \pi} |K_n(x)|\, dx \to o$ as $n \to \infty$

Convolution : Let f and g be 2π periodic integrable functions then the convolution of function f and g on interval $[-\pi, \pi]$ is denoted and defined as

$$(f*g)(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,g(x-y)\,dy$$

OR

$$(f*g)(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\,g(y)\,dy$$

Note : $(f*g)=(g*f)$

Theorem : Let $\{K_n\}_{n=1}^{\infty}$ be a family of Good Kernels and f is an integrable periodic function defined on the circle then

$\lim_{n\to\infty}(f*K_n)(x)=f(x)$ whenever, f is continuous at x.

If f is continuous everywhere then the above limit is uniform.

Proof : If $\in>0$ and f is continuous at x then we can choose δ, So that $|y|<\delta$.

$$\Rightarrow |f(x-y)-f(x)|<\in \qquad (1)$$

Consider,

$$(f*K_n)(x)-f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\,f(x-y)\,dy-f(x)$$

(Definition of convolution)

As K_n is a good Kernel $\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\,dy=1$

$$\therefore (f*K_n)(x)-f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\,f(x-y)\,dy-\frac{1}{2\pi}f(x)\int_{-\pi}^{\pi} K_n(y)\,dy$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\left[f(x-y)-f(x)\right]dy$$

$$\therefore \left|(f*K_n)(x)-f(x)\right|=\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\left[f(x-y)-f(x)dy\right]\right|$$

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi}|K_n(y)|\,|f(x-y)-f(x)|\,dy$$

$$\le \frac{1}{2\pi}\int_{|y|<\delta}|K_n(y)|\,|f(x-y)-f(x)|\,dy$$

$$+\frac{1}{2\pi}\int_{\delta\le|y|\le\pi}|K_n(y)|\,|f(x-y)-f(x)|\,dy$$

Note that

$$|y| < \delta \Rightarrow -\delta < y < \delta$$

$$\delta < |y| < \pi \Rightarrow -\pi < y < -\delta \quad \& \quad \delta < y < \pi$$

$$\therefore \left|\left(f * K_n\right)(x) - f(x)\right| \le \frac{\in}{2\pi} \int_{|y|<\delta} \left|K_n(y)\right| dy + \frac{2B}{2\pi} \int_{\delta \le |y| \le \pi} \left|K_n(y)\right| dy \qquad (2)$$

Clearly, 1st term is bounded by $\frac{\in M}{2\pi}$ (by 2nd property of good Kernel) and by 3rd property of Good Kernel for large value of n, 2nd term will be less than $\in$.

Hence for some constant C we have, $\left|\left(f * K_n\right)(x) - f(x)\right| \le C \in$

$$\Rightarrow \left(f * K_n\right)(x) \to f(x) \quad \text{as} \quad n \to \infty.$$

If f is continuous everywhere then is it uniformly continuous.

Hence, δ can be chosen independent of x which proves desired conclusion.

$$f * K_n \to f$$

i.e. $\lim_{n \to \infty} \left(f * K_n\right)(x) = f(x)$

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3

DIRICHLET KERNEL

Unit Structure

3.1 Dirichlet's Kernel

3.2 Properties of Dirichlet's Kernel

3.3 Dirichlet Theorem on point wise convergence of Fourier series

3.1 DIRICHLET'S KERNEL :

We have complex form of a Fourier series expansion of a periodic function f of a period 2π defined on $[-\pi, \pi]$.

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta} \qquad (1)$$

The **Nth partial sum of Fourier series** expansion of a series (1) is denoted and defined as,

$$S_N f(\theta) = \sum_{n=-N}^{N} \hat{f}(n) e^{in\theta} \qquad (2)$$

We have Fourier series coefficient.

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\Psi) e^{-in\Psi} d\Psi \qquad (3)$$

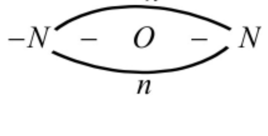
Using equation (3) in equation (2) we have,

$$S_N f(\theta) = \sum_{n=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\Psi) e^{-in\Psi} d\Psi \;\; e^{in\theta}$$

$$= \frac{1}{2\pi} \sum_{-N}^{N} \left(\int_{-\pi}^{\pi} f(\Psi) e^{-in\Psi} d\Psi \right) \;\; e^{in\theta}$$

$$= \frac{1}{2\pi} \sum_{N=-N}^{N} \left(\int_{-\pi}^{\pi} f(\Psi) e^{-in(\Psi-\theta)} d\Psi \right)$$

$$\therefore S_N f(\theta) = \frac{1}{2\pi} \sum_{-N}^{N} \left(\int_{-\pi}^{\pi} f(\Psi) e^{-in(\Psi-\theta)} d\Psi \right)$$



Put $\Psi - \theta = \Phi$, $d\Psi = d\Phi$
When $\Psi = -\pi$, $\Phi = -\pi - \theta$
When $\Psi = \pi$, $\Phi = \pi - \theta$

$$S_N f(\theta) = \frac{1}{2\pi} \sum_{-N}^{N} \int_{-\pi-\theta}^{\pi-\theta} f(\theta + \Phi) e^{-in\Phi} d\Phi$$

Since f is periodic function of period 2π

$$\therefore S_N f(\theta) = \frac{1}{2\pi} \sum_{-N}^{N} \int_{-\pi}^{-\pi} f(\theta + \Phi) e^{-in\Phi} d\Phi$$

$$S_N f(\theta) = \frac{1}{2\pi} \int_{-\pi}^{-\pi} f(\theta + \Phi) D_N(\Phi) d\Phi \qquad (4)$$

where $D_N(\Phi) = \sum_{-N}^{N} e^{in\Phi}$ (5)

and it is known as **N**$^{\mathbf{th}}$ **Dirichlet Kernel**.

Equation (4) represents Nth partial sum of Fourier series in terms of Dirichlet Kernel.

3.2 PROPERTIES OF DIRICHLET'S KERNEL :

Theorem 1: The Nth Dirichlets's kernel is given by

$$D_N(\Phi) = \sum_{-N}^{N} e^{in\Phi} = \frac{\sin\left(N + \frac{1}{2}\right)\Phi}{\sin\frac{1}{2}\Phi}$$

Proof : We have

$$D_N(\Phi) = \sum_{-N}^{N} e^{in\Phi}$$

$$D_N(\Phi) = \left(e^{-iN\Phi} + e^{-i(N-1)\Phi} + e^{-i(N-2)\Phi} + ... + e^0 + e^{i\Phi} + e^{2i\Phi} + ... + e^{iN\Phi}\right)$$

$$= e^{-iN\Phi}\left(1 + e^{i\Phi} + e^{2i\Phi} + ... + e^{iN\Phi} + e^{(nH)i\Phi} + e^{i(N+2)\Phi} + ... + e^{2iN\Phi}\right)$$

$$= e^{-iN\Phi} \sum_{n=0}^{2N} e^{in\Phi}$$

$$= e^{-iN\Phi} \sum_{n=0}^{2N} \left(e^{i\Phi}\right)^n$$

The above series is a geometric series with first term *a=1* and common ratio $= r = e^{i\Phi}$, $\forall r \neq 1$.
we have

$$\sum_{n=0}^{K} r^n = \frac{r^{K+1} - 1}{r - 1}$$

$$\therefore D_N(\Phi) = e^{-iN\varnothing}\left(\frac{\left(e^{i\Phi}\right)^{2N+1} - 1}{e^{i\Phi} - 1}\right)$$

$$\therefore D_N(\Phi) = \left(\frac{e^{i(N+1)\Phi} - e^{-iN\Phi}}{e^{i\Phi} - 1}\right)$$

Multiply Numerator as well as Denominator by $e^{-i\Phi/2}$

$$\therefore D_N(\Phi) = \left(\frac{e^{i(N+1)\Phi} - e^{-iN\Phi}}{e^{i\Phi} - 1}\right) \times \frac{e^{-i\Phi/2}}{e^{-i\Phi/2}}$$

$$= \frac{e^{i\left(N+\frac{1}{2}\right)\Phi} - e^{-i\left(N+\frac{1}{2}\right)\Phi}}{e^{i\Phi/2} - e^{-i\Phi/2}}$$

$$D_N(\Phi) = \frac{\dfrac{e^{i\left(N+\frac{1}{2}\right)\Phi} - e^{-i\left(N+\frac{1}{2}\right)\Phi}}{2i}}{\dfrac{e^{i\Phi/2} - e^{-i\Phi/2}}{2i}}$$

$$\therefore D_N(\Phi) = \frac{\sin\left(N+\frac{1}{2}\right)\Phi}{\sin\frac{1}{2}\Phi} \qquad \ldots\left\{\frac{e^{i\Phi} - e^{-i\Phi}}{2i} = \sin\Phi\right\}$$

Theorem 2: Suppose that f is periodic and integrable then nth partial sum of Fourier series expansion of f is given by

$$S_N(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x-y)\, f(y)\, dy = \frac{1}{2\pi} D_N(y)\, f(x-y)\, dy$$

i.e. $\quad S_N(f)(x) = (D_N * f)(x) = (f * D_N)(x)$

Proof : The Nth partial sum of Fourier series is given by

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx} \qquad (1)$$

where $\hat{f}(n)$ is a Fourier coefficient given by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, e^{-iny}\, dy \qquad (2)$$

Put (2) in (1) we get

$$S_N(f)(x) = \sum_{-N}^{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, e^{-iny}\, dy\right) e^{inx}$$

$$S_N(f)(x) = \frac{1}{2\pi}\sum_{-N}^{N}\left(\int_{-\pi}^{\pi} f(y)\, e^{in(x-y)}\, dy\right) \qquad (3)$$

Put $x-y=z$
$dy=-dz$
When $y=\pi, \quad z=x-\pi$
When $y=-\pi, \quad z=x+\pi$

$$\therefore S_N(f)(x)=\frac{1}{2\pi}\sum_{-N}^{N}\left(\int_{x+\pi}^{x-\pi} f(x-z)\,e^{inz}\,(-dz)\right)$$
$$=\frac{1}{2\pi}\sum_{-N}^{N}\left(\int_{x-\pi}^{x+\pi} f(x-z)\,e^{inz}\,dz\right)$$

Since f is periodic function of period 2π defined on the interval $[-\pi,\pi]$

$$\therefore S_N(f)(x)=\frac{1}{2\pi}\sum_{-N}^{N}\left(\int_{-\pi}^{+\pi} f(x-z)\,e^{inz}\,dz\right)$$
$$\therefore S_N(f)(x)=\frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi}^{\pi} f(x-y)\,e^{iny}\,dy \qquad (4)$$

Put (3) and (4) we get,

$$S_N(f)(x)=\frac{1}{2\pi}\sum_{-N}^{N}\left(\int_{-\pi}^{\pi} f(y)\,e^{in(x-y)}\,dy\right)=\frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi}^{\pi} f(x-y)\,e^{iny}\,dy$$

Since $D_N(x)=\sum_{-N}^{N} e^{inx}$

$$\therefore S_N f(x)=\frac{1}{2}\int_{-\pi}^{\pi} f(y)\sum_{N=-N}^{N} e^{in(x-y)}\,dy=\frac{1}{2}\int_{-\pi}^{\pi} f(x-y)\sum_{-N}^{N} e^{iny}\,dy$$
$$=\frac{1}{2}\int_{-\pi}^{\pi} f(y)\,D_N(x-y)\,dy=\frac{1}{2}\int_{-\pi}^{\pi} f(x-y)\,D_N(y)\,dy$$

$\therefore$ By definition of convolution,

$$S_N(f)(x)=(f*D_N)(x)=(D_N*f)(x)$$

Theorem 3: $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\theta)\,d\theta=1$ where, D_N Nth Dirichlet Kernel.

Proof : We have Nth Dirichilet Kernel

$$D_N(\theta)=\sum_{n=-N}^{N} e^{in\theta}$$
$$D_N(\theta)=\sum_{n=-N}^{N}(\cos n\theta+i\sin n\theta)$$

$$D_N(\theta) = (\cos 0 + i\sin 0) + \left[(\cos\theta + i\,\text{si}\,\not{n}\theta) + (\cos(-\theta) + i\,\text{si}\,\not{n}(-\theta))\right]$$
$$+\left[(\cos 2\theta + i\sin \not{2}\theta) + (\cos(-2\theta) + i\sin(\not{-}2\theta))\right] + ...$$
$$+\left[(\cos N\theta + i\sin \not{N}\theta) + (\cos(-N\theta) + i\,\text{si}\,\not{n}(-N\theta))\right]$$

$$D_N(\theta) = 1 + 2\cos\theta + 2\cos 2\theta + ... + 2\cos N\theta$$

$$D_N(\theta) = 1 + 2\sum_{n=1}^{N}\cos n\theta$$

On Integrating both side from $-\pi$ *to* π

$$\int_{-\pi}^{\pi} D_N(\theta)\,d\theta = \int_{-\pi}^{\pi} 1\,d\theta + 2\sum_{n=1}^{N}\int_{-\pi}^{\pi}\cos n\,\theta\,d\theta$$

$$\int_{-\pi}^{\pi} D_N(\theta)\,d\theta = 2\pi + 2\times 0.....\left\{\int_{-\pi}^{\pi}\cos n\,\theta\,d\theta = 0\right\}$$

$$\therefore \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\theta)\,d\theta = 1$$

Theorem 4: $\int_{-\pi}^{\pi}|D_N(x)|\,dx \geq c\log N$ as $N\to\infty$ where, C is any constant and $D_N(x)$ is N^{th} Dirichlet Kernel

Proof : Step (1)

We have $\|f\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|\,dx$

Similarly $\|D_N(x)\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(x)|\,dx$

Since $|D_N(x)|$ is even,

$$\therefore \|D_N(x)\|_1 = \frac{2}{2\pi}\int_{o}^{\pi}|D_N(x)|\,dx$$

We have, $D_N(x) = \dfrac{\sin\left(N+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}$

$$\|D_N(x)\|_1 = \frac{1}{\pi}\int_{o}^{\pi}\left|\frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}\right|dx$$

Put $\frac{x}{2} = y$

$\therefore dx = 2dy$

When $x = 0,\quad y = 0$

When $x=\pi, \quad y=\pi/2$

$$\therefore \left\|D_N(x)\right\|_1 = \frac{1}{\pi}\int_o^{\pi/2}\left|\frac{\sin\left(N+\frac{1}{2}\right)(2y)}{\sin y}\right|2dy$$

$$=\frac{2}{\pi}\int_o^{\pi/2}\left|\frac{\sin(2N+1)y}{\sin y}\right|dy \qquad (1)$$

Sin y can be approximated as y

i.e. $\sin y \approx y$

$$\therefore \left\|D_N(x)\right\|_1 = \frac{2}{\pi}\int_o^{\pi/2}\left|\frac{\sin(2N+1)y}{y}\right|dy + o(1)$$

Step (2) :

Put $(2N+1)y=t$

$(2N+1)dy=dt$

When $y=0, \quad t=0$

When $y=\pi/2, \quad t=\left(\frac{2N+1}{2}\right)\pi$

$$\therefore \left\|D_N(x)\right\|_1 = \frac{2}{\pi}\int_0^{\left(\frac{2N+1}{2}\right)\pi}\left|\frac{\frac{\sin t}{t}}{2N+1}\right|\frac{dt}{2N+1}+o(1)$$

$$\therefore \left\|D_N(x)\right\|_1 = \frac{2}{\pi}\int_0^{\left(\frac{2N+1}{2}\right)\pi}\left|\frac{\sin t}{t}\right|dt+o(1)$$

$$\left\|D_N(x)\right\|_1 = \frac{2}{\pi}\sum_{K=0}^{2N}\int_{\frac{1}{2}K\pi}^{\frac{1}{2}(K+1)\pi}\left|\frac{\sin t}{t}\right|dt+o(1)$$

Step (3) :

Put $t=S+\frac{K\pi}{2}$

$\therefore dt=ds$

When $t=\frac{1}{2}K\pi \quad , \quad S=0$

When $t=\frac{1}{2}(K+1)\pi \quad , \quad S=\frac{\pi}{2}$

$$\left\|D_N(x)\right\|_1 = \frac{2}{\pi}\sum_{K=0}^{2N}\int_0^{\pi/2}\left|\frac{\sin\left(S+\frac{K\pi}{2}\right)}{S+\frac{K\pi}{2}}\right| ds + o(1) \quad (2)$$

We have,

$$\sin\left(S+\frac{K\pi}{2}\right) - \begin{cases} \sin s \quad if\ K\ is\ even\ \{\sin(S+n\pi) = \sin S\} \\ \cos s \ \ if\ K\ is\ odd\ \ \left\{\sin\left(S+\frac{n\pi}{2}\right) = \cos\right\} \end{cases}$$

$$u_K(S) = \begin{cases} \sin s \quad if\ K\ is\ even \\ \cos s \ \ if\ K\ is\ odd \end{cases}$$

$$\left\|D_N(x)\right\|_1 = \frac{2}{\pi}\sum_{K=0}^{2N}\int_0^{\pi/2}\left|\frac{u_K(S)}{S+\frac{K\pi}{2}}\right| ds + o(1) \quad (3)$$

The value $S+\frac{K\pi}{2}$ can be approximated to $\frac{K\pi}{2}$.

Since $0 \le \frac{1}{\frac{K\pi}{2}} - \frac{1}{S+\frac{K\pi}{2}}$

$$0 \le \frac{S+\frac{K\pi}{2}-\frac{K\pi}{2}}{\frac{K\pi}{2}\left(S+\frac{K\pi}{2}\right)} = \frac{S}{\frac{K\pi}{2}\left(S+\frac{K\pi}{2}\right)}$$

$$0 \le \frac{S}{\frac{K\pi S}{2}+\frac{K^2\pi^2}{4}}$$

The maximum value of $\left(\frac{S}{\frac{K\pi}{2}}+\frac{K^2\pi^2}{4}\right)$ is $\frac{\pi}{\left(\frac{K\pi}{2}\right)^2} = \frac{S}{K^2\pi}$

$$\therefore 0 \le \frac{S}{\frac{K\pi}{2}} + \frac{K^2\pi^2}{4} \le \frac{S}{K^2\pi}$$

Also $\sum_{K=1}^{\infty}\frac{1}{K^2}$ is convergent and Hence bounded.

$$\therefore \left\| D_N(x) \right\|_1 = \frac{2}{\pi} \sum_{K=0}^{2N} \int_0^{\pi/2} \left| \frac{u_K(S)}{S + \frac{K\pi}{2}} \right| ds + o(1)$$

$\therefore$ This equation can be written as

$$\left\| D_N(x) \right\|_1 = \frac{2}{\pi} \sum_{K=0}^{2N} \frac{1}{K\frac{\pi}{2}} \int_0^{\pi/2} u_K(S)\, ds + o(1) \qquad (4)$$

3) Step (4) :

Consider, $\int_o^{\pi/2} u_K(S)\, ds = \int_o^{\pi/2} \sin s\, ds = 1$ if K is even and $\int_o^{\pi/2} u_K(S)\, ds = \int_o^{\pi/2} \cos s\, ds = 1$ if K is odd use this value in (4).

$$\left\| D_N(x) \right\|_1 = \frac{2}{\pi} \sum_{K=0}^{2N} \frac{1}{K\frac{\pi}{2}} (1) + O\,(1)$$

$$\therefore \left\| D_N(x) \right\|_1 = \frac{4}{\pi^2} \sum_{K=0}^{2N} \frac{1}{K} + O\,(1)$$

Now we have, $\sum_{K=0}^{2N} \frac{1}{K} = \log N$

$$\therefore \left\| D_N(x) \right\|_1 = \frac{4}{\pi^2} \log N + O(1)$$

$\therefore$ By using definition of L^1 nom

$$\therefore \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| D_N(x) \right| dx = \frac{4}{\pi^2} \log N + O(1)$$

$$\therefore \int_{-\pi}^{\pi} \left| D_N(x) \right| dx = \frac{8}{\pi^2} \log N + O(1)$$

$$\int_{-\pi}^{\pi} \left| D_N(x) \right| dx \geq C \log N$$

Theorem 5: Dirichlet Kernel is not good Kernel.

