F.Y.B.SC (I.T)

SEMESTER - II (CBCS)

NUMERICAL AND STATISTICAL METHODS

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Books and References:					
Sr. No.	**Title**	**Author/s**	**Publisher**	**Edition**	**Year**
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UNIT 1

1

MATHEMATICAL MODELING AND ENGINEERING PROBLEM SOLVING

Unit Structure

1.0 Objectives

The objective of this chapter is to introduce the reader to mathematical modeling and its application in engineering problem solving. This chapter also discuss the conservation laws and how numerical methods are useful in engineering problem solving.

1.1 Introduction

Mathematics is widely used in solving real-world problems especially due to the increasing computational power of digital computers and computing methods, which have facilitated the easy handling of lengthy and complicated problems.

Numerical methods are techniques by which mathematical problems are formulated so that they can be solved with arithmetic operations. Although there are many kinds of numerical methods, they have one common characteristic: they invariably involve large numbers of tedious arithmetic calculations and with the development of fast, efficient digital computers, the role of numerical methods in solving engineering problems has increased.

Translating a real-life problem into a mathematical form can give a better representation of certain problems and helps to find a solution for the problem.

There are many advantages of translating a real-life problem into a mathematical form like

- Mathematics helps us to formulate a real-world problem.
- Computers can be used to perform lengthy numerical calculations.
- Mathematics has well defined rules for manipulations.

In the coming sections we will discuss the concepts of mathematical modelling and how to model a given problem. We will also discuss how mathematical modeling can be used in engineering problem solving.

1.2 Mathematical Modeling

The method of translating a real-life problem into a mathematical form is called mathematical modeling. Mathematical modeling is a tool widely used in science and engineering. They provide a rigorous description of various real-world phenomena. Mathematical modeling helps to understand and analyse these real-world phenomena.

In mathematical modelling, we consider a real-world problem and write it as a mathematical problem. We then solve this mathematical problem, and interpret the solution in terms of the real-world problem which was considered. We will see how to formulate a given problem using a very simple example.

Example 1.2.1: To travel 500 kms if 50 litres of petrol is required. How much petrol is needed to go to a place which is 200 kms away?

Solution. *Step I*: We will first mathematically formulate this given problem:

Let x be the litres of petrol needed and y be the kilometres travelled. Then we know that more the distance travelled, more is the petrol required. That is x is directly proportional to y.

Thus, we can write

$$x = ky$$

where k is a positive constant.

Step II: Solving the model using the given data:

Now, since it is given that to travel 500 kms if 50 litres of petrol are required, we get

$$50 = k \times 500$$

That is

$$k = \frac{1}{10}$$

Step III: Using the solution obtained in Step II to interpret the given problem:

To find the litres of petrol required to travel 200kms, we substitute the values in the mathematical model. That is

$$x = \frac{1}{10} \times 200$$

which gives

$$x = 20$$

That is, we need 20 litres of petrol to travel 200kms away.

Now, we will see one more example to get a better understanding of mathematical modeling.

Example 1.2.2: A motorboat goes upstream on a river and covers the distance between two towns on the riverbank in 12 hours. It covers this distance downstream in 10 hours. If the speed of the stream is 3 km/hr, find the speed of the boat in still water.

Solution. *Step I*: We will first mathematically formulate this given problem:

Let x be the speed of the boat, t be the time taken and y be the distance travelled. Then using the formula for speed, distance and time we get

$$y = tx$$

Step II: Solving the model using the given data:

We know that while going upstream, the actual speed of the boat is equal to

$$speed\ of\ the\ boat - speed\ of\ the\ river$$

Hence, in upstream, the actual speed of the boat is $x - 3$.

Similarly, while going downstream, the actual speed of the boat is equal to

$$speed\ of\ the\ boat + speed\ of\ the\ river$$

Hence, in downstream, the actual speed of the boat is $x + 3$.

Since, the time taken to travel upstream is 12 hours, we have

$$y = 12(x - 3)$$

and as the time taken to travel downstream is 12 hours, we have

$$y = 10(x + 3)$$

Thus,

$$12(x - 3) = 10(x + 3)$$

which gives $x = 33$.