Proof: By above property of Dirichlet Kernel, the 2nd property of good Kernel fails and hence Dirichlet Kernel is not good Kernel.

3.3 DIRICHLET'S THEOREM :

Statement : The Fourier series of real continuous periodic function f which has only finite number of relative maxima and minima converges everywhere to f (and hence converges uniformly)

OR

Suppose that f is an integrable periodic function that is differentiable at $x = x_0$ then $\lim_{N\to\infty} S_N f(x_0) = f(x_0)$.

Proof : We have N^th^ partial sum of integrable periodic function f as

$$S_N f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\, D_N(y)\, dy$$

at $x = x_0$

$$S_N f(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0-y)\, D_N(y)\, dy$$

Consider,

$$S_N f(x_0) - f(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0-y)\, D_N(y)\, dy - f(x_0).$$

By property of Dirichlet Kernel,

$$S_N f(x_0) - f(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0-y)\, D_N(y)\, dy - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0) D_N(y)\, dy\right)$$

$$\left\{\therefore \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(y)\, dy = 1\right\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0-y)\, D_N(y)\, dy - \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(y)\, f(x_0)\, dy$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x_0-y)\, D_N(y) - D_N(y)\, f(x_0)\right) dy$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} \left(f(x_0-y) - f(x_0)\right) D_N(y)\, dy\right]$$

as again by property of Dirichlet Kernel.

$$S_N f(x_0) - f(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[f(x_0-y)\, f(x_0)\right] \left(\frac{\sin\left(N+\frac{1}{2}\right)y}{\sin\frac{1}{2}y}\right) dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\left[\left(N+\frac{1}{2}\right)y\right] g(y)\, dy$$

where, $g(y) = \dfrac{f(x_0-y) - f(x_0)}{\sin y/2}$

i.e. $g(y)=\frac{f(x_0-y)-f(x_0)}{y}\times\frac{y/2}{\sin y/2}\times 2$

clearly, $\sin\left[\left(N+\frac{1}{2}\right)y\right]$ is bounded near zero and hence integrable on $[-\pi,\pi]$. Also 2nd factor $g(y)$ is bounded and hence integrable on $[-\pi,\pi]$ {since f is diff at x_0 & $\lim_{y\to 0}\frac{\sin y/2}{y/2}=1$}

Hence it follows that

$$S_N f(x_0)-f(x_0)\to 0 \text{ as } N\to\infty$$

$$\Rightarrow \lim_{N\to\infty} S_N f(x_0)=f(x_0)$$

Ex : If f is 2π periodic and piecewise smooth on $\mathbb{R}$ then show that $\lim_{N\to\infty} S_N f(\theta)=\frac{1}{2}\left[f(\theta^-)+f(\theta^+)\right]$ and hence show that $\lim_{N\to\infty} S_N f(\theta)=f(\theta)$ for every θ where f is continuous.

Solution : We have,

Step (1) :

$$\int_{-\pi}^{o} D_N(\Phi)\,d\Phi=\pi \;......\left\{\because \int_{-\pi}^{\pi} D_N(\Phi)\,d\Phi=2\pi\right\}$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{0} D_N(\Phi)\,d\Phi=\frac{1}{2}$$

$$\Rightarrow \frac{f(\theta^-)}{2\pi}\int_{-\pi}^{0} D_N(\Phi)\,d\Phi=\frac{f(\theta^-)}{2} \qquad (1)$$

Also $\int_{o}^{\pi} D_N(\Phi)\,d\Phi=\pi$

$$\therefore \frac{f(\theta^+)}{2\pi}\int_{o}^{\pi} D_N(\Phi)\,d\Phi=\frac{f(\theta^+)}{2} \qquad (2)$$

Step (2) :

We have Nth partial sum of Fourier series

$$S_N f(\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta+\Phi)\,D_N(\Phi)\,d\Phi$$

$$=\frac{1}{2\pi}\int_{-\pi}^{0} f(\theta+\Phi)\,D_N(\Phi)\,d\Phi+\frac{1}{2\pi}\int_{0}^{\pi} f(\theta+\Phi)\,D_N(\Phi)\,d\Phi$$

Consider,

$$S_N f(\theta)-\frac{1}{2}\left[f\left(\theta^-\right)+f\left(\theta^+\right)\right]$$

$$=\frac{1}{2\pi}\int_{-\pi}^{0} f(\theta+\Phi)\,D_N(\Phi)\,d\Phi+\frac{1}{2\pi}\int_{0}^{\pi} f(\theta+\Phi)\,D_N(\Phi)\,d\Phi$$

$$-\frac{1}{2}\left[\frac{f\left(\theta^-\right)}{\pi}\int_{-\pi}^{0} D_N(\Phi)\,d\Phi+\frac{f\left(\theta^+\right)}{\pi}\int_{0}^{\pi} D_N(\Phi)\,d\Phi\right]$$

$$=\frac{1}{2\pi}\int_{-\pi}^{o}\left[f(\theta+\Phi)-f\left(\theta^-\right)\right]D_N(\Phi)\,d\Phi+\frac{1}{2\pi}\left[f(\theta+\Phi)-f\left(\theta^+\right)\right]D_N(\Phi)\,d\Phi \tag{4}$$

Step 3 :

We have,

$$D_N(\Phi)=\frac{e^{i(N+1)\Phi}-e^{iN\Phi}}{e^{i\Phi}-1}$$

Consider,

$$\int_{-\pi}^{\pi} g(\Phi)\left[e^{i(N+1)\Phi}-e^{iN\Phi}\right]d\Phi$$

where $g(\Phi)=\begin{cases}\dfrac{f(\theta+\Phi)-f\left(\theta^-\right)}{e^{i\Phi}-1} & -\pi<\Phi<0\\[2ex] \dfrac{f(\theta+\Phi)-f\left(\theta^+\right)}{e^{i\Phi}-1} & 0<\Phi<\pi\end{cases}$

g is well defined function defined on $[-\pi,\pi]$ and also g is smooth except at $\Phi=0$

Also, $f(\theta+\Phi)-f(\theta)=0$ at $\Phi=0$.

Hence, $g(\Phi)$ is in $\frac{0}{0}$ form at $\Phi=0$.

$\therefore$ By applying L^1 Hospital rule,

$$\lim_{\Phi\to 0^+} g(\Phi)=\lim_{\Phi\to 0^+}\frac{f(\theta+\Phi)-f\left(\theta^+\right)}{e^{i\Phi}-1}$$

$$=\lim_{\Phi\to 0^+}\frac{f'(\theta+\Phi)}{ie^{i\Phi}}=\frac{f'\left(\theta^+\right)}{i}$$

Similarly,

$$\lim_{\Phi\to 0^-} g(\Phi)=\lim_{\Phi\to 0^-}\frac{f(\theta+\Phi)-f\left(\theta^-\right)}{e^{i\Phi}-1}$$

$$=\lim_{\Phi\to 0^-}\frac{f^1(\theta+\Phi)-o}{ie^{i\Phi}}=\frac{f'\left(\theta^-\right)}{i}$$

Thus R.H.S. & L.H.S. limit exist.

Hence g is piecewise continuous on $[-\pi,\pi]$.

Step (4) : Using equation (4) we have,

$$S_N f(\theta) - \frac{1}{2}\left[f(\theta^-) + f(\theta^+) \right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{o} g(\Phi)\left[e^{i(N+1)\Phi} - e^{iN\Phi} \right] d\Phi + \frac{1}{2\pi}\int_{o}^{\pi} g(\Phi)\left[e^{i(N+1)\Phi} - e^{-iN\Phi} \right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\Phi)\left[e^{i(N+1)\Phi} - e^{-iN\Phi} \right] d\Phi \qquad (5)$$

We have, Fourier coefficient $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\Phi)\, e^{-in\Phi}\, d\Phi$

By Riemann Lebesque lemma , $\hat{f}(n) \to o$ as $|n| \to \infty$.

Consider,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\Phi)\left(e^{i(N+1)\Phi} - e^{iN\Phi} \right) d\Phi = \hat{f}(N+1) - \hat{f}(N)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\Phi)\left[e^{i(N+1)} - e^{iN\Phi} \right] d\Phi \to o \quad as\ N \to \infty$$

$$\therefore \lim_{N\to\infty} S_N f(\theta) = \frac{1}{2}\left[f(\theta^-) + f(\theta^+) \right]$$

whenever if f is continuous at θ then $\lim_{N\to\infty} S_N f(\theta) = f(\theta)$

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4

FEJER KERNEL

Unit Structure

4.1 CESARO MEAN AND CESARO SUMMATION :

Let $C_0 + C_1 + C_2 + ... + ... = \sum_{K=0}^{\infty} C_K$ be a series of complex numbers.

Define n^{th} partial sum by $S_n = \sum_{K=0}^{n} C_k$.

This series converges to S if $\lim_{N \to \infty} S_N = S$.

The average of 1^{st} N partial sum is denoted and defined by

$$\sigma_N = \frac{S_0 + S_1 + S_2 + ... + S_{N-1}}{N}$$

i.e. $\sigma_N = \frac{1}{N} \sum_{n=0}^{N-1} S_n$ is called N^{th} **Cesaro Mean** of the series $\sum_{K=0}^{\infty} C_k$.

If σ_N converges to σ as $N \to \infty$ then we say that $\sum_{N=\infty}^{\infty} C_n$ is **Cesaro summable** to σ.

Example : Consider $1-1+1-1+1-1+..... = \sum_{K=0}^{\infty} (-1)^K$

Partial sum of the sequence $\{1-1+1-1+1........\}$ is $\{1, 0, 1, 0, ...\}$ which has no limit since partial sum fluctuate between 0 and 1.

So average value $\sigma_N = \frac{1+0}{2} = \frac{1}{2}$.

Therefore, above series is Cesaro summable to $\frac{1}{2}$.

4.2 FEJER'S KERNEL

The **N^th Cesaro mean of Fourier series** is given by

$$\sigma_N f(x) = \frac{S_0 f(x) + S_1 f(x) + ... + S_{N-1} f(x)}{N}$$

We have, N^th partial sum of Fourier series given by $S_N f = f * D_N$.

$$\sigma_N f(x) = \frac{\left[f * D_0(x)\right] + \left[f * D_1(x)\right] + ... + \left[f * D_{N-1}(x)\right]}{N}$$

$$\sigma_N f(x) = \frac{f * \left\{D_0(x) + D_1(x) + ... + D_{N-1}(x)\right\}}{N}$$

$$\Rightarrow \sigma_N f(x) = f * F_N(x)$$

where $F_N(x) = \dfrac{D_0(x) + D_1(x) + ... + D_{N-1}(x)}{N}$

i.e. $F_N(x) = \dfrac{1}{N}\sum_{n=0}^{N-1} D_n(x)$ is called **the N^th Fejer's kernel**.

4.3 PROPERTIES OF FEJER'S KERNEL

Theorem 1: The N^th is Fejer's kernel is given by

$$F_N(x) = \frac{1}{N}\frac{\sin^2\left(\frac{Nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}$$

Proof : We have,

$$F_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} D_n x$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{\sin\left(n + \frac{1}{2}\right)x}{\sin 1/2\, x}\right)$$

$$= \frac{1}{N\sin\frac{1}{2}x}\left[\sum_{n=0}^{N-1}\left(\frac{e^{i\left(n+\frac{1}{2}\right)x} - e^{-i\left(n+\frac{1}{2}\right)x}}{2i}\right)\right]$$

$$F_N(x) = \frac{1}{N\sin\frac{1}{2}x}\left[\sum_{n=0}^{N-1}\left(\frac{e^{inx}\, e^{\frac{ix}{2}} - e^{-inx}.e^{\frac{-ix}{2}}}{2i}\right)\right]$$

$$= \frac{1}{2iN\sin\frac{1}{2}x}\left(e^{\frac{ix}{2}}\sum_{n=0}^{N-1} e^{inx} - e^{\frac{-ix}{2}}\sum_{n=0}^{N-1} e^{-inx}\right)$$

Both of above series are in geometric progressive

for 1st series, Common ratio $= r = e^{ix}$,
for 2nd series, Common ratio $r = e^{-ix}$
Note that, $|r| = 1$

$\therefore$ Using $\displaystyle\sum_{n=0}^{K} r^n = \frac{r^{K+1}-1}{r-1}$

$$F_N(x) = \frac{1}{2iN\sin\frac{1}{2}x}\left(e^{\frac{ix}{2}}\left(\frac{\left(e^{ix}\right)^N - 1}{e^{ix}-1}\right) - e^{\frac{-ix}{2}}\left(\frac{\left(e^{-ix}\right)^N - 1}{e^{-ix}-1}\right)\right)$$

$$= \frac{1}{2iN\sin\frac{x}{2}}\left[\left(\left(\frac{e^{iNx}-1}{e^{\frac{-ix}{2}}\left(e^{ix}-1\right)}\right) - \left(\frac{e^{-iNx}-1}{e^{\frac{ix}{2}}\left(e^{-ix}-1\right)}\right)\right)\right]$$

$$= \frac{1}{2iN\sin\frac{x}{2}}\left[\frac{e^{iNx}-1}{e^{\frac{ix}{2}} - e^{\frac{-ix}{2}}} - \frac{e^{-iNx}-1}{e^{\frac{-ix}{2}} - e^{\frac{ix}{2}}}\right]$$

$$= \frac{1}{2iN\sin\frac{x}{2}}\left[\frac{e^{iNx}-1}{e^{\frac{ix}{2}} - e^{\frac{-ix}{2}}} + \frac{e^{-iNx}-1}{e^{\frac{ix}{2}} - e^{\frac{-ix}{2}}}\right]$$

$$= \frac{1}{(2i)^2 N\sin\frac{x}{2}}\left[\frac{e^{iNx}+e^{-iNx}-2}{\sin\frac{x}{2}}\right]$$

$$= \frac{1}{(2i)^2 N\sin^2 x/2}\left(e^{iNx}+e^{-iNx}-2\right)$$

$$F_N(x) = \frac{1}{(2i)^2 N\sin^2 x/2}\left(e^{\frac{iNx}{2}} - e^{\frac{-iNx}{2}}\right)^2$$

$$= \frac{1}{N\sin^2 x/2}\left(\frac{e^{\frac{iNx}{2}} - e^{\frac{-iNx}{2}}}{2i}\right)^2$$

$$\therefore F_N(x) = \frac{1}{N}\left(\frac{\sin^2\frac{Nx}{2}}{\sin^2\frac{x}{2}}\right)$$

Theorem 2: The Nth Cesaro sum of Fourier series of continuous periodic function f is given by

$$\sigma_N f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x-y)\, f(y)\, dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(y)\, f(x-y)\, dy$$

where, F_N is Nth Fejer's kernel.

i.e. $\sigma_N f(x) = (F_N * f)(x) = (f * F_N)(x)$

Proof : We have Nth partial sum of Fourier series is given by

$$S_N f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x-y)\, f(y)\, dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(y)\, f(x-y)\, dy$$

where, D_N is Nth Dirichlet Kernel.

Taking summation on both side.

$$\sum_{n=0}^{N-1} S_n f(x) = \frac{1}{2\pi}\sum_{n=0}^{N-1}\int_{-\pi}^{\pi} D_n(x-y) f(y)\, dy$$

$$= \frac{1}{2\pi}\sum_{n=0}^{N-1}\int_{-\pi}^{\pi} D_n(y)\, f(x-y)\, dy$$

$$\sum_{n=0}^{N-1} S_n f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=0}^{N-1} D_n(x-y)\, f(y)\, dy = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=0}^{N-1} D_n(y)\, f(x-y)\, dy$$

We have Nth Cesaro sum of Fourier series f $\sigma_N f = \frac{1}{N}\sum_{n=0}^{N-1} S_n$ and also we have, Fejer's Kernel $F_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} D_n(x)$.

$$\therefore N\sigma_N f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} N\, F_N(x-y)\, f(y)\, dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} N\, F_N(y)\, f(x-y)\, dy$$

$$\therefore \sigma_N f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x-y)\, f(y)\, dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(y)\, f(x-y)\, dy$$

Thus $\sigma_N f(x) = F_N * f(x) = f * F_N(x)$

Theorem 3: $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\, dx = 1$ where $F_N(x)$ is Nth is Fejer's kernel

Proof : Nth Fejer's Kernel is given by, $F_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} D_n(x)$.

Now integrating using limit $-\pi\ to\ \pi$

$$\int_{-\pi}^{\pi} F_N(x)\,dx = \frac{1}{N}\int_{-\pi}^{\pi}\sum_{n=0}^{N-1} D_n(x)\,dx$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\int_{-\pi}^{\pi} D_n(x)\,dx$$

$$\therefore \int_{-\pi}^{\pi} F_N(x)\,dx = \frac{1}{N}\sum_{n=0}^{N-1}(2\pi) = \frac{1}{N}2\pi N = 2\pi$$

$$\therefore \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\,dx = 1$$

Hence proved,

Theorem 4: $\lim_{N\to\infty} \int_{\delta<|x|<\pi} F_N(x)\,dx = 0$ if $0<\delta<\pi$.

Proof : We have, N^{th} Fejer Kernel

$$F_N(x) = \frac{\sin^2 \frac{Nx}{2}}{N\sin^2 \frac{x}{2}}$$

The maximum value of $\sin^2 \frac{x}{2}$ is one.

Also, $\sin^2 \frac{x}{2}$ increases as x goes away from the origin in $[-\pi,\pi]$.

Hence, $F_N(x) \le \frac{1}{N\sin^2 \frac{\delta}{2}}$ where $\delta \le |x| \le \pi$

$$\Rightarrow \int_{\delta<|x|\le\pi} F_N(x)\,dx \to 0 \text{ as } N\to\infty.$$

Theorem 5: Fejer Kernel $F_N(x)$ is good kernel

Proof : Since we have

1) $F_N(x) \ge 0 \quad \forall x$

2) $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\,dx = 1$

3) $\exists M>0$ such that $\int_{-\pi}^{\pi} |F_N(x)|\,dx \le M$

4) for every $\delta>0$, $\int_{\delta\le|x|\le\pi} |F_N(x)|\,dx \to 0$ as $N\to\infty$

Thus Fejer's Kernel is good kernel.

4.4 FEJER'S THEOREM :

Theorem: If f is integrable on the circle then Fourier Series of f is Cesaro summable to f at every point of continuity of f. Moreover, if is continuous on the circle then Fourier series of f is uniformly Cesaro summable to f

Proof :

Step (1) : If f is integrable function defined on the circle then it can be approximated as a Fourier series

$$f(x) \approx \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

The Nth Cesaro mean of Fourier Series is given by

$$\sigma_N f(x) = \frac{1}{N} \sum_{n=0}^{N-1} S_n f(x)$$

Where, $S_N f(x)$ is N^{th} Partial sum of Fourier series.

N^{th} Cesaro mean of Fourier series of f can be written as convolution

$$\sigma_N f(x) = (f * F_N)(x)$$

where, F_N is N^{th} Fejer kernel

Step(2) :

We have property of good kernel i.e. let $\{K_n\}_{n=1}$ be a family of good kernel and f is integrable function defined on the circle then

$$\lim_{n\to\infty} (f * \mathrm{K}_n)(x) = f(x)$$

Whenever, f is continuous at x.

Moreover, if f is continuous everywhere then above limit is uniform.

Step(3): We know that N^{th} Fejer kernel F_N is good kernel

$\therefore$ By property mention in step (2) we can write

$$\lim_{N\to\infty} (f * F_N)(x) = f(x)$$

$$\Rightarrow \lim_{N\to\infty} \sigma_N f(x) = f(x)$$

Hence, Fourier series of an integrable function defined on the circle is Cesaro summable to f at every point of continuity Also, by step(2), if f is continuous on the circle then the Fourier series of f is uniformly Cesaro summable to f.

Fejer's Theorem: Alternative Form
Alternatively the statement of Fejers theorem may be written as

Statement : If f is continuous and periodic then averages $\sigma_N f$ of partial sum of Fourier series of f converges uniformly to f as $N \to \infty$.
i.e. $\lim_{N\to\infty} \sigma_N f(x) = f(x)$

Proof : Claim : $\sigma_N f \to f$ as $N \to \infty$
i.e. $\lim_{N\to\infty} \sigma_N f(x) = f(x)$
We have N^th Cesaro mean of Fourier series of f is given by,

$$\sigma_N f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(y)\, f(x-y)\, dy$$

Consider,

$$\sigma_N f(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(y)\, f(x-y)\, dy - \frac{f(x)}{2\pi}\int_{-\pi}^{\pi} F_N(y)\, dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(y)\, f(x-y)\, dy - \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(y)\, f(x)\, dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(y)\left[f(x-y) - f(x)\right] dy$$

$$\sigma_N f(x) - f(x) = \frac{1}{2\pi}\int_{|y|<\delta} F_N(y)\left[f(x-y) - f(x)\right] dy$$

$$+\frac{1}{2\pi}\int_{\delta<|y|<\pi} F_N(y)\left[f(x-y) - f(x)\right] dy \qquad (1)$$

For any choice of δ such that $o < \delta < \pi$. By the properties of Fejer Kernel, the 1^st integral,

$\frac{1}{2\pi}\int_{|y|<\delta} F_N(y)\left[f(x-y) - f(x)\right] dy$ has modulus bounded by $\frac{1}{2\pi}$ sup

$$\left\{\left|f(x-y) - f(x) / |y| < \delta\right|\right\} \qquad (2)$$

A continuous periodic function is uniformly continuous so given $\in > o$, we fix δ so small so that the bound of equation (2) is $\in\!/2 \quad \forall N$.

The modulus of 2^nd integral $\frac{1}{2\pi}\int_{\delta|y|<\pi} F_N(y)\left[f(x-y) - f(x)\right] dy$ is

bounded by $\frac{1}{2\pi} 2 \sup\left\{\left|f(y)\right|\right\} \int_{\delta<|y|<\pi} F_N(y)\, dy \qquad (3)$

For large N, the bound of equation (3) is $\in/2$.

Since $\lim_{N\to\infty} \int_{\delta|y|<\pi} F_N(y)\,dy = 0$

Now using equation (1), (2) and (3),

$$\sigma_N f(x) - f(x) < \frac{\in}{2} + \frac{\in}{2} = \in \text{ as } N \to \infty$$
$$\lim_{N\to\infty} \sigma_N f(x) = f(x)$$

Alternative Proof of Fejer's Theorem

Step 1: We have theorem

Let $\{K_n\}_{n=1}^{\infty}$ be a family of Good Kernels and f is an integrable periodic function defined on the circle then

$\lim_{n\to\infty}(f * K_n)(x) = f(x)$ whenever, f is continuous at x.

If f is continuous everywhere then the above limit is uniform.

Step 2: We know that Fejer Kernel is a good kernel and hence by above theorem, we have

$\lim_{n\to\infty}(f * F_n)(x) = f(x)$ whenever, f is continuous at x.

If f is continuous everywhere then the above limit is uniform.

Step 3: We also know that, $\sigma_N f(x) = F_N * f(x) = f * F_N(x)$

Hence by above step 2, we have

$\lim_{n\to\infty}\sigma_N f(x) = f(x)$ whenever, f is continuous at x.

If f is continuous everywhere then the above limit is uniform.

4.5 UNIQUENESS OF FOURIER SERIES

Theorem : If f is integrable periodic function defined on the circle and $\hat{f}(n) = 0\ \forall n$ then $f = 0$ at all points of continuity of a function f.

Proof : We have N[th] partial sum of Fourier series of f

$$S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

Since $\hat{f}(n) = 0 \ \forall n$

$\therefore S_N f(x) = 0 \ \forall n$ (1)

i.e. all partial sum of Fourier series of function f are zero

Also, we have N[th] Cesaro mean of Fourier series of function f.

$$\sigma_N f(x) = \frac{1}{N}\sum_{n=0}^{N-1} S_n f(x)$$

By equation (1)

$$\sigma_N f(x) = 0 \ \forall n \qquad (2)$$

i.e. N[th] Cesaro mean of Fourier series of f are zero we have, property of Fejer Kernel.

$$\sigma_N f(x) = f * F_N(x)$$

By equation (2)

$$f * F_N(x) = 0$$

$$\Rightarrow f(x) = 0 \qquad (\because F_N > 0)$$

Uniqueness of Fourier Series :

Since Fourier series of a continuous periodic function f converges to f, the function f is uniquely determined by its Fourier coefficients.

If f and g are two functions having same Fourier coefficients then functions f and g are necessarily equal i.e. if $\widehat{f}(n) = \widehat{g}(n)$ then

$\widehat{f-g} = 0$

$\therefore f - g = 0$ {By above then i.e. if $\widehat{f}(n) = 0 \Rightarrow f = 0$ }

$\Rightarrow f = g$

4.6 THE WEIERSTRASS APPROXIMATION THEOREM :

Statement :

Any continuous periodic function f can be approximated by trigonometric polynomial.

OR

If f is continuous function defined on the interval $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$ and $\in > 0$ then there exist trigonometric polynomial P such that $|f(x) - p(x)| < \in$, $-\pi \leq x \leq \pi$

Proof :

By Fejer's Theorem, if f is continuous and periodic then averages $\sigma_N f$ of partial sum of Fourier series of function f converges uniformly to f.

i.e. $|\sigma_N f(x) - f(x)| < \in$ for $\in > 0 \quad -\pi \leq x \leq \pi$

Here, $\sigma_N f(x)$ itself proves existence of trigonometric polynomial $P(x)$.

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5

POISSON KERNEL

Unit Structure

5.1 ABEL MEAN AND SUMMATION :

Definition : A series of complex number $\sum_{k=0}^{\infty} C_k$ is said to be **Abel Summable** to S if for every $0 \le r < 1$ the series $A(r) = \sum_{k=0}^{\infty} C_k\, r^k$ is convergent and if $\lim_{r \to 1} A(r) = S$. The quantity $A(r)$ is called **Abel mean** of the series.

Example : consider the Series

1-2+3-4+5- ---------- $= \sum_{k=0}^{\infty} (-1)^k (k+1)$

$$\Rightarrow A(x) = \sum_{k=0}^{\infty} (-1)^k (k+1)\, r^k$$

$$= \frac{1}{(1+r)^2}$$

$$\lim_{r \to 1} A(r) = \frac{1}{4}$$

Hence Series 1-2+3-4+5-6+........ is Abel summable to $\frac{1}{4}$.

5.2 POISSON KERNEL

The **Poisson kernel** is denoted and defined as $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$

Definition : Let us define **Abel Mean of the Fourier series**

$$f(\theta) \approx \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

where, a_n is Complex Fourier coefficient, is given by

$$A_r f(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}$$

Since n takes positive and negative integer value, we consider $|n|$ here. Here f is integrable and $|a_n|$ a complex Fourier coefficient which is uniformly bounded. Hence Series $A_r f(\theta)$ converges absolutely and uniformly for each r, $0 \le r < 1$.

Theorem: The Abel Mean can be written as convolution of periodic integrable function f and the Poisson kernel $P_r(\theta)$ as

$$A_r f(\theta) = (f * P_r)(\theta)$$

Proof : We have,

$$A_r f(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}$$

where, complex Fourier coefficient

$$a_n = \hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) e^{-in\phi} \, d\phi$$

$$\therefore A_r f(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) e^{-in\phi} \, d\phi \right) e^{in\theta}$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) e^{-in(\phi-\theta)} \, d\phi \right)$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} \int_{-\pi}^{\pi} f(\phi) e^{-in(\phi-\theta)} d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-\phi)} \, d\phi \right)$$

since we have, Poisson Kernel $P_r(\theta) = \sum_{n=\infty}^{\infty} r^{|n|} e^{in\theta}$

$$\therefore A_r f(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) P_r(\theta - \phi) \, d\phi$$

$$A_r f(\theta) = (f * P)_r(\theta) = (P_r * f)(\theta)$$

5.3 PROPERTIES OF POISSON KERNEL

Theorem 1: If $0 \le r < 1$ then Poisson kernel $P_r(\theta) = \dfrac{1 - r^2}{1 - 2r\cos\theta + r^2}$

Proof: We have by definition of poisson kernel

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

$$P_r(\theta) = \sum_{n=0}^{\infty} r^n e^{in\theta} + \sum_{n=1}^{\infty} r^n e^{-in\theta} \qquad (1)$$

Both of above Series are geometric Series.
For 1st Series,
First term = a =1 and Common Ratio = R = $re^{i\theta}$

$$|R| = |re^{i\theta}| = |r||e^{i\theta}| < 1$$

Since $0 \le r < 1 \Rightarrow |r| < 1$ & $|e^{i\theta}| = 1$

For 2nd Series,
First team = $a = re^{-i\theta}$ and Common ratio = $R = re^{-i\theta}$

$$|R| = |re^{i\theta}| = |r||e^{-i\theta}| < 1$$

We have sum of infinite term of geometric Series whose 1st team is *a* and common ratio is R is given by $S_\infty = \frac{a}{1-R}$, provided $|R| < 1$.

Use this in equation (1)

$$P_r(\theta) = \frac{1}{1-re^{i\theta}} + \frac{re^{-i\theta}}{1-re^{-i\theta}}$$

$$= \frac{1-re^{-i\theta} + re^{-i\theta} - r^2}{1-re^{-i\theta} - re^{i\theta} + r^2}$$

$$= \frac{1-r^2}{1-2r\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)+r^2}$$

$$= \frac{1-r^2}{1-2r\cos\theta + r^2} \qquad \because \left\{\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}\right\}$$

Theorem 2: The Poisson kernel $P_r(\theta) \ge 0$

Proof:

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}, \quad 0 \le r < 1$$

Since $0 \le r < 1 \Rightarrow 1 - r^2 > 0$
Also $-1 \le \cos\theta \le 1$. Hence in any case

$$1 - 2r\cos\theta + r^2 > 0$$

Hence $P_r(\theta) \ge 0$.

Theorem 3: $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1$ where $P_r(\theta)$ is the Poisson kernel

Proof:

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

$$\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos\theta + r^2}\,d\theta$$

Since $P_r(\theta)$ is even function

$$\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 2\int_{0}^{\pi} \frac{1-r^2}{1-2r\cos\theta + r^2}\,d\theta$$

Also we can write

$$\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 2\left(\frac{1}{2}\int_{0}^{2\pi} \frac{1-r^2}{1-2r\cos\theta + r^2}\,d\theta\right)$$

$$= \int_{0}^{2\pi} \frac{1-r^2}{1-2r\cos\theta + r^2}\,d\theta \qquad (1)$$

By applying contour integration Method

Put $Z = e^{i\theta} \Rightarrow \quad |z| = 1$

$dz = ie^{i\theta}\,d\theta = i\,z\,d\theta$

$$\frac{dz}{iz} = d\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}$$

Put in (1)

$$I = \int_{-\pi}^{\pi} P_r(\theta)\,d\theta = \int_c \frac{1-r^2}{1-2r\left(\frac{z+\frac{1}{z}}{2}\right)+r^2}\,\frac{dz}{iz}$$

$$= \int_c \frac{1-r^2}{1-r\left(z+\frac{1}{z}\right)+r^2} \times \frac{1}{iz}\,.dz$$

$$= \int_c \frac{(1-r^2)z}{z-r\left(z^2+1\right)+r^2 z} \times \frac{1}{iz}\,.dz$$

$$= \frac{1}{i} \int_c \frac{1-r^2}{z - rz^2 - r + r^2 z} dz$$

$$I = \frac{1}{i} \int_c \frac{1-r^2}{-rz^2 + (1+r^2)z - r} dz \qquad (2)$$

To Find poles and residues :

Let $-rz^2 + (1+r^2)z - r = o$

$$-rz^2 + z + r^2 z - r = o$$

$$rz(r-z) - (r-z) = o$$

$$(rz-1)(r-z) = o$$

$$\Rightarrow z = r \text{ and } z = \frac{1}{r} \text{ are poles}$$

Since $|z| = |r| < 1$, $z = r$ lies inside circle $C(|z| = 1)$.

$|z| = \left|\frac{1}{r}\right| > 1$, so $z = \frac{1}{r}$ lies outside circle C.

By Cauchy Residue theorem,

$$I = \frac{1-r^2}{i} \times 2\pi i \times \left(\lim_{z \to r} (z-r) \cdot \frac{1}{(z-r)(1-rz)} \right)$$

$$= \frac{1-r^2}{i} \times 2\pi i \times \frac{1}{1-r^2}$$

$$= 2\pi$$

From (1), $\int_{-\pi}^{\pi} P_r(\theta) d\theta = 2\pi$

$$\Rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) d\theta = 1$$

Theorem 4: For $\delta > 0$, $\int_{\delta \le |\theta| \le \pi} |P_r(\theta)| d\theta \to 0 \text{ as } r \to 1$

Proof :

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}, \quad 0 \le r < 1$$

$$1 - 2r\cos\theta + r^2 = (1-r)^2 + 2r(1-\cos\theta)$$

As $r \to 1$, $1 - 2r\cos\theta + r^2 = 2(1-\cos\theta)$

which is bounded as $\cos\theta$ is bounded.
Hence

$$P_r(\theta) \le \frac{1-r^2}{C_\delta}$$

(as θ approaches towards π, $\cos\theta$ decreases)

$$\therefore \int_{\delta \le |\theta| \le \pi} P_r(\theta)\,d\theta \le \int \frac{1-r^2}{C_\delta} \to 0 \ \text{as}\ r \to 1$$

Theorem 5: The Poisson Kernel is a good kernel.

Proof: Since we have proved

1) $P_r(\theta) \ge 0$

2) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1$

3) $\exists\ M > 0$ Such that $\forall\ 0 \le r < 1$

$$\int_{-\pi}^{\pi} |P_r(\theta)|\,d\theta \le M$$

4) for every $\delta > 0$, $\int_{\delta \le |\theta| \le \pi} |P_r(\theta)|\,d\theta \to 0 \quad \text{as}\ r \to 1$

Hence Poisson Kernel is a good kernel.

5.4 ABEL SUMMABILITY OF FOURIER SERIES:

Theorem: The Fourier Series of an integrable function on circle is Abel summable to f at every point of continuity, Moreover, if f is continuous on the circle then the Fourier series of f is uniformly Abel summable to f.

Proof: Step 1: We have, Abel mean of the function $f(\theta)$ which is approximated by the Fourier series where f is integrable function defined on the circle.

$$f(\theta) \approx \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

$$A_r f(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}$$

Abel Mean of Fourier Series of f can be written as convolution

$$A_r f(\theta) = (f * P_r)(\theta)$$

Where, $P_r(\theta)$ is the Poisson kernel

Step 2 : We have property of a good kernel,
Let $\{K_n\}_{n=1}^{\infty}$ be a family of good kernel and f is integrable function defined on the circle then

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

whenever, f is continuous at x. If f is continuous everywhere then above limit is uniform.

Step 3: We know that Poisson kernel $P_r(\theta)$ is a good kernel Therefore by above property mention in step (2)

$$\lim_{r\to 1}(f * P_r)(\theta) = f(\theta) \qquad 0 \le r < 1$$

$$\Rightarrow \lim_{r\to 1} A_r\, f(\theta) = f(\theta)$$

Hence, Fourier series of an integrable function defined on the circle is Abel summable to f at every point of continuity.

Also, by step (2)
If f is continuous on the circle then the Fourier series of f is uniformly Abel summable to f.

Ex: If $P_r(\theta)$ denotes the Poisson kernel, show that the function

$$u(r,\theta) = \frac{\partial\{P_r(\theta)\}}{\partial\theta}, \quad 0 \le r < 1,\ \theta \in R \text{ satisfies}$$

(i) $\Delta u = 0$ in the disc where $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$

(ii) $\lim_{r\to 1} u(r,\theta) = 0$ for each θ

However u is not identically zero.

Solution: (i) We have $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$

On differentiating w.r.t θ, we have

$$\frac{\partial\{P_r(\theta)\}}{\partial\theta} = \sum_{n=-\infty}^{\infty} inr^{|n|} e^{in\theta}$$

$$u(r,\theta) = \frac{\partial\{P_r(\theta)\}}{\partial\theta} = \sum_{n=-\infty}^{\infty} inr^{|n|} e^{in\theta} \qquad (1)$$

Consider $\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$

On differentiating (1) term by term, we obtain

$$\Delta u = in\left|n\left(n-1\right)\right| r^{|n-2|} e^{in\theta} + \frac{in|n|}{r} r^{|n-1|} e^{in\theta} + \frac{\left(in\right)^3}{r^2} r^{|n|} e^{in\theta}$$

$$\Delta u = \left\{ in\left|n\left(n-1\right)\right| + \frac{in|n|}{r} r + \frac{\left(in\right)^3}{r^2} r^2 \right\} r^{|n-2|} e^{in\theta}$$

$$\Delta u = \left\{ in\left|n\left(n-1\right)\right| + in|n| - in^3 \right\} r^{|n-2|} e^{in\theta}$$

$$\Delta u = 0$$

(ii)We have $P_r\left(\theta\right) = \dfrac{1-r^2}{1-2r\cos\theta + r^2}, \quad 0 \le r < 1$

$$u\left(r,\theta\right) = \frac{\partial\left\{P_r\left(\theta\right)\right\}}{\partial\theta} = \frac{\partial}{\partial\theta}\left\{\frac{1-r^2}{1-2r\cos\theta + r^2}\right\}$$

$$u\left(r,\theta\right) = -\frac{\left(1-r^2\right)\left(2r\sin\theta\right)}{\left(1-2r\cos\theta + r^2\right)^2}$$

Consider

$$\lim_{r\to 1} u(r,\theta) = \lim_{r\to 1} -\frac{\left(1-r^2\right)\left(2r\sin\theta\right)}{\left(1-2r\cos\theta + r^2\right)^2}$$

$$\lim_{r\to 1} u(r,\theta) = 0$$

Since $0 \le r < 1$ u is not identically zero.

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6

DIRICHLET PROBLEM

Unit Structure

6.1 LAPLACIAN OPERATOR AND HARMONIC FUNCTIONS:

Two dimensional transient (time dependent) heat equation is given by

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\sigma}{k}\ \frac{\partial u}{\partial t}$$

where $u(x,y,t)$ is the temperature at point (x,y) at time t.

Transient means temperature depends on time. The $\sigma \& k$ are physical quantities namely specific heat and thermal conductivity of the material respectively.

If temperature is independent of time then $\frac{\partial u}{\partial t}=0$ and such a physical situation is known as steady state. Hence above Heat Equation can be written as

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

This equation is known as **Laplace equation**.
Laplace equation can be written as :

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

The Operator $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is known as Laplacian operator.

The Solution of Laplace equation $\Delta u=0$ is known as **Harmonic function.**

6.2 DIRICHLET'S PROBLEM FOR UNIT DISC:

Consider unit disc in the plane $D=\{(x,y)\in\mathbb{R}^2 / x^2+y^2<1\}$ whose boundary is unit circle $C=\{(x,y)\in\mathbb{R}^2 / x^2+y^2=1\}$.

In polar co - ordinate (r,θ) with $0\le r<1$ & $0\le\theta<2\pi$, we have unit disc $D=\{(r,\theta) / 0\le r<1,\ 0\le\theta<2\pi\}$ whose boundary is a unit circle $C=\{(r,\theta) / r=1, 0\le\theta<2\pi\}$.

The boundary value problem $\Delta u=0$ with $u=f(\theta)$ at $r=1,\ \ 0\le\theta<2\pi$ is known as **Dirichlet problem** in the unit disc.

Note: The Laplace equation $\Delta u=0$ where $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ which is in Cartesian form can be convert in terms of polar form (r,θ) as

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0$$

i.e. $\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$

6.3 SOLUTION OF DIRICHLET PROBLEM FOR UNIT DISC:

Problem Statement:

Consider unit disc $D=\{(r,\theta) / 0\le r<1,\ 0\le\theta<2\pi\}$

whose boundary is unit circle

$C=\{(r,\theta) / r=1, 0\le\theta<2\pi\}$

The governing steady-state heat equation given by the Laplace equation

$\Delta u=0$

i.e. $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0$ (1)

subjected to boundary condition.,

$u=f(\theta)$ at $r=1,\ 0\le\theta<2\pi$ (2)

Solution: Let us apply separation of variables method to solve Dirichlet problem.

Let $u(r,\theta)=F(r)G(\theta)$ (3)

where , $F(r)$ is some function of r and $G(\theta)$ is some function of θ

Using equation (3) in equation (1)

$$\frac{\partial^2}{\partial r^2}(FG)+\frac{1}{r}\frac{\partial}{\partial r}(FG)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(FG)=0$$

$$F''G+\frac{1}{r}F'G+\frac{1}{r^2}FG''=0$$

$$F''G+\frac{1}{r}F'G=-\frac{1}{r^2}FG''$$

Divide both sides by FG

$$\frac{F''G+\frac{1}{r}F'G}{FG}=\frac{-\frac{1}{r^2}FG''}{FG}$$

$$\therefore\quad \frac{F''+\frac{1}{r}F'}{F}=\frac{-\frac{1}{r^2}G''}{G}$$

$$\therefore\quad \frac{rF''+F'}{rF}=\frac{-G''}{r^2G}$$

$$\therefore\quad \frac{rF''+F'}{F}=\frac{-G''}{rG}$$

$$\therefore\quad \frac{r^2F''+rF'}{F}=\frac{-G''}{G}$$

which is separation form of given D.E.

Since r and θ are independent variables we can write

$$\frac{r^2F''+rF'}{F}=\frac{-G''}{G}=\lambda \qquad (4)$$

Where λ is constant

Consider, $\dfrac{-G''(\theta)}{G(\theta)}=\lambda$

$$\Rightarrow G''(\theta)+\lambda G(\theta)=0$$

$$\Rightarrow (D^2+\lambda)G(\theta) \qquad \omega here,\ D=\frac{d}{d\theta} \qquad (5)$$

Consider Auxiliary equation $\quad D^2+\lambda=0$

$$\Rightarrow D^2=-\lambda$$

Since G is a function of θ and $0\le\theta<2\pi$ i.e. G is defined on a circle i.e. G is periodic of paired 2π

$\Rightarrow \lambda\ge 0$

$let\ \lambda=m^2 \quad , \quad m\in z$

$D^2=-m^2$

$\therefore D=\pm mi$

Hence solution of equation (5) can be written as

$$G(\theta) = A\cos m\theta + B\sin m\theta$$

Or $$G(\theta) = Ae^{im\theta} + Be^{-im\theta} \qquad (6)$$

where $A \,\&\, B$ are constants.

Now consider,

$$\frac{r^2F''(r) + rF'(r)}{F(r)} = \lambda$$

$$r^2\, F''(r) + r\, F'(r) - \lambda F(r) = 0 \qquad (7)$$

$$put \;\; r = e^z \quad i.e.\;\; z = \log r$$

$$\Rightarrow \qquad r.F'(r) = DF(z)$$

$$r^2.F''(r) = D(D-1)F(z)$$

where $D = \dfrac{d}{dr}$

Put these values in equation (7)

$$D(D-1)F(z) + DF(z) - \lambda F(z) = 0$$

$$(D^2 - D + D - \lambda)F(z) = 0$$

$$(D^2 - \lambda)F(z) = 0$$

Auxiliary equation

$$D^2 - \lambda = 0$$

$$D^2 = \lambda = m^2$$

$$D = \pm m$$

$$\therefore\; F(x) = Ce^{mz} + De^{-mz}$$

where C and D arbitrary constants.

Put $Z = \log r$

$$F(r) = Ce^{m\log r} + De^{-m\log r}$$

$$\therefore F(r) = Cr^m + Dr^{-m} \qquad (8)$$

$$\therefore\; F(r) = Cr^m + \frac{D}{r^m}$$

Using equation (6) and (8) in (3) i.e. $u(r,\theta) = F(r)G(\theta)$ we have

$$u(r,\theta) = \left(cr^m + \frac{D}{r^m}\right)\left(Ae^{im\theta} + Be^{-im\theta}\right) \qquad (9)$$

Since $0 \le r < 1$

as $r \to 0$ then $\dfrac{D}{r^m} \to \infty$ and F will be unbounded at center and hence arbitrary constant $D = 0$.

$\therefore$ Solution (9) can be written as

$u(r,\theta)=Cr^m(Ae^{im\theta}+Be^{-im\theta})$

$$u(r,\theta)=Er^{|m|}\,e^{im\theta}, \qquad m\in z \tag{10}$$

where E is new constant combining all the solutions

$$u(r,\theta)=\sum_{m=-\infty}^{\infty} a_m r^{|m|}\,e^{im\theta} \tag{11}$$

where a_m is arbitrary constant.

Equation (11) gives general solution of Dirichlet problem to find particular solution we need to find constants a_m which can be determined by boundary condition given by equation (2), $u=f(\theta)$ at $r=1$.

$$\therefore\ u(1,\theta)=\sum_{m=-\infty}^{\infty} a_m\,e^{im\theta} \tag{12}$$

The above equation is complex form of Fourier series of periodic function $f(\theta)$ of period 2π.

Hence, a_m is a Fourier coefficient which is given by,

$$a_m=\frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-im\theta}\,d\theta \tag{13}$$

6.4 THE POISSON INTEGRAL:

Theorem: Let f be integrable function define on the unit circle then the function u defined in the unit disc given by the Poisson integral as $u(r,\theta)=(f*P_r)(\theta)$ has the following property

1) u has two continuous derivatives in the unit disc and satisfies $\Delta u=0$ (i.e. u satisfies Laplace equation)

2) If θ is any point of continuity of function f then
$\lim_{r\to 1} u(r,\theta)=f(\theta)$

If f is continuous everywhere then this limit is uniform .

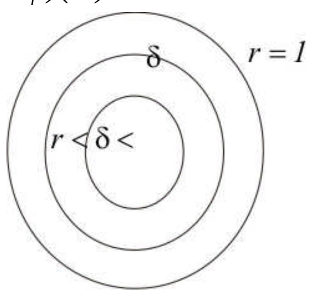
3) If f is continuous then $u(r,\theta)$ is the unique solution to the steady state heat equation equation in the disc which satisfies above condition (1) & (2).

Proof :

Step (1) :

Claim : $u(r,\theta)$ has two continuous derivatives in unit disk and it satisfies Laplace equation

we have, $u(r,\theta)=(f*P_r)(\theta)$



Fix $\rho<1$ inside each disc $r<\rho<1$ centered at origin.

The Series u Can be differentiated term by term and the differentiated series is uniformly and absolutely convergent. Thus, u can be differentiated twice. (Infact, u can be differentiated infinitely many times) and since this holds for for all $\rho<1$, we can conclude that u is twice differentiable inside the unit disc.

In polar co- ordinates we have $\Delta u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

Put $u=(f*P_r)(\theta)$

Term by term differentiation gives us $\Delta u=0$

Step (2) :

Claim :

a) $\lim_{r\to 1} u(r,\theta)=f(\theta)$, whenever f is continues at θ.

b) If f is continuous everywhere then above limit is uniform.

We have, property of a good kernel,

Let $\{K_n\}_{n=1}^{\infty}$ be a family of good kernel and f is integrable function defined on the circle then

$$\lim_{n\to\infty}(f*K_n)(x)=f(x)$$

whenever, f is continuous at θ. If f is continuous everywhere then above limit is uniform.

We know that Poisson kernel $P_r(\theta)$ is a good kernel Therefore by above property mention in step (2)

$$\lim_{r\to 1}(f * P_r)(\theta) = f(\theta) \qquad 0 \le r < 1$$

$$\Rightarrow \lim_{r\to 1} u(r,\theta) = f(\theta)$$

whenever, f is continuous at x. If f is continuous everywhere then above limit is uniform.

Hence claim.

Step (3) :

Suppose $v(r,\theta)$ is another solution of steady state heat equation $\Delta v = 0$ in the unit disc and converges to f as $r \to 1$

i.e. $\lim_{r\to 1} V(r,\theta) = f(\theta)$

Sub claim : $V(r,\theta) = u(r,\theta)$

For each fix r with $0 < r < 1$ the function $V(r,\theta)$ has a Fourier series expansion

$$V(r,\theta) = \sum_{n=-\infty}^{\infty} a_n(r) e^{in\theta}$$

$$a_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(r,\theta) e^{-in\theta} d\theta$$

Since $V(r,\theta)$ satisfies Laplace equation

i.e. $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0$ (1)

Put $v = a_n(r) e^{in\theta}$ $\quad -\infty \le n < \infty$

$$a_n''(r) e^{in\theta} + \frac{1}{r} a_n'(r) e^{in\theta} + \frac{-n^2}{r^2} a_n(r) e^{in\theta} = 0$$

$$a_n''(r) e^{in\theta} + \frac{1}{r} a_n'(r) e^{in\theta} - \frac{n^2}{r^2} a_n(r) e^{in\theta} = 0$$

$$\therefore a_n''(r) + \frac{1}{r} a_n'(r) - \frac{n^2}{r^2} a_n(r) = 0 \qquad (2)$$

The solution of above equation (2) is given by,

$a_n(r) = A_n r^n + B_n r^{-n}$ $\therefore n \ne 0$ {see solution of Dirchlet problem}

where A_n & B_n are arbitrary constants.

To evaluate constant A_n *and* B_n we observe that $A_n(r)$ is bounded because v is bounded

Since,

$$a_n(r) = A_n r^n + \frac{B_n}{r^n}$$

Since $a_n(r)$ bounded $B_n = 0$

Hence, $B_n = 0$

Also to find A_n if we take limit $r \to 1$ Since v converges uniformly to f, we can write A_n as a Fourier coefficient

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta$$

By similar arguments above formula holds for $n = 0$ Hence, for each $0 < r < 1$, the Fourier Series of v is given by $u(r,\theta)$. So by the uniqueness of Fourier series of continuous function, we must have, $v(r,\theta) = u(r,\theta)$

Note: If u Satisfies Laplace equation $\Delta u = 0$ in the unit disc and converges to zero uniformly as $r \to 1$ then u must be identically zero. However if uniform convergence is replaced by pointwise convergence then this conclusion may fail.

Ex 1: In a semicircular plate of radius 1 cm, the bounding diameter is kept at 0^0C and the circumference is at fixed temperature u_0^0C until steady state condition revels. Find the temperature distribution in the semi - circular plate.

Solution : The steady state temperature with the semi - circular plate is given by Laplace equation (Polar form)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad (1)$$

where, $u(r,\theta)$ represent temperature within semi-circular plate with boundary condition

$$u(r,0) = u(r,\pi) = 0 \qquad (2)$$

$$u(1,\theta) = u_0 \qquad (3)$$

We have general solution of dirichelet problem as

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} a_m \, r^{|m|} \, e^{im\theta}$$

This solution may be written as

$$u(r,\theta) = \sum_{m=0}^{\infty}(A_m \cos m\theta + B_m \sin m\theta)r^{|m|} \qquad (4)$$

where A_m & B_m are arbitrary constants.

$u(r,0) = 0$ i.e. $u = 0 \; at \; \theta = 0$

$$\Rightarrow 0 = A_m \cos 0$$

$$\Rightarrow A_m = 0$$

put in (4)

$$u(r,\theta)=\sum_{m=0}^{\infty} B_m\, r^m \sin m\theta \qquad (5)$$

Now $u(r,\pi)=0$ i.e. at $u=0$ at $\theta=\pi$

$0=B_m \, Sin(m\pi)\, r^m$

$\Rightarrow Sin(m\pi)=0$

$\Rightarrow m\pi=n\pi \qquad n=0,1,2,.........$

$i.e.\, m=n$

Also from (3)

$u(1,\theta)=u_0$ Where $r=1,\ u=u_0$

put in (5)

$$u_0=\sum_{m=0}^{\infty} B_m \sin m\theta$$

Which represents the sine series and B_m represent the Fourier coefficient of sine series.

$$B_m=\frac{2}{\pi}\int_0^{\pi} u_0 \sin m\theta\, d\theta$$

$$B_m=\frac{-2u_0}{\pi}\left(\frac{\cos m\theta}{m}\right)_0^{\pi}$$

$$=\frac{-2u_0}{\pi m}[(-1)^m-1]$$

Put this value of B_m in equation (5)

$$u(r,\theta)=\sum_{m=0}^{\infty}\frac{2u_0}{\pi m}[1-(-1)^m]\sin m\theta\, r^m$$

The solution is not defined at $m=0$

$$B_m=\frac{2}{\pi}\int_0^{\pi} u_0 \sin m\theta\, d\theta$$

Put $m=0$

$$B_0=\frac{2}{\pi}\int_0^{\pi} u_0 \sin m\theta\, d\theta=0$$

$$\therefore u(r,\theta)=\sum_{m=1}^{\infty}\frac{2u_0}{\pi m}[1-(-1)^m]\sin m\theta\, r^m$$

$1-(-1)^m=0$ if m is even

$=2$ if m is odd

$$\therefore u(r,\theta)=\frac{4u_0}{\pi}\sum_{m=1}^{\infty}\frac{\sin[(2m-1)\theta]\, r^{2m-1}}{2m-1}$$

Which gives temperature distribution $u(r,\theta)$ within the semicircular plate.

Ex. 2: Solve Dirichlet Problem on unit disc defined by
$D = \{(r,\theta) / 0 \le r < 1, \quad 0 \le \theta < 2\pi\}$
Whose boundary is unit circle $C = \{(r,\theta) / r = 1, \ 0 \le \theta < 2\pi\}$
Subject to boundary condition $u = \sin\theta$ on C.

Solution : Consider Dirichlet Problem on unit disc D whose boundary is unit circle C given by $\Delta u = 0$ subject to $u = \sin\theta \ on\, C.$. We have general solution of Dirichlet problem,

$$u(r,\theta) = \sum_{m=0}^{\infty} (A_m \cos m\theta + B_m \sin m\theta) r^m \qquad (1)$$

On the boundary C we have $u = \sin\theta$ at $r = 1$

$$\sin\theta = \sum_{m=0}^{\infty} (A_m \cos m\theta + B_m \sin m\theta)$$

Which is a Fourier series expansion where, $A_m \ \& \ B_m$ represents fourier coefficients.

$$A_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos m\theta \, d\theta$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sin\theta \cos m\theta \, d\theta$$

$\therefore \ A_m = 0$ ...... { By Orthogonality property of circular function}

$$B_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin m\theta \, d\theta$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sin\theta \sin m\theta \, d\theta = 0$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin\theta \cdot \sin(m\theta) \, d\theta$$

$$= \begin{cases} 0 & m \ne 1 \\ 1 & m = 1 \end{cases}$$

$$\therefore B_1 = 1 \quad \& \quad B_m = 0 \qquad \forall \ m \ne 1$$

$$\& \ A_m = 0 \qquad \forall \ m$$

$$\therefore \ u(r,\theta) = B_1 r \sin\theta = r \sin\theta$$

Ex. 3: Find the solution of Dirichelet problem on unit disc D whose boundary is unit circle C as defined before subjected to boundary conditions.

$$f(\theta) = \begin{cases} u_0 & 0 < \theta < \pi \\ -u_0 & \pi < \theta < 2\pi \end{cases}$$

Solution : We have dirichlet problem $\Delta u = 0$

Where $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

on unit disc $D = \{(r,\theta) / 0 \le r < 1, \; 0 \le \theta < 2\pi\}$

Whose boundary is unit circle $C = \{(r,\theta) / r = 1, \; 0 \le \theta < 2\pi\}$ subject to boundary condition

$$f(\theta) = \begin{cases} u_0 & 0 < \theta < \pi \\ -u_0 & \pi < \theta < 2\pi \end{cases}$$

We have general solution of Dirichlet problem

$$u(r,\theta) = \sum_{m=0}^{\infty} (A_m \cos\theta + B_m \sin m\theta) r^m$$

$$at \; r = 1 \qquad (1)$$

$$u(1,\theta) = f(\theta)$$

$$\therefore f(\theta) = \sum_{m=0}^{\infty} (A_m \cos m\theta + B_m \sin m\theta)$$

which is Fourier Series expansion of $f(\theta)$ where $A_m \; \& \; B_m$ are Fourier coefficients

we have , $A_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cdot\cos(m\theta)\, d\theta$

$$A_m = \frac{1}{\pi}\int_0^{\pi} u_0 \cos m\theta + \frac{1}{\pi}\int_{\pi}^{2\pi} -\mu_0 \cos m\theta \, d\theta$$

$$= \frac{u_0}{\pi}\left[\frac{\sin m\theta}{m}\right]_0^{\pi} - \frac{u_0}{\pi}\left[\frac{Sin\, m\theta}{m}\right]_{\pi}^{2\pi}$$

$$= 0$$

$$B_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin m\theta\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} u_0 \sin m\theta + \frac{1}{\pi}\int_{\pi}^{2\pi} -u_0 \sin m\theta \, d\theta$$

$$= \frac{u_0}{\pi}\left[\frac{-\cos m\theta}{m}\right]_0^{\pi} - \frac{u_0}{\pi}\left[\frac{-\cos m\theta}{\pi}\right]_{\pi}^{2\pi}$$

$$= \frac{-u_0}{m\pi}\left[(-1)^m - 1\right] + \frac{u_0}{m\pi}\left[1 - (-1)^m\right]$$

$$= \frac{2u_0}{m\pi}\left(1 - (-1)\right)^m$$

at $m = 0$

$$B_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\ 0.d\theta = 0$$

Also,

$$1-(-1)^2 = 2 \quad \text{if } m \text{ is odd}$$
$$= 0 \quad \text{if } m \text{ is even}$$

$$\therefore \quad B_m = \frac{4u_0}{(2m+1)\pi}$$

$$\therefore \quad f(\theta) = \sum_{m=1}^{\infty} \frac{4u_0}{(2m-1)\pi}\ \sin\left[(2m-1)\theta\right] r^{2m-1}$$
$$= \frac{4u_0}{\pi}\ \sum_{m=1}^{\infty} \frac{\sin\left[(2m-1)\theta\right]}{2m-1} r^{2m-1}$$

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7

HILBERT SPACES

Unit Structure

7.1 DEFINITION: HILBERT SPACE

Definition 1 :

Let H be a complex Banach space then H is called Hilbert space if $\langle x, y\rangle$ associated to each of two vectors x & $y \in H$ in such a way that

i) $\overline{\langle x, y\rangle} = \langle y, x\rangle$

ii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$

iii) $\langle x, x\rangle = \|x\|^2 \quad \forall x, y, z \in H$

for all scalars α, β

Definition 2 :

The vector space with their inner product and norm satisfying :

i) The inner product is strictly positive definite.

i.e. $\|x\| = 0 \Rightarrow x = 0$

ii) The vector space is complete.

i.e. Every Cauchy sequence in the norm converges to a limit in the vector space, is called Hilbert Space.

Definition 3 :

A set H is called Hilbert Space if it satisfied the following properties

i) H is a vector space over $\mathbb{C}\left(or\ \mathbb{R}^2\right)$

ii) H is an inner product space satisfying.

a) $\langle f,g\rangle=\overline{\langle g\ f\rangle}$ (conjugate symmetry)

b) $\langle \alpha f+\beta g,h\rangle=\alpha\langle f,h\rangle+\beta\langle g,h\rangle$ (linearity property)

c) $\langle f,f\rangle\geq 0 \quad \forall f\in H,\ f,g,h\in H,\alpha,\beta\in\mathbb{C}$

iii) Let $\|f\|=\langle f,f\rangle^{1/2}$

$\|f\|=0$ if and only if $f=0$ i.e. Inner product is strictly positive definite.

iv) The Cauchy - Schwarz inequality and Triangle inequality

Cauchy - Schwarz inequality

$|\langle f,g\rangle|\leq\|f\|\|g\|$

Triangle inequality

$\|f+g\|\leq\|f\|+\|g\| \quad \forall f,g\in H$.

V) H is complete in the metric $d(f,g)=\|f-g\|$

Note : In the above definition of Hilbert space, the Cauchy-Schwarz inequality and triangle inequality are direct consequence of property (I) & (II).

7.2 EXAMPLES OF HILBERT SPACE :

1) The space R^d

Let $X=\left(x_1,x_2,.......,x_d\right)$

$Y=\left(y_1,y_2,.......,y_d\right)$

Then inner product of X & Y

$\langle X,Y\rangle=x_1y_1+x_2y_2+.....+x_dy_d$ and

$\|X\|=\langle X,X\rangle^{1/2}$

$=\sqrt{x_1^2+x_2^2+....+x_d^2}$

Which is usual Euclidean distance .

2) The space C^d

Let $Z = (z_1, z_2, ..., z_d)$

$W = (w_1, w_2, ..., w_d) \in C^d$

Then $\langle z, w\rangle = z_1\overline{w}_1 + z_2\overline{w}_2 + ... + z_d\overline{w}_d$ and

$\|Z\| = \langle Z, Z\rangle^{1/2} = (z_1\overline{z}_1 + ... + z_d\overline{z}_d)^{1/2}$

$= \left(|z_1|^2 + ... + |z_d|^2\right)^{1/2}$

$z\overline{z} = |Z|^2$

3) The sequence space $\ell^2(\mathbb{Z})$

The sequence space $\ell^2(\mathbb{Z})$ over $\mathbb{C}$ is set of all infinite sequences of complex number as $(...... a_{-n}, ...a_{-1}, a_0, a_1, a_2, ... a_n, ...)$ such that $\sum_{n\in\mathbb{Z}} |a_n|^2 < \infty$

Let $A = (....a_{-1}, a_0, a_1, ...)$

$B = (....b_{-1}, b_0, b_1, ...)$ be the elements in $\ell^2(\mathbb{Z})$

Then $\langle A, B\rangle = \sum_{n\in\mathbb{Z}} a_n\overline{b}_n$

$$\|A\| = \langle A, A\rangle^{\frac{1}{2}} = \left(\sum_{n\in\mathbb{Z}} a_n\overline{a}_n\right)^{\frac{1}{2}} = \left(\sum_{n\in\mathbb{Z}} |a_n|^2\right)^{\frac{1}{2}}$$

4) The sequence space $\ell^2(N)$

The sequence space $\ell^2(N)$ over $\mathbb{C}$ is set of all infinite sequence of complex number as $(a_1, a_2, ... a_n, ...)$ one sided such that $\sum_{n\in N} |a_n|^2 < \infty$

Let $A = (a_1, a_2 ......)$

$B = (b_1, b_2 ......)$

$\langle A, B\rangle = \sum_{n=1}^{\infty} a_n\overline{b}_n$

$$\|A\| = \langle A, A\rangle^{\frac{1}{2}} = \left(\sum_{n=1}^{\infty} a_n\overline{a}_n\right)^{\frac{1}{2}} = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{\frac{1}{2}}$$

5) Square Integrable function $L^2(E)$.

Let E be measurable subset of $\mathbb{R}^d$ with $m(E) > 0$. Let $L^2(E)$ denote the space of square integrable function that are supported on E.

i.e. $L^2(E) =$ {f supported on E such that $\int_E |f(x)|^2 dx < \infty$ }

The norm & Inner product is defined as

$$\langle f, g\rangle = \int_E f(x)\,\bar{g}(x)\,dx$$

$$\|f\| = \left(\int_E f(x)\,\overline{f(x)}\,dx\right)^{\frac{1}{2}} = \left(\int_E |f(x)|^2\,dx\right)^{\frac{1}{2}}$$

7.3 PROPERTIES OF HILBERT SPACE:

Theorem 1: Let $X, Y, Z \in H$ α, β, γ are scalars then

i) $\langle \alpha X - \beta Y, Z\rangle = \alpha\langle X, Z\rangle - \beta\langle Y, Z\rangle$

ii) $\langle X, \beta Y + \gamma Z\rangle = \bar{\beta}\langle X, Y\rangle + \bar{\gamma}\langle X, Z\rangle$

iii) $\langle X, \beta Y - \gamma Z\rangle = \bar{\beta}\langle X, Y\rangle - \bar{\gamma}\langle X, Z\rangle$

iv) $\langle X, 0\rangle = 0 = \langle 0, X\rangle, \forall X \in H$

Proof :

i)Consider

$$\begin{aligned}\langle \alpha X - \beta Y, Z\rangle &= \langle \alpha X + (-\beta)Y, Z\rangle \\ &= \langle \alpha X, Z\rangle + \langle (-\beta)Y, Z\rangle \\ &= \alpha\langle X, Z\rangle + (-\beta\langle Y, Z\rangle) \\ &= \alpha\langle X, Z\rangle - \beta\langle Y, Z\rangle\end{aligned}$$

ii) $$\begin{aligned}\langle X, \beta Y + \gamma Z\rangle &= \langle \overline{\beta Y + \gamma Z, X}\rangle \\ &= \overline{\beta\langle Y, X\rangle + \gamma\langle Z, X\rangle} \\ &= \overline{\beta\langle Y, X\rangle} + \overline{\gamma\langle Z, X\rangle} \\ &= \bar{\beta}\overline{\langle Y, X\rangle} + \bar{\gamma}\overline{\langle Z, X\rangle} \\ &= \bar{\beta}\langle X, Y\rangle + \bar{\gamma}\langle X, Z\rangle\end{aligned}$$

iii) $$\begin{aligned}\langle X, \beta Y - YZ\rangle &= \langle X, \beta Y + (-\gamma Z)\rangle \\ &= \bar{\beta}\langle X, Y\rangle + \overline{(-\gamma)}\langle X, Z\rangle \\ &= \bar{\beta}\langle X, Y\rangle + \overline{(-1)\gamma}\langle X, Z\rangle \\ &= \bar{\beta}\langle X, Y\rangle - \bar{\gamma}\langle X, Z\rangle \\ &= \bar{\beta}\langle X, Y\rangle - \bar{\gamma}\langle X, Z\rangle\end{aligned}$$

iv) Consider $\langle 0, X\rangle = \langle 0.0, X\rangle = 0\langle 0, X\rangle = 0$

$$\langle X, 0\rangle = \overline{\langle 0, X\rangle} = \bar{0} = 0$$

Definition : Orthogonality : Let V be vector space over $\mathbb{R}(\mathbb{C})$ with inner product and associated norm $\|\bullet\|$. The two element X and Y are said to be orthogonal if $\langle X, Y \rangle = 0$ and we write $X \perp Y$.

Theorem 2: The Pythagorean Theorem :
If X & Y $\in H$ are orthogonal then $\|X+Y\|^2 = \|X\|^2 + \|Y\|^2 = \|X-Y\|^2$

Proof :

$$\|X+Y\|^2 = \langle X+Y, X+Y \rangle = \langle X, X \rangle + \langle X, Y \rangle + \langle Y, X \rangle + \langle Y, Y \rangle$$

$$= \|X\|^2 + \langle X, Y \rangle + \langle Y, X \rangle + \|Y\|^2$$

Since $X \perp Y \therefore \langle X, Y \rangle = \langle Y, X \rangle = 0$

$\therefore \|X+Y\|^2 = \|X\|^2 + \|Y\|^2$

$\|X-Y\|^2 = \langle X-Y, X-Y \rangle = \langle X, X \rangle - \langle X, Y \rangle - \langle Y, X \rangle + \langle Y, Y \rangle$

$$= \|X\|^2 - 0 - 0 + \|Y\|^2$$

Since $X \perp Y$

$$\langle X, Y \rangle = \langle Y, X \rangle = 0$$

$\therefore \|X-Y\|^2 = \|X\|^2 + \|Y\|^2$

7.4 THE CAUCHY - SCHWARZ INEQUALITY :

Theorem 3: For any $X, Y \in H$

$$|\langle X, Y \rangle| \le \|X\| \; \|Y\|$$

Proof : Case (i) if $Y = 0$ $\|Y\| = 0$ and

$$\langle X, Y \rangle = \langle X, 0 \rangle = 0.$$

and obviously Cauchy - Schwarz inequality holds.

Case (ii) If $Y \neq 0$

For any scalar λ we have

$\langle X + \lambda Y, X + \lambda Y \rangle \ge 0$ ……. {+ve definite prop.}

$\langle X, X + \lambda Y \rangle + \lambda \langle Y, X + \lambda Y \rangle \ge 0$ …… {Linearity prop.}

$\langle X, X \rangle + \bar{\lambda} \langle X, Y \rangle + \lambda \langle Y, X \rangle + \lambda \bar{\lambda} \langle Y, Y \rangle \ge 0$

$\|X\|^2 + \bar{\lambda} \langle X, Y \rangle + \lambda \langle Y, X \rangle + |\lambda|^2 \|Y\|^2 \ge 0$

Since $Y \neq 0$ put $\lambda = \dfrac{-\langle X, Y \rangle}{\|y\|^2}$

$$\|X\|^2 + \frac{-\overline{\langle X,Y\rangle}\langle X,Y\rangle}{\|Y\|^2} - \frac{\langle X,Y\rangle}{\|Y\|^2}\langle Y,X\rangle + \frac{|\langle X,Y\rangle|^2}{\left(\|Y\|^2\right)^2}\|Y\|^2 \geq 0$$

$$\|X\|^2 - \frac{|\langle X,Y\rangle|^2}{\|Y\|^2} - \frac{|\langle X,Y\rangle|^2}{\|Y\|^2} + \frac{|\langle X,Y\rangle|^2}{\|Y\|^2} \geq 0$$

$$\therefore \|X\|^2 \geq \frac{|\langle X,Y\rangle|^2}{\|Y\|^2}$$

$$\therefore \|X\|^2\|Y\|^2 \geq |\langle X,Y\rangle|^2$$

$$\therefore |\langle X,Y\rangle| \leq \|X\|\|Y\|$$

Theorem 4: Triangle Inequality :

For any $X, Y \in H$, $\|X+Y\| \leq \|X\| + \|Y\|$

Proof : $\|X+Y\|^2 = \langle X+Y, X+Y\rangle$

$$= \langle X,X\rangle + \langle X,Y\rangle + \langle Y,X\rangle + \langle Y,Y\rangle$$

$$\left(\langle X,X\rangle = \|X\|^2,\ \langle Y,Y\rangle = \|Y\|^2\right)$$

By Cauchy Schwarz inequality us have, $|\langle X,Y\rangle| \leq \|X\|\ \|Y\|$

$$\Rightarrow |\langle X,Y\rangle + \langle Y,X\rangle| \leq \|X\|\ \|Y\| + \|X\|\ \|Y\|$$

$$\Rightarrow \|X+Y\|^2 \leq \|X\|^2 + 2\|X\|\ \|Y\| + \|Y\|^2$$

$$\Rightarrow \|X+Y\|^2 \leq \left(\|X\| + \|Y\|\right)^2$$

$$\Rightarrow \|X+Y\| \leq \|X\| + \|Y\|$$

Theorem 5: Parallelogram Law

If $X, Y \in H$ then

$$\|X+Y\|^2 + \|X-Y\|^2 = 2\|X\|^2 + 2\|Y\|^2$$

Proof :
Consider,

$$\|X+Y\|^2 + \|X-Y\|^2 = \langle X+Y, X+Y\rangle + \langle X-Y, X-Y\rangle$$

$$= \langle X,X\rangle + \langle X,Y\rangle + \langle Y,X\rangle + \langle Y,Y\rangle$$

$$+ \langle X,X\rangle - \langle X,Y\rangle - \langle Y,X\rangle + \langle Y,Y\rangle$$

$$= 2\|X\|^2 + 2\|Y\|^2$$

7.5 ORTHONORMAL BASIS :

Definition : A finite or countably infinite subset $\{e_1, e_2........\}$ of Hilbert Space H is said to be orthonormal if

$$\langle e_k, e_\ell \rangle = \begin{cases} 1 & when\, k = \ell \\ 0 & when\, k \neq \ell \end{cases}$$

and $\|e_k\| = 1 \quad \forall k$

i.e. Each e_k has unit norm and is orthogonal to e_ℓ whenever $k \neq \ell$.

Property: Let H be a non-zero Hilbert space so that the class of all its orthonormal set is non-empty. This class is a partially ordered set w.r.t. set inclusion relation.

Definition :

An orthonormal set $\{e_i\}$ in Hilbert space H is said to be **complete** if it is maximal in partial order set i.e. if it is impossible to adjoin the vector e to collection $\{e_i\}$ in such a way that $\{e, e_i\}$ is an orthonormal set which properly contains $\{e_i\}$.

Theorem : Every non-zero Hilbert space contains a complete orthonormal set.

Proof :

We know that

i) An orthonormal set $\{e_i\}$ in Hilbert space H is said to be complete if it is maximal in partial order set w.r.t. set inclusion relation.

ii) Zorn's Lemma states that if P is partially ordered set in which every chain has an upper bound then P posses a maximal element.

iii) Since the union of any chain of orthonormal set is clearly an upper bound for the chain in the partially ordered set of all orthonormal set.

The above three statements shows that every non-zero Hilbert space contains complete orthonormal set.

Theorem : If $\{e_k\}_{k=1}^{\infty}$ is orthonormal and $f = \sum a_k e_k \in H$ where sum is finite then $\|f\|^2 = \sum |a_k|^2$.

Proof :

$$\|f\|^2 = \langle f, f \rangle = \left\langle \sum a_k e_k, \sum a_l e_l \right\rangle$$
$$= \sum a_k \overline{a}_\ell \langle e_k, e_\ell \rangle$$
$$= \sum |a_k|^2 \quad .....\{\langle e_k, e_\ell \rangle = 1 \; k = l$$
$$= 0 \quad k \neq l$$

Orthonormal Basis:

Given an orthonormal subset $\{e_1, e_2 ...\} = \{e_k\}_{k=1}^{\infty}$ of Hilbert Space H Spans H i.e. Linear Combination of elements in $\{e_1, e_2 ......\}$ are dense in H and $\{e_1, e_2 ......\}$ are linearly independent then we say that $\{e_1, e_2 ......\}$ is an orthonormal basis for H.

Note : For any $f \in H$ and $\{e_k\}_{k=1}^{\infty}$ is orthonormal basis for H then

$$f = \sum_{k=1}^{\infty} a_k e_k, \quad a_k \in C$$

i.e. f can be written as linear combination of elements in $\{e_1, e_2 ......\}$.

Consider,

$$\langle f, e_j \rangle = \left\langle \sum_{k=1}^{\infty} a_k e_k, \quad e_j \right\rangle$$
$$= \sum_{k=1}^{\infty} a_k \langle e_k, e_j \rangle$$

When $\{ \text{for } k = j, \langle e_k, e_k \rangle = 1 \; \& \; \text{for } k \neq j, \; \langle e_k, e_j \rangle = 0 \}$

i.e. $\langle f, e_j \rangle = a_j$

Hence, whenever $f = \sum_{k=1}^{\infty} a_k e_k$ then $a_k = \langle f, e_k \rangle$.

7.6 EQUIVALENT CHARACTERIZATION :

Theorem : The following property of an orthonormal set $\{e_k\}_{k=1}^{\infty}$ are equivalent.

1) Finite linear combination of elements in $\{e_k\}_{k=1}^{\infty}$ are dense in H.
2) If $f \in H$ and $\langle f, e_j \rangle = 0 \; \forall \; j$ then $f \equiv 0$.

3) If $f \in H$ and $S_N f = \sum_{k=1}^{N} a_k e_k$ then $S_N(f) \to f$ as $N \to \infty$ in norm of Hilbert space H.

4) If $a_k = \langle f, e_k \rangle$ then $\|f\|^2 = \sum_{k=1}^{\infty} |a_k|^2$.

Proof : Step (1) : $(1) \Rightarrow (2)$

Given : finite linear combination of elements in $\{e_k\}_{k=1}^{\infty}$ are dense in H.

Let $f \in H$ and $\langle f, e_j \rangle = 0 \; \forall \; j$

Claim : $f \equiv 0$

Proof : Since finite linear combination of elements in $\{e_k\}_{k=1}^{\infty}$ are dense in H, there exist a sequence $\{g_n\}$ of elements in H which is finite linear combination of elements in $\{e_k\}_{k=1}^{\infty}$ such that $\|f - g_n\| \to o$ as $n \to \infty$.

Since $\langle f, e_j \rangle = 0 \quad \forall \; j$

$\Rightarrow \langle f, g_n \rangle = 0 \;\; \forall n ....$ $\{ \therefore g_n$ is finite linear combination of elements in $\{e_k\}_{k=1}^{\infty} \}$

By Cauchy - Schwarz inequality.

Consider,

$\|f\|^2 = \langle f, f \rangle = \langle f, f - g_n \rangle \le \|f\| \|f - g_n\|$

$\{ \langle f, f - g_n \rangle = \langle f, f \rangle + \langle f, -g_n \rangle$

$= \langle f, f \rangle - \langle f, g_n \rangle$

$= \langle f, f \rangle \qquad ...\{ \langle f, g_n \rangle = 0 \}$

Letting $n \to \infty$

$\|f\|^2 = 0 \qquad \{ \|f - g_n\| \to 0 \; as \; n \to \infty \}$

$\|f\| = 0$

$\therefore f = 0$

Step 2 : $(2) \Rightarrow (3)$

Given $f \in H, \langle f, e_j \rangle = 0, \; \forall \; j$ then $f = 0$

Also we have $S_N f = \sum_{k=1}^{N} a_k e_k$ where, $a_k = \langle f, e_k \rangle$.

Claim : $\|S_N(f)-f\| \to 0$ AS $N \to \infty$

Consider,

$$\langle f-S_N(f),S_N(f)\rangle = \langle f,S_N(f)\rangle - \langle S_N(f),S_N(f)\rangle$$
$$= \left\langle f,\sum_{K=1}^{N} a_k e_k \right\rangle - \left\langle \sum_{k=1}^{N} a_k e_k, \sum_{\ell=1}^{N} a_\ell e_\ell \right\rangle$$
$$= \sum_{K=1}^{N} \overline{a}_k \langle f,e_k\rangle - \sum_{\ell,k=1}^{N} a_k \overline{a}_\ell \langle e_k,e_\ell\rangle$$
$$= \sum_{K=1}^{N} \overline{a}_k \left\langle \sum_{\ell=1}^{\infty} a_\ell e_\ell, e_K \right\rangle - \sum_{\ell,k=1}^{N} a_k \overline{a}_\ell \langle e_k,e_\ell\rangle$$
$$= \sum_{K=1}^{N} a_k \overline{a}_k - \sum_{K=1}^{N} a_k \overline{a}_k \ldots\ldots \begin{Bmatrix} \langle e_k,e_\ell\rangle = 1 & \ell = k \\ = o & \ell \neq k \end{Bmatrix}$$
$$= \sum_{K=1}^{N} |a_k|^2 - \sum_{K=1}^{N} |a_k|^2$$
$$= 0$$
$$\langle f-S_N(f),S_N(f)\rangle = 0$$

$$\Rightarrow f - S_N(f) \perp S_N(f)$$

By Pythagorean theorem,

$$\|f\|^2 = \|f-S_N(f)\|^2 + \|S_N(f)\|^2$$
$$= \|f-S_N(f)\|^2 + \sum_{K=1}^{N} |a_k|^2$$

$$\Rightarrow \|f\|^2 \geq \sum_{K=1}^{N} |a_k|^2$$

Letting $N \to \infty$

$\sum_{K=1}^{\infty} |a_k|^2 \leq \|f\|^2$ {This is known as **Bessel's Inequality**}

Bessel's inequality implies that series $\sum_{K=1}^{\infty} |a_k|^2$ is convergent.

Therefore, partial sum $\{S_N(f)\}_{N=1}^{\infty}$ forms Cauchy seq. in H.

Since $\|S_N(f)-S_M(f)\| = \left\| \sum_{K=1}^{N} a_k e_k - \sum_{K=1}^{M} a_k e_k \right\|$

$$= \left\| \sum_{K=M+1}^{N} a_k e_k \right\| \qquad N > M$$
$$= \sum_{K=M+1}^{N} |a_k|^2 \qquad \textit{whenever } N > M$$

Since H is complete $\exists\, g \in H$ such that $S_N(f) \to g$ as $N \to \infty$.

Fix j and note that for all sufficiently larger N,

$$\begin{aligned}\langle f - S_N(f), e_j\rangle &= \langle f, e_j\rangle - \langle S_N(f), e_j\rangle \\ &= a_j - \langle \sum a_k e_k, e_j\rangle \\ &= a_j - a_j \ldots (orthonormality) \\ &= 0\end{aligned}$$

Since $S_N(f) \to g$ we can write

$$\langle f - g, e_j\rangle = 0 \quad \forall\, j$$

$\Rightarrow f - g = o$ ………… {By given hypothesis (2)}

$\therefore f = g$ $\qquad\qquad \langle f, e_j\rangle = 0, \forall j \;\; \Rightarrow f = 0$

Hence $S_N(f) \to f$ as $N \to \infty$

i.e. $\|S_N(f) - f\| \to 0$ as $N \to \infty$

Step 3 : (3) $\Rightarrow$(4)

Given $f \in H \quad S_N(f) = \sum_{k=1}^{N} a_k e_k$

$\|S_N(f) - f\| \to o$ as $N \to \infty$

Claim : $\|f\|^2 = \sum_{k=1}^{\infty} |a_k|^2$

We have $\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{K=1}^{N} |a_k|^2$

Letting $N \to \infty$ and using $\|S_N(f) - f\| \to 0$ as $N \to \infty$

$\|f\|^2 = \sum_{K=1}^{\infty} |a_k|^2$

This is known as **Parseval's Identity**.

Step 4 : (4) $\Rightarrow$(1)

$\|f\|^2 = \sum_{k=1}^{\infty} |a_k|^2$

Claim : finite $\ell.c.$ of elements in $\{e_k\}_{k=1}^{\infty}$ are dense in H.

We have from equation

$\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{K=1}^{N} |a_k|^2$

as $N \to \infty$, we have Parseval's identity.

$$\|f\|^2 = \sum_{K=1}^{\infty} |a_k|^2$$

$\Rightarrow \|f - S_N(f)\| \to 0$ as $N \to \infty$

Since each $S_N(f)$ is finite linear combination of elements in $\{e_k\}_{k=1}^{\infty}$.

Hence finite linear combination of elements in $\{e_k\}_{k=1}^{\infty}$ are dense in H.

Ex 1: Let H be Hilbert Space. Show that for any $x, y \in H$

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

Solution:

Consider $\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$

$= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle$

$- \left[\|x\|^2 + \|y\|^2 - \langle x, y\rangle - \langle y, x\rangle\right]$

$+ i\left[\{x+iy, x+iy\}\right]$

$- i\left[\{x-iy, x-iy\}\right]$

$= 2\langle x, y\rangle + 2\langle y, x\rangle + i\left[\|x\|^2 + \langle x, iy\rangle + \langle iy, x\rangle + \langle iy, iy\rangle\right]$

$- i\left[\|x\|^2 + \langle x, -iy\rangle + \langle -iy, x\rangle + \langle -iy, -iy\rangle\right]$

$= 2\langle x, y\rangle + 2\langle y, x\rangle + i\left[\|x\|^2 - i\langle x, y\rangle + i\langle y, x\rangle + \|y\|^2\right]$

$- i\left[\|x\|^2 + i\langle x, y\rangle - i\langle y, x\rangle + \|y\|^2\right]$

$= 2\langle x, y\rangle + 2\langle y, x\rangle + \langle x, y\rangle - \langle y, x\rangle + \langle x, y\rangle - \langle y, x\rangle$

$= 4\langle x, y\rangle$

Ex 2: Let $\{e_1, e_2, ..., e_n\}$ be a finite orthonormal set in a Hilbert space H. If *x* is any vector in H. Then show that

$$\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \le \|x\|^2$$

Also show $x - \sum_{i=1}^{n} \langle x, e_i\rangle e_i \perp^{er} e_j$ for each j.

Solution : Consider

$$0 \le \left\| x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \right\|^2$$

$$= \left\langle x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i ,\ x - \sum_{j=1}^{n} \langle x, e_j \rangle e_j \right\rangle$$

$$= \|x\|^2 + \left\langle x, -\sum_{j=1}^{n} \langle x, e_j \rangle e_j \right\rangle - \left\langle \sum_{i=1}^{n} \langle x, e_i \rangle e_i, x \right\rangle + \left\langle \sum_{i=1}^{n} \langle x, e_i \rangle e_i, \sum_{j=1}^{n} \langle x, e_j \rangle e_j \right\rangle$$

$$= \|x\|^2 - \sum_{j=1}^{n} \overline{\langle x, e_j \rangle} \langle x, e_j \rangle - \sum_{i=1}^{n} \langle x, e_i \rangle \langle e_i, x \rangle + \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x, e_i \rangle \overline{\langle x, e_j \rangle} \langle e_i, e_j \rangle$$

$$= \|x\|^2 - \sum_{j=1}^{n} \langle x, e_j \rangle \overline{\langle x, e_j \rangle} - \sum_{i=1}^{n} \langle x, e_i \rangle \overline{\langle x, e_i \rangle} + \sum_{i=1}^{n} \langle x, e_i \rangle \overline{\langle x, e_i \rangle}$$

$$= \|x\|^2 - \sum_{i=1}^{n} \langle x, e_i \rangle \overline{\langle x, e_i \rangle} - \sum_{i=1}^{n} \langle x, e_i \rangle \overline{\langle x, e_i \rangle} + \sum_{i=1}^{n} \langle x, e_i \rangle \overline{\langle x, e_i \rangle}$$

$$= \|x\|^2 - \sum_{i=1}^{n} \left| \langle x, e_i \rangle^2 \right| \Rightarrow \sum_{i=1}^{n} \left| \langle x, e_i \rangle \right|^2 \le \|x\|^2$$

Consider $\left\langle x - \sum_{i=1}^{n} \langle x_1, e_i \rangle e_i, e_j \right\rangle$

$$= \langle x, e_j \rangle + \left\langle -\sum_{i=1}^{n} \langle x, e_i \rangle e_i, e_j \right\rangle$$

$$= \langle x, e_j \rangle - \sum_{i=1}^{n} \langle x, e_i \rangle \langle e_i, e_j \rangle$$

$$= \langle x, e_j \rangle - \langle e_j, e_j \rangle \langle x, e_j \rangle$$

$$= \langle x, e_j \rangle - \langle x, e_j \rangle$$

$$= 0$$

$$\therefore x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \perp e_j \quad \forall\ j$$

Ex 3: Let H be Hilbert space. Let $\{e_i\}$ be an orthonormal set in H. Then show that the following conditions are equivalent.

1) $\{e_i\}$ is complete

2) $x \perp \{e_i\}$ then $x = 0$

3) If $x \in H$ then $x = \sum_i \langle x, e_i \rangle e_i$

4) If $x \in H$ then $\|x\|^2 = \sum_i \left| \langle x, e_i \rangle \right|^2$

Solution :

Step - I : $(1) \Rightarrow (2)$

Let $\{e_i\}$ be complete.

Suppose $x \perp e_i \ \forall i$

Sub claim **-** $x = 0$

Suppose that $x \neq 0$

Define $e = \dfrac{x}{\|x\|}$

Clearly $\langle e, e_i \rangle = 0 \left(\because x \perp e_i \right) \forall i$. Thus $\{e, e_i\}$ is orthonormal set which properly contains $\{e_i\}$

Which is contradiction to $\{e_i\}$ be complete.

Hence our assumption is wrong.

$\Rightarrow x = 0$

Step - II : $(2) \Rightarrow (3)$

Suppose $x \perp e_i \ \forall i$ then $x = 0$

Sub claim : $x = \sum\limits_i \langle x, e_i \rangle e_i$

We know that $x - \sum \langle x, e_i \rangle e_i$ is orthogonal to $\{e_i\}$

By hypothesis, $x - \sum \langle x, e_i \rangle e_i = 0$

$$\Rightarrow x = \sum \langle x, e_i \rangle e_i$$

Step III : $(3) \Rightarrow (4)$

Suppose for $x \in H$, $x = \sum \langle x, e_i \rangle e_i$

Sub claim : $\|x\|^2 = \sum \left| \langle x, e_i \rangle \right|^2$

Consider $\|x\|^2 = (x, x)$

$$= \left\langle \sum \langle x, e_i \rangle e_i, \sum \langle x, e_j \rangle e_j, \right\rangle$$

$$= \sum_i \langle x, e_i \rangle \sum_j \overline{\langle x, e_j \rangle} \langle e_i, e_j \rangle$$

$$= \sum_i \langle x, e_i \rangle \overline{\langle x, e_i \rangle} 1$$

$$\left(\because \{e_i\} \ orthonormal\ set \right)$$

$$= \sum \left| \langle x, e_i \rangle \right|^2$$

Step IV : $(4) \Rightarrow (1)$

Suppose $x \in H, \|x\|^2 = \sum |\langle x, e_i \rangle|^2$

Sub claim : $\{e_i\}$ is complete.

Suppose $\{e_i\}$ is not complete then it is proper subset of an orthonormal set $\{e_i, e\}$. Since $e \perp^{er} e_i \quad \forall i$

Put $x = e$ in above identity.

$$\Rightarrow \|e\|^2 = \sum |\langle e, e_i \rangle|^2$$
$$= \sum 0^2$$
$$= 0$$

This is contradiction to e is a unit vector

Hence our assumption is wrong.

Thus $\{e_i\}$ is complete.

Note : Let $\{e_i\}$ be complete orthonormal set in Hilbert space H. Let x be an arbitrary vector in H. Then $\langle x, e_i \rangle$ are Fourier coefficients of x and the expression $x = \sum_i \langle x, e_i \rangle e_i$ is called Fourier series expansion of x and the equation, $\|x\|^2 = \sum |\langle x, e_i \rangle|^2$ is called Parseval's identity. (all w.r.t. complete orthonormal set $\{e_i\}$ under consideration.)

Ex 4: If $\{e_i\}_{i=1}^{n}$ is an orthonormal set in Hilbert space H and if x is any vector in H then $S = \{e_i \,|\langle x, e_i \rangle \neq 0\}$ is either empty or countable.

Solution :

For each +ve integer n, consider $S_n = \left\{ e_i \,\middle|\, |\langle x, e_i \rangle|^2 > \frac{\|x\|^2}{n} \right\}$. We have Bessel's inequality.

$$\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2$$

Bessel's inequality gives us, S_n contains at most $(n-1)$ vectors since $S = \bigcup_{n=1}^{\infty} S_n$. S is either empty or countable.

Ex 5: Show that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

Solution : We recall from the definition in Problem 32-5 that since C is convex, it is non-empty and contains $(x+y)/2$ whenever it contains x and y. Let d = inf $\{\|x\| : x \in C\}$. There clearly exists a sequence $\{x_n\}$ of vectors in C such that $\|x_n\| \to d$. By the convexity of C, $(x_m + x_n)/2$ is in C and $\|(x_m + x_n)/2\| \geq d$, so $\|(x_m + x_n)\| \geq 2d$. Using the parallelogram law, we obtain

$$\begin{aligned}\|x_m - x_n\|^2 &= 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2 \\ &\leq 2\|x_m\|^2 + 2\|x_n\|^2 - 4d^2;\end{aligned}$$

and since $2\|x_m\|^2 + 2\|x_n\|^2 - 4d^2 \to 2d^2 + 2d^2 - 4d^2 = 0$, it follows that $\{x_n\}$ is a Cauchy sequence in C. Since H is complete and C is closed C is complete, and there exists a vector x in C such that $x_n \to x$. It is clear by the fact that $\|x\| = \|\lim x_n\| = \lim\|x_n\| = d$ that x is a vector in C with smallest norm. To see that x is unique, suppose that x' is a vector in C other than x which also has norm d. Then $(x + x')/2$ is also in C, and another application of the parallelogram law yields.

$$\begin{aligned}\left\|\frac{x+x'}{2}\right\|^2 &= \frac{\|x\|^2}{2} + \frac{\|x'\|^2}{2} - \left\|\frac{x-x'}{2}\right\|^2 \\ &< \frac{\|x\|^2}{2} + \frac{\|x'\|^2}{2} = d^2,\end{aligned}$$

which contradicts the definition of d.

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8

HILBERT SPACE $L^2[-\pi,\pi]$

Unit Structure

8.1 HILBERT SPACE $L^2[-\pi,\pi]$

Consider the Hilbert space L^2, associated with measure space $[0,2\pi]$ where measure is Lebesgue measure and integrals are Lebesgue integrals. This space essentially consist of all complex functions f defined on $[0,2\pi]$ which are Lebesgue measurable and square integrable.

i.e. $\int_0^{2\pi} |f(x)|^2\, dx < \infty$

Its norm and inner product is defined as $\|f\|_2 = \left(\int_0^{2\pi} |f(x)|^2\, dx\right)^{1/2}$

$$\langle f,g\rangle = \int_0^{2\pi} f(x)\cdot\overline{g(x)}\,dx$$

The function $\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}$ where $n = 0,\pm1,\pm2,....$ forms an orthonormal basis for H since $\int_0^{2\pi} e^{imx}.e^{-inx}\, dx = \begin{cases} 2\pi & m=n \\ 0 & m\neq n\end{cases}$

This gives us $e_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}, n\in\mathbb{Z}$

For any $f\in L^2$, the number, $C_n = \langle f, e_n\rangle = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} f(x)e^{-inx}\,dx$ gives Fourier coefficient of the Fourier series expansion of f given by, $f(x) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} C_n e^{inx}$.

Definition:

The Hilbert Space $L^2[0,2\pi]$ or $L^2[-\pi,\pi]$.

Let R denote set of complex valued Riemann integrable functions defined on a circle then the inner product and norm is defined as

$$\langle f,g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{g(\theta)}\,d\theta \text{ and } \|f\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi}|f(\theta)|^2\,d\theta\right)^{\frac{1}{2}}.$$

Similarly, for interval $[-\pi,\pi]$.

$$\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}\,d\theta \text{ and } \|f\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2\,d\theta\right)^{\frac{1}{2}}.$$

8.2 EXISTENCE OF ORTHONORMAL BASIS OF HILBERT SPACE

Theorem : Any Hilbert Space has on orthonormal basis.

Proof : The proof of this theorem is follows from gram Schmidt process.

Given finite family of elements $\{f_1, f_2......, f_k\}$, the span of this family is set of all elements which are finite linear combination of elements $\{f_1, f_2......, f_k\}$ We denote it by span $\{f_1, f_2......, f_k\}$. Now we construct a sequence of orthonormal vectors say $e_1, e_2......$ such that span $(\{e_1, e_2......, e_n\})$= span $\{f_1, f_2......, f_n\}$ $\forall$ $n \geq 1$.

Let us prove this by induction on n.

Step 1 : By Linear independent hypothesis, $f_1 \neq 0$ then we can take $e_1 = \frac{f_1}{\|f_1\|}$.

Step 2 : Assume that orthonormal vectors $\{e_1, e_2......, e_k\}$ has been found such that span $(\{e_1, e_2......, e_k\})$ = span $\{f_1, f_2......, f_k\}$.
Claim : span $(\{e_1, e_2......, e_{k+1}\})$ = span $\{f_1, f_2......, f_{k+1}\}$

i.e. $e'_{k+1} = f_{k+1} + \sum_{j=1}^{k} a_j e_j$

$$\Rightarrow \left(e'_{k+1}, e_j\right) = \left\langle f_{k+1} + \sum_{i=1}^{k} a_i e_i , e_j \right\rangle$$

$$= \left\langle f_{k+1}, e_j \right\rangle + \sum_{j=1}^{k} \left\langle a_i e_i , e_j \right\rangle$$

$$= \left\langle f_{k+1}, e_j \right\rangle + \sum_{i=1}^{k} a_i \left\langle e_i , e_j \right\rangle$$

$$\left\langle e'_{k+1}, e_j \right\rangle = \left\langle f_{k+1}, e_j \right\rangle + a_j$$

To have : $\left\langle e'_{k+1}, e_j \right\rangle = 0 \quad \forall j$

We must have $\left\langle f_{k+1}, e_j \right\rangle = -a_j$

This choice of a_j, for $1 \le j \le k$ assure that e'_{k+1} is orthogonal to $\{e_1, ...e_k\}$.

Moreover, our linear independent hypothesis assure that $e'_{k+1} \neq 0$

Hence, the choice of e_{k+1} is $e_{k+1} = \dfrac{e'_{k+1}}{\left\| e'_{k+1} \right\|}$.

Hence span$\left(\{e_1, e_2......, e_n\}\right) =$ span $\{ f_1, f_2......, f_n\}$.

Thus, Every Hilbert space has an orthonormal Basis.

Example: Consider, Hilbert space H. Transform Basis $\{ f_1, f_2......, f_n\}$ into orthonormal basis where, $f_1 = (1,-1,1)$, $f_2 = (2,1,0)$, $f_3 = (-1,-1,1)$. (Take Euclidean inner product)

Solution:

1) $e_1 = \dfrac{f_1}{\| f_1 \|} = \dfrac{(1,-1,1)}{\sqrt{3}} = \left(\dfrac{1}{\sqrt{3}}, \dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}} \right)$

2) Using $e'_{k+1} = f_{k+1} + \sum_{j=1}^{k} a_j e_j$

$$e'_2 = f_2 - \left\langle f_2, e_1 \right\rangle e_1$$

$$e'_2 = \left(\frac{5}{3}, \frac{4}{3}, -\frac{1}{3} \right), \quad a_j = -\left\langle f_{K+1}, e_j \right\rangle$$

$$e_2 = \frac{e'_2}{\left\| e'_2 \right\|} = \left(\frac{5}{\sqrt{42}}, \frac{4}{\sqrt{42}}, -\frac{1}{\sqrt{42}} \right)$$

3) Using $e'_{k+1} = f_{k+1} + \sum_{j=1}^{k} a_j e_j$

$$e'_3 = f_3 - \langle f_3\, e_1 \rangle e_1 - \langle f_3\, e_2 \rangle e_2$$

$$e'_3 = \left(\frac{-1}{7}, \frac{2}{7}, \frac{3}{7} \right)$$

$$e_3 = \frac{e'_3}{\|e'_3\|} = \left(\frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)$$

8.3 ORTHONORMAL BASIS OF $L^2[-\pi, \pi]$:

Theorem 1: The sets $\{e^{inx}\}_{n=-\infty}^{\infty}$ & $\{\cos nx\}_{n=-\infty}^{\infty} \cup \{\sin nx\}_{n=-\infty}^{\infty}$ are complete orthonormal basis for $L^2[-\pi, \pi]$. Also the sets $\{\cos nx\}_{n=-\infty}^{\infty}$ & $\{\sin nx\}_{n=-\infty}^{\infty}$ are complete orthogonal basis for $L^2[0, \pi]$.

Proof : Consider, $\Psi_n(x) = e^{inx}$

Let $f \in L^2[-\pi, \pi]$

Let $\in > 0$ (small)

Claim : N^{th} partial sum of Fourier series of f approximate f in norm within $\in$ when N is sufficiently large.

i.e. $\|S_N f - f\| < \in$ as $N \to \infty$.

We can find 2π periodic function $\tilde{f}$ possessing derivatives of all order such that $\|f - \tilde{f}\| < \in/3$.

Let $C_n = (2\pi)^{-1} \langle f, \Psi_n \rangle$

$$\left\{ C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = (2\pi)^{-1} \langle f, e_n \rangle \right\}$$

and $\tilde{C}_n = (2\pi)^{-1} \langle \tilde{f}, \psi_n \rangle$ be Fourier coefficients of f & $\tilde{f}$ respectively.

We know that Fourier series $\sum \tilde{C}_n \Psi_n \to \tilde{f}$ uniformly.

Hence it converges to $\tilde{f}$ in norm.

If we take N sufficiently large then $\left\| \tilde{f} - \sum_{-N}^{N} \tilde{C}_n \Psi_n \right\| < \frac{\in}{3}$

By Bessel's inequality

$$\left\| \sum_{-N}^{N} \tilde{C}_n \Psi_n - \sum_{-N}^{N} C_n \Psi_n \right\|^2 \le \sum_{-N}^{N} \left| \tilde{C}_n - C_n \right|^2$$

$$\leq \sum_{\infty}^{\infty} \left| \tilde{C}_n - C_n \right|^2 \quad \left(|N| \to \infty \right)$$

$$= \left\| \tilde{f} - f \right\|^2 < \left(\frac{\in}{3} \right)^2$$

Consider,

$$f - \sum_{-N}^{N} C_n \Psi_n = \left(f - \tilde{f} \right) + \left(\tilde{f} - \sum_{-N}^{N} \tilde{C}_n \Psi_n \right) + \left(\sum_{-N}^{N} \tilde{C}_n \Psi_n - \sum_{-N}^{N} C_n \Psi_n \right)$$

Taking norm on both side
Now using triangle inequality.

$$\left\| f - \sum_{-N}^{N} C_n \Psi_n \right\| \leq \left\| f - \tilde{f} \right\| + \left\| \tilde{f} - \sum_{-N}^{N} \tilde{C}_n \Psi_n \right\| + \left\| \sum_{-N}^{N} \tilde{C}_n \Psi_n - \sum_{-N}^{N} C_n \Psi_n \right\|$$

$$< \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3} = \in$$

This proves completeness of set $\{\Psi_n\} = \{e^{inx}\}$ in $L^2[-\pi, \pi]$. Completeness of $\{\cos nx\}_{n=-\infty}^{\infty} \cup \{\sin nx\}_{n=-\infty}^{\infty}$ in $L^2[-\pi, \pi]$ can be derived by completeness of $\{e^{inx}\}$.

Similarly, completeness of $\{\cos nx\}$ & $\{\sin nx\}$ in $L^2[-\pi, \pi]$ can be prove by considering even & odd extension of $f \in L^2[0, 2\pi]$ to $[-\pi, \pi]$.

Theorem 2: Let $H = L^2[-\pi, \pi]$ and $f_n(t) = e^{\text{int}}$ for $n = 0, \pm 1, \pm 2, ...$ and $t \in [-\pi, \pi]$ then $\{ f_n(t) | n = 0, \pm 1, \pm 2, ...\}$ is an orthonormal basis for $L^2[-\pi, \pi]$.

Proof :

Step 1 : Lets verify $\{ f_n(t) | n = 0, \pm 1, \pm 2, ...\}$ is orthonormal

$$\langle f_n, f_m \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(t) \overline{f_m(t)}\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\text{int}} . \overline{e^{imt}}\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\text{int}} . e^{-imt}\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\,i(n-m)t}\, dt$$

$$= \frac{1}{2\pi}\left[\frac{e^{i(n-m)t}}{i(n-m)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi i(n-m)}\left[e^{i(n-m)\pi} - e^{-i(n-m)\pi}\right]$$

$$= \frac{1}{2\pi i(n-m)}\left[\cos(n-m)\pi + i\sin(n-m)\pi - \cos(n-m)\pi + i\sin(n-m)\pi\right]$$

$$= \frac{2i\sin(n-m)\pi}{2\pi i(n-m)}$$

$$= \frac{\sin(n-m)\pi}{\pi(n-m)}$$

$$= 0$$

{Since $n \neq m \in \mathbb{Z}$ and $\sin k\pi = 0 \quad k \in \mathbb{Z}$}

$$\therefore \langle f_n, f_m \rangle = 0, \quad n \neq m$$

Now consider,

$$\| f_n \|_2^2 = \langle f_n, f_n \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(t)\overline{f_n(t)}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\text{int}}\, e^{-\text{int}} dt$$

$$= \frac{1}{2\pi}[\pi + \pi] = 1$$

$$\langle f_n, f_n \rangle = 1 \quad \forall n$$

Hence,

$$\langle f_n, f_m \rangle = 0, \quad n \neq m \text{ and } \| f_n \|_2 = 1 \quad \forall\, n.$$

Thus, set $\{ f_n(t) | n = 0, \pm 1, \pm 2, ... \}$ is orthonormal.

Step 2 : Claim : $\{ f_n(t) | n = 0, \pm 1, \pm 2, ... \}$ is basis for $H = L^2[-\pi, \pi]$.

Since $\{ f_n(t) | n = 0, \pm 1, \pm 2, ... \}$ is linearly independent and it spans $H = L^2[-\pi, \pi]$, hence $\{ f_n(t) | n = 0, \pm 1, \pm 2, ... \}$ is basis for $H = L^2[-\pi, \pi]$.

Theorem 3: The set $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos nt}{\sqrt{\pi}}, \frac{\sin nt}{\sqrt{\pi}} \middle| n = 1, 2, ..., t \in [-\pi, \pi]\right\}$ is an orthonormal basis for $L^2[-\pi, \pi]$.

Prove of this theorem is similar to above theorem so left as an exercise

8.4 MEAN SQUARE CONVERGENCE:

Consider space R of integrable functions define on the circle.

Let $e_n(\theta) = e^{in\theta}$, n is an integer then clearly, Set $\{e_n\}_{n\in z}$ is orthonormal.

Consider, $\langle f, e_n \rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{e^{in\theta}}\, d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-in\theta}\, d\theta$$

$$= \hat{f}(n) = a_n \text{ \{Fourier coefficient\}}$$

where, $\hat{f}(n)$ or a_n is Fourier coefficient of complex Fourier series of function f.

Consider the N[th] partial Sum , $S_N(f) = \sum_{|n|\le N} a_n e_n$

Then orthonormal property of family $\{e_n\}$ and the fact that $\langle f, e_n \rangle = a_n$ gives that the difference $f - S_N(f)$ is orthogonal to e_n i.e. $f - S_N(f) \perp e_n \ \forall |n| \le N$.

Since $\langle f - S_N(f), e_n \rangle = \langle f, e_n \rangle - \langle S_N(f), e_n \rangle$

$$= a_n - \left\langle \sum_{|m|\le N} a_m e_m, e_n \right\rangle$$

$$= a_n - \sum a_m \langle e_m, e_n \rangle$$

$$= a_n - a_n \quad \left\{ \begin{aligned} \langle e_m, e_n \rangle &= 1\, m = n \\ &= 0\, m \ne n \end{aligned} \right\}$$

$$= 0$$

Hence, $(f - S_N(f))$ is orthogonal to e_n, $\forall |n| \le N$

$\Rightarrow \left(f - \sum_{|n|\le N} a_n e_n \right)$ is orthogonal to $\sum_{|n|\le N} b_n e_n$ where, b_n is complex.

We have, $f = f - \sum_{|n|\le N} a_n e_n + \sum_{|n|\le N} b_n e_n$

$\therefore$ By Pythagorean theorem, $\|f\|_2^2 = \left\| f - \sum_{|n|\le N} a_n e_n \right\|_2^2 + \left\| \sum_{|n|\le N} b_n e_n \right\|_2^2$

when $a_n = b_n$ the orthogonal property of family $\{e_n\}_{n\in\mathbb{Z}}$ gives us $\left\| \sum_{|n|\le N} a_n e_n \right\|_2^2 = \sum_{|n|\le N} |a_n|^2$.

$\|f\|_2^2 = \|f - S_N(f)\|_2^2 + \sum_{|n|\le N} |a_n|^2$.

This is called **mean square approximation**.

8.5 BEST APPROXIMATION LEMMA :

Statement: If f is integrable function defined on a circle with Fourier co-efficient a_n then $\|f - S_N(f)\| \le \left\| f - \sum_{|n|\le N} c_n e_n \right\|$ for any complex number c_n. Moreover, equality holds when $c_n = a_n \ \forall |n| \le N$.

Proof :

Consider

$$f - \sum_{|n|\le N} c_n e_n = f - \sum_{|n|\le N} (a_n - b_n) e_n \quad \text{where } a_n - b_n = c_n$$

$$f - \sum_{|n|\le N} c_n e_n = f - \sum_{|n|\le N} a_n e_n + \sum_{|n|\le N} b_n e_n$$

Taking norm on both sides.

$$\left\| f - \sum_{|n|\le N} c_n e_n \right\| = \left\| f - \sum_{|n|\le N} a_n e_n + \sum_{|n|\le N} b_n e_n \right\|$$

Since a_n is Fourier coefficient $\sum_{|n|\le N} a_n e_n = S_N(f)$

$$\left\| f - \sum_{|n|\le N} c_n e_n \right\| = \left\| f - S_N(f) + \sum_{|n|\le N} b_n e_n \right\|.$$

Also we have $f - S_N(f)$ is orthogonal to $\sum_{|n|\le N} b_n e_n$.

By Pythagorean theorem.

$$\left\| f - \sum_{|n| \le N} c_n e_n \right\|^2 = \left\| f - S_N(f) \right\|^2 + \left\| \sum_{|n| \le N} b_n e_n \right\|^2$$

This statement gives us, $\left\| f - \sum_{|n| \le N} c_n e_n \right\| \ge \left\| f - S_N(f) \right\|$

when $c_n = a_n$ where, a_n is Fourier coefficient given.
$c_n = a_n - b_n \Rightarrow b_n = 0$

$$\Rightarrow \left\| f - S_N(f) \right\| = \left\| f - \sum_{|n| \le N} c_n e_n \right\|$$

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9

RIESZ FISHER THEOREM

Unit Structure :

9.1 COMPLETENESS OF $L^2\left(\mathbb{R}^d\right)$:

Theorem : The space $L^2\left(\mathbb{R}^d\right)$ is complete in its metric.

Proof : Let $\left\{f_n\right\}_{n=1}^{\infty}$ be a Cauchy sequence in L^2.

Consider $\left\{f_{n_k}\right\}_{k=1}^{\infty}$ be a subsequence of $\left\{f_n\right\}_{n=1}^{\infty}$ with the property

$$\left\|f_{n_{k+1}}-f_{n_k}\right\|\le 2^{-k} \quad \forall k\ge 1 \tag{1}$$

Let $$f(x)=f_{n_1}(x)+\sum_{k=1}^{\infty}\left(f_{n_{k+1}}(x)-f_{n_k}(x)\right) \tag{2}$$

and $$g(x)=\left|f_{n_1}(x)\right|+\sum_{k=1}^{\infty}\left|f_{n_{k+1}}(x)-f_{n_k}(x)\right| \tag{3}$$

Consider partial sum

$$S_k f(x)=f_{n_1}(x)+\sum_{k=1}^{K}\left(f_{n_{k+1}}(x)-f_{n_k}(x)\right)$$

and $$S_k\ g(x)=\left|f_{n_1}(x)\right|+\sum_{k=1}^{K}\left|f_{n_{k+1}}(x)-f_{n_k}(x)\right|$$

The triangle inequality implies that

$$\left\|S_k(g)\right\|\le\left\|f_{n_1}\right\|+\sum_{k=1}^{K}\left\|f_{n_{k+1}}-f_{n_k}\right\|$$

$$\le\left\|f_{n_1}\right\|+\sum_{k=1}^{K}2^{-k} \qquad \{\text{by } (1)\}$$

Letting $K \to \infty$ and applying monotone convergence theorem we have,

$$\int |g|^2 < \infty$$

Since $k \to \infty \quad 2^{-k} \to 0$

$$\therefore \|S_k(g)\| \le \|f_{n_1}(x)\|$$

$\therefore k^{th}$ partial sum of g is finite

$\therefore g$ is square summable & hence square integrable

$$\Rightarrow \int |g|^2 < \infty$$

{by (2) & (3)

$$\Rightarrow |f| \le g$$

$\Rightarrow \int |f|^2 < \infty$

$\Rightarrow f \in L^2(\mathbb{R}^d)$

In particular, the series defining f converges almost everywhere and since $(k-1)^{th}$ partial sum of this series is precisely f_{n_k}, we have, $f_{n_k} \to f(x)$ almost everywhere for all x.

To show $f_{n_k} \to f$ in $L^2(\mathbb{R}^d)$

We have, $|f - S_k(f)|^2 \le (2g)^2 \quad \forall \ k$

Applying dominated convergence theorem, we obtain, $\|f_{n_k} - f\| \to 0$ as $k \to \infty$.

Since $\{f_n\}_{n=1}^{\infty}$ is Cauchy sequence for given $\in > 0, \exists\ N$ such that $n, m \ge N \ |f_n - f_m| < \in/2$.

If n_k is chosen, so that $n_k > N$

$$\|f_{n_k} \to f\| < \in/2$$

$\therefore$ By triangle inequality

$\|f_n - f\| \le \|f_n - f_{n_k}\| + \|f_{n_k} - f\| < \in/2 + \in/2 = \in$

$\|f_n - f\| < \in$ whenever $n > N$

Hence sequence $\{f_n\} \to f$ in $L^2(\mathbb{R}^d)$

$\therefore L^2(\mathbb{R}^d)$ is complete.

9.2 BESSEL'S INEQUALITY FOR $L^2[-\pi, \pi]$:

If f is L^2-periodic function then $\sum\left|\hat{f}(n)\right|^2 \le \|f\|_2^2$.

Proof : Let $f - S_N(f) = g$ where $S_N(f)$ is N[th] partial sum of f i.e. $S_N(f) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$.

Consider,

$$
\begin{aligned}
\langle g, e_n \rangle &= \langle f - S_N(f), e_n \rangle \\
&= \langle f, e_n \rangle - \langle S_N(f), e_n \rangle \\
&= \hat{f}\langle n \rangle - \left\langle \sum_{-N}^{N} \hat{f}(m)e^{imx}, e_n \right\rangle \\
&= \hat{f}\langle n \rangle - \sum_{-N}^{N} \hat{f}(m)\left\langle e^{imx}, e_n \right\rangle \\
&= \hat{f}\langle n \rangle - \sum_{-N}^{N} \hat{f}(m)\left\langle e^{imx}, e^{inx} \right\rangle \\
&= \hat{f}\langle n \rangle - \hat{f}\langle n \rangle \qquad e_n = e^{inx} \\
&= 0
\end{aligned}
$$

$$
\{\langle e_n, e_m \rangle = 0 \;\; m \ne m
$$

$$
= 1 \;\; m = n \;\}
$$

Consider,

$$
\begin{aligned}
\|f\|_2^2 &= \|S_N(f) + g\|_2^2 \\
&= \langle S_N(f) + g, S_N(f) + g \rangle \\
&= \langle S_N(f), S_N(f) \rangle + \langle S_N(f), g \rangle + \langle g, S_N(f) \rangle + \langle g, g \rangle \\
&= \langle S_N(f), S_N(f) \rangle + \langle g, g \rangle
\end{aligned}
$$

$$
\{\langle g, e_n \rangle = 0
$$

$$
\Rightarrow \left\langle g, \sum \hat{f}(n)e_n \right\rangle = 0
$$

$$
\Rightarrow \langle g, S_N(f) \rangle = 0
$$

$$
\therefore \|f\|_2^2 = \|S_N(f)\|_2^2 + \|g\|_2^2
$$

$$
\therefore \|f\|_2^2 \ge \|S_N(f)\|_2^2 \qquad (1)
$$

Consider,

$$\left\|S_N(f)\right\|_2^2 = \left\langle S_N(f), S_N(f)\right\rangle$$

$$= \left\langle \sum_{-N}^{N} \hat{f}(n)e^{inx}, \sum_{-N}^{N} \hat{f}(m)e^{imx}\right\rangle$$

$$= \sum_{n,m=-N}^{N} \hat{f}(n)\,\overline{\hat{f}(m)}\left\langle e^{inx}, e^{imx}\right\rangle$$

$$= \sum_{-N}^{N} \hat{f}(n)\,\overline{\hat{f}(n)} = \sum_{-N}^{N}\left|\hat{f}(n)\right|^2 \qquad (2)$$

Substituting (2) in (1) we get

$$\|f\|_2^2 \geq \sum_{-N}^{N}\left|\hat{f}(n)\right|^2$$

Writing $N \to \infty$

$$\sum_{n=-\infty}^{\infty}\left|\hat{f}(n)\right|^2 \leq \|f\|_2^2$$

Thus we proved,

$$\sum_{n=-\infty}^{\infty}\left|\hat{f}(n)\right|^2 \leq \|f\|_2^2$$

9.3 THE RIESZ FISHER THEOREM:

Statement : Suppose that f is L^2-periodic function then the Nth partial sum of its Fourier Series $S_N(f)$ converges to f in L^2 (I) where $I = [-\pi, \pi]$.

i.e. $\lim\limits_{N\to\infty}\left\|S_N(f) - f\right\|_2 = 0$

Moreover, $\sum\limits_{n=-\infty}^{\infty}\left|\hat{f}(n)\right|^2 = \|f\|_2^2$ {**Parseval's identity**}

Conversely, suppose that $\{a_n\}_{n\in\mathbb{Z}}$ is two sided complex sequence which is square summable i.e. $\sum|a_n|^2 < \infty$ then there is unique function f in L^2 (I) that has a_n as its Fourier coefficient.

Proof : Step (1) : Let $f \in L^2$(I)

Given $\in > o$ choose a continuous periodic function g such $\|f - g\| < \in$ ……(1)

Then $\left\|S_N(f) - f\right\|_2 = \left\|S_N(f-g) + S_N(g) - (f-g) - g\right\|_2$

$\Rightarrow \left\|S_N(f) - f\right\|_2 \leq \left\|S_N(f-g)\right\|_2 + \left\|S_N(g) - g\right\|_2 + \|g - f\|_2$

We have $\left\|S_N(f)\right\| \le \|f\|$

$\Rightarrow \left\|S_N(f-g)\right\| \le \|f-g\|$

$\therefore \left\|S_N(f)-f\right\|_2 \le \|f-g\|_2 + \left\|S_N(g)-g\right\|_2 \le \|g-f\|_2$

$< \left\|S_N(g)-g\right\|_2 + 2\in ..... \; from(1)$

$\therefore \left\|S_N(f)-f\right\|_2 < 2\in + \left\|S_N(g)-g\right\|_2$

Since g is continuous periodic function,

$\therefore \left\|S_N(g)-g\right\|_2 \le \in$ for large N

$\left\|S_N(f)-f\right\|_2 < 3\in$

$\therefore \left\|S_N(f)-f\right\|_2 \to 0 \; as \; N \to \infty$

$\therefore \lim_{N\to\infty} \left\|S_N(f)-f\right\|_2 = 0$

Step (2) We have

$f - S_N(f) \perp S_N(f)$ i.e. $f - S_N(f)$ is orthogonal to $S_N(f)$.

$\therefore$ By Pythagorean theorem,

$\|f\|_2^2 = \left\|f - S_N(f)\right\|_2^2 + \left\|S_N(f)\right\|_2^2$

Also we have $\left\|S_N(f)\right\|_2^2 = \sum_{-N}^{N} \left|\hat{f}(n)\right|^2$

We get $\|f\|_2^2 = \left\|f - S_N(f)\right\|_2^2 + \sum_{-N}^{N} \left|\hat{f}(n)\right|^2$.

Letting $N \to \infty$ and using $\lim_{N\to\infty} \left\|f - S_N(f)\right\|^2{}_2 = 0$

$$\|f\|_2^2 = \sum_{-\infty}^{\infty} \left|\hat{f}(n)\right|^2$$

(This is known as **Parseval's Identity**)

Step (3) Converse part :

Suppose that $(a_n)_{n=-\infty}^{\infty}$ is square summable two sided sequence of complex numbers.

Let $f_N(x) = \sum_{n=-N}^{N} a_n e^{inx}$.

The orthonormality of exponential function e_n implies that for M<N.

$\|f_N - f_M\|_2^2 = \sum_{M<|n|\le N} |a_n|^2 \; \left\{\|f\|_2^2 = \sum |a_n|^2 \; parseval's\, identity\right\}$ and $a_n = \hat{f}_n$.

By the assumption of square summability i.e. $\sum |a_n|^2 < \infty$.

The right side of above equation converges to zero as $M, N \to \infty$. i.e. $\| f_N - f_M \|_2 \to 0$ as $N, M \to \infty$.

$\therefore \{ f_n \}$ is Cauchy sequence in $L^2(I)$.

Let f be the limit

By orthonormality, $\langle f_N, e_n \rangle = a_n \quad \forall N \,\&\, n$

Letting $N \to \infty$

$$\lim_{N\to\infty} \langle f_N, e_n \rangle = a_n \quad \forall n$$

$$\therefore \langle f, e_n \rangle = a_n \qquad \forall n$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = a_n \quad \forall n$$

$\Rightarrow a_n$ is Fourier coefficient of Fourier series of function f. Also by uniqueness of Fourier series, we can conclude that there exists unique f whose Fourier coefficient is a_n.

9.4 UNITARY ISOMORPHISM :

Unitary Mappings : Suppose H & H' be two given Hilbert spaces with respect to inner product $(\cdot,\cdot)_H$ & $(\cdot,\cdot)_{H^1}$ and corresponding norm $\|\cdot\|_H$ & $\|\cdot\|_{H^1}$.

A mapping $U : H \to H^1$ is called unitary mapping if

1) U is linear

i.e. $U(\alpha f + \beta g) = \alpha U(f) + \beta U(g)$ where α, β are scalars & $f, g \in H$.

2) U is bijection

3) $\|Uf\|_{H^1} = \| f \|_H \quad \forall f \in H$

Note :

1) Since unitary mapping U is bijective, its inverse $U^{-1} : H^1 \to H$ is also unitary mapping. (prove it)
2) Property (3) of unitary mapping implies that $(Uf, Ug)_{H^1} = (f, g)_H \quad \forall f, g \in H$

Unitary Isomorphism : Two Hilbert spaces $H \,\&\, H^1$ are said to be unitarily equivalent or unitary isomorphic if $\exists$ a unitary mapping $U : H \to H^1$.

Note : Unitary isomorphism of Hilbert spaces is an equivalence relation

Theorem : Any two infinite dimensional Hilbert spaces are unitary equivalent or unitary isomorphic.

Proof : If $H \,\&\, H'$ are two infinite dimensional Hilbert spaces.

We may select orthonormal basis i.e. $\{e_1, e_2, ...\}$ of H & $\{e_1', e_2', ...\}$ of H'.

Consider the mapping $U : H \to H'$ defined as if $f = \sum_{k=1}^{\infty} a_k e_k$ then $U(f) = g$ where, $g = \sum_{k=1}^{\infty} a_k e_k'$, $g \in H', f \in H$.

Claim : $U : H \to H^1$ is unitary

1) $U(\alpha f + \beta h) = \alpha U(f) + \beta U(h)$, $f, h \in H, \alpha, \beta$ are scalars.

Let $f = \sum_{k=1}^{\infty} a_k e_k$, $h = \sum_{k=1}^{\infty} b_k e_k$

Consider

$$U(\alpha f + \beta h) = U\left(\alpha \sum_{k=1}^{\infty} a_k e_k + \beta \sum_{k=1}^{\infty} b_k e_k\right)$$
$$= U\left(\sum_{k=1}^{\infty} (\alpha a_k + \beta b_k) e_k\right)$$
$$= \sum_{k=1}^{\infty} (\alpha a_k + \beta b_k) e_k'$$
$$= \sum \alpha a_k e_k' + \sum \beta b_k e_k'$$
$$= \alpha\left(\sum a_k e_k'\right) + \beta\left(\sum b_k e_k'\right)$$
$$= \alpha\, U(f) + \beta\, U(h)$$

2) Claim U is bijective
   Clearly, $U(f) = U(h)$

$$U\left(\sum a_k e_k\right) = U\left(\sum b_k e_k\right)$$
$$\sum a_k e_k' = \sum b_k e_k'$$
$$\Rightarrow a_k = b_k \qquad \forall k$$
$$\Rightarrow f = h$$
$$\Rightarrow U \text{ is one} - \text{one}$$

For any $g = \sum a_k e_k' \in H'$, we have $f = \sum a_k e_k \in H$ such that $U(f) = g \Rightarrow U$ is onto .

Clearly U is invertible

3) Claim $\|Uf\|_{H^1} = \|f\|_H$

Consider, $f = \sum_{k=1}^{\infty} a_k e_k \Rightarrow U(f) = \sum_{k=1}^{\infty} a_k e'_k$

$$\|Uf\|_{H^1} = \left\| \sum_{k=1}^{\infty} a_k e'_k \right\|_{H'}$$

$$= \left\| \sum_{k=1}^{\infty} |a_k^2| \right\|^{1/2} \;.....\{By\ parseval's\ identity\}$$

$$= \left\| \sum_{k=1}^{\infty} a_k e_k \right\|_H \;....\{again\ by\ parseval's\ identity\}$$

$$= \|f\|_H$$

Hence by (1), (2) & (3) $U : H \to H'$ is unitary and hence $H \,\&\, H'$ are unitary isomorphic.

Theorem : Suppose $f \in L^2[-\pi,\pi]$ then the mapping $f \to \{a_n\}$ is unitary correspondence between $L^2[-\pi,\pi]$ & square summable sequence $\ell^2(Z)$.

Proof :

Step (1) : Let $H = L^2[-\pi,\pi]$ with inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$$

Let $f \in L^2[-\pi,\pi]$

Let $[e_k]_{k=1}^{\infty}$ is an orthonormal basis for H.

$f = \sum_{k=1}^{\infty} a_k e_k, \quad a_k \in C$

Step (2) : Let $H^1 = \ell^2(Z)$ (sequence space) defined as

$\ell^2(Z) = \left\{ (.....a_{-1}, a_0, a_1....) \middle| a_j \in C \,\&\, \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$ with inner product.

$$\langle a,b\rangle = \sum_{k=+\infty}^{\infty} a_k \overline{b_k}$$

Step (3) Consider a mapping $U : H \to H'$ such that $f \to \{a_n\},\ f \in H \,\&\, \{a_n\} \in H'$

$$U\left(\sum a_k e_k\right) = \{a_k\}$$

Claim : $U : H \to H'$ is unitary

1) Sub-claim : U is linear

i.e. $U(\alpha f+\beta g)=\alpha U(f)+\beta U(g) \quad \alpha,\beta$ scalar $f,g\in H$

Let $f=\sum_{k=1}^{\infty}a_k e_k, \quad g=\sum_{k=1}^{\infty}b_k e_k$

$$\begin{aligned} U(\alpha f+\beta g) &= U(\alpha f+\beta g)\\ &= U\left(\alpha\sum a_k e_k+\beta\sum b_k e_k\right)\\ &= U\left(\sum(\alpha a_k+\beta b_k)e_k\right)\\ &= \{\alpha a_k+\beta b_k\}\\ &= \alpha\{a_k\}+\beta\{b_k\}\\ &= \alpha U(f)+\beta U(g) \end{aligned}$$

2) Sub-claim : U is bijective

i.e. U is one-one and onto.

Clearly, U is one-one

Since $U(f)=U(g)$

$$U\left(\sum a_k e_k\right)=U\left(\sum b_k e_k\right)$$

$$\Rightarrow \{a_k\}=\{b_k\}$$

$$\Rightarrow a_k=b_k \qquad \forall K$$

$$\Rightarrow \sum a_k e_k=\sum b_k e_k \Rightarrow f=g \Rightarrow U \text{ is one-one}$$

To Prove U is onto

Consider, $\|f-S_N f\|^2=\left\|\sum_{k=1}^{\infty}a_k e_k-\sum_{k=1}^{N}a_k e_k\right\|^2$

$$=\left\|\sum_{k=N+1}^{\infty}a_k e_k\right\|^2$$

$$=\sum_{n=N+1}^{\infty}|a_n|^2$$

If $\{a_n\}\in\ell^2(z)$ then

$$\|S_N(f)-S_M(f)\|^2=\left\|\sum_{k=1}^{N}a_k e_k-\sum_{k=1}^{M}a_k e_k\right\|^2 \quad N>M$$

$$=\left\|\sum_{k=M+1}^{N}a_k e_k\right\|^2$$

$$=\sum_{k=M+1}^{N}|a_k|^2<\infty$$

$\therefore \|S_N(f)-S_M(f)\|\to 0$ as $N,M\to\infty$.

Hence completeness of L^2 guarantee that, there is $f \in L^2$ such that $\| f - S_N f \| \to 0$ as $N \to \infty$.

As f has $\{a_n\}$ as its Fourier coefficient we can conclude that $f \to \{a_n\}$ is onto (By the uniqueness of Fourier coefficient)

Hence U is bijective

3) Claim : $\|Uf\|_{H^1} = \| f \|_H$

Consider, $\|Uf\|^2_{H^1} = \|a_n\|^2_{H^1}$

$$= \langle a_n, a_n \rangle$$
$$= \sum a_n \, \overline{a}_n$$
$$= \sum |a_n|^2$$
$$= \| f \|^2_H$$

Hence by (1) , (2) & (3) , $U : H \to H^1$ is unitary mapping.

9.5 SEPARABLE HILBERT SPACE:

Definition : The space H is said to be separable if their exist countable collection $\{ f_k \}$ of elements in the space H such that there linear combination are dense in space H.

Theorem : A Hilbert Space H is separable if and only if it has countable orthonormal basis.

Proof : Step 1: Suppose that Hilbert space H is separable.

Claims : Hilbert space H has countable orthornormal basis.
Suppose Hilbert space H has uncountable orthornormal basis say $\{e_\alpha\}_{\alpha \in \Delta}$

Then $\|e_\alpha - e_\beta\| > 1, \ \forall \alpha, \beta \in \Delta \ \& \ \alpha \neq \beta$

$$\Rightarrow S\left(e_\alpha, \frac{1}{2}\right) \cap S\left(e_\beta, \frac{1}{2}\right) = \Phi \qquad \forall \alpha, \beta \in \Delta \,\&\, \alpha \neq \beta .$$

Hence there exist an uncountable family of disjoint open sphere with radius ½.

$\Rightarrow$ H is not separable which is a contradiction to our assumption.
Hence Hilbert space H has countable orthonormal basis.

Step (2) Converse part
Hilbert Space H has countable orthonormal basis

Claim : Hilbert space H is separable.
Let H has a countable orthornormal basis say $\{e_n\}$.
Let $f \in H$

$$f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n \quad \forall f \in H$$

$$\begin{cases} \langle f, e_n \rangle = a_n \\ f = \sum_{n=1}^{\infty} a_n e_n \end{cases}$$

$\Rightarrow f \in H$ is a cluster point (i.e. limit point) of set of linear combination of elements of $\{e_n\}$.

Since $\{e_n\}$ is complete orthonormal basis, set of linear combination of elements of $\{e_n\}$ contains countable dense set of linear combination of $\{e_n\}$ with rational coefficients.

Hence H is separable Hilbert space.

Theorem : Hilbert Space $L^2[-\pi, \pi]$ is separable.

Proof : Step (1) : Let $H = L^2[-\pi, \pi]$
We know that Hilbert space $L^2[-\pi, \pi]$ has an orthonormal basis $\{ f_n \,|n = 0, \pm 1|, \pm 2, ...\}$.

Where, $f_n(t) = \frac{e^{\text{int}}}{\sqrt{2\pi}}, n \in \mathbb{Z}, t \in [-\pi, \pi]$.

Since set of integer is countable, hence set of orthonormal basis $\{ f_n \,|n = 0, \pm 1|, \pm 2, ...\}$ is countable.

Step (2) : If Hilbert Space H has a countable orthonormal basis then H is separable.

Step (3) : Hilbert Space $L^2[-\pi, \pi]$ has a countable orthonormal basis. Hence H is separable.

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