Step III: Using the solution obtained in Step II to interpret the given problem:

From Step II it is clear that the speed of the boat in still river is 33 km/hr.

Our next example, will show how mathematical modeling can be used to study population of a country.

Example 1.2.3: Suppose the current population is 200,000,000 and the birth rate and death rates are 0.04 and 0.02 respectively. What will be the population in 5 years?

Solution. *Step I*: We will first mathematically formulate this given problem:

We know that the population of a country increases with birth and decreases with death. Let t denote time in years where $t = 0$ implies the present time. Let $p(t)$ denote the population at time t.

If $B(t)$ denote the number of births and $D(t)$ denote the number of deaths in year t then

$$p(t + 1) = p(t) + B(t) - D(t)$$

Let $b = \frac{B(t)}{p(t)}$ be the birth rate for the interval $t, t + 1$ and $d = \frac{D(t)}{p(t)}$ be the death rate for the interval $t, t + 1$.

Then

$$p(t + 1) = p(t) + bp(t) - dp(t)$$
$$= (1 + b - d)p(t)$$

Then, when $t = 0$ we get

$$p(1) = (1 + b - d)p(0)$$

Similarly, taking $t = 1$ we get

$$p(2) = (1 + b - d)p(1)$$
$$= (1 + b - d)^2 p(0)$$

Continuing in this manner we get

$$p(t) = (1 + b - d)^t p(0)$$

Taking $(1 - b - d) = c$ we get

$$p(t) = c^t p(0)$$

where c is called the growth rate.

Step II: Solving the model using the given data:

From the given data, we have $b = 0.04$ and $d = 0.02$. Thus

$$c = 1 + b - d$$
$$= 1.02$$

Step III: Using the solution obtained in Step II to interpret the given problem:

The population in 5 years can be estimated as

$$p(5) = (1.02)^5 \times 200{,}000{,}000$$
$$= 220816160.6$$

Since population cannot be in decimal, we round it to 220816161.

1.3 Conservation Law and Engineering Problems

Conservation laws of science and engineering basically deals with

$$Change = Increase - Decrease$$

This equation incorporates the fundamental way in which conservation laws are used in engineering that is to determine the change with respect to time.

Another way to use conservation laws is the case in which the change is 0, that is

$$Increase = Decrease$$

This case is also known as $Steadty - State$ computation.

Let A be the quantity of interest defined on a domain Ω. Then the rate of change of Q is equal to the total amount of Q produced or destroyed in $\omega \subseteq \Omega$ and the flux of Q across the boundary $\partial\omega$ that is the amount of Q that either goes in or comes out of ω.

This can be mathematically expressed as

$$\frac{d}{dt}\int_{\omega} Q\,dx = \int_{\omega} S dx - \int_{\partial\omega} Fv\,d\sigma(x)$$

where v is the unit outward normal, $d\sigma(x)$ is the surface measure and F, S are flux and the quantity produced or destroyed respectively.

On simplifying, using integration by parts rule, we get

$$U_t + div(F) = S$$

where $div(F)$ is the divergence.

When S is taken to be zero, we get

$$U_t + div(F) = 0$$

which is called the conservation law, as the only change in U comes from the quantity entering or leaving the domain of interest.

As an example, consider the following:

Heat Equation:

Assume that a hot rod is heated at one end and is left to cool, without providing any further source of heat. The heat spreads uniformly (that is diffuses out) and the temperature of the rod becomes uniform after some time.

Here, let U be the temperature of the material. The diffusion of the heat is given by Fourier's law as

$$F(U) = -k\,\nabla U$$

where k is the conductivity of the medium.

Thus, the heat equation is obtained as

$$U_t - div(k\,\nabla U) = 0$$

Conservation laws arise in many models in science and numerical methods play a very important role in approximating or simulating the solutions of conservation laws. Most of the engineering problems deals with conservation laws.

- Chemical engineering focus on mass balance reactors.
- In civil engineering force balances are utilized to analyse structures.
- In mechanical engineering the same principles are used to analyse the transient up-and down motion or vibrations of an automobile.
- Electrical engineering uses conservation of energy voltage balance.

1.4 Summary

This chapter

- gives an introduction to mathematical modeling.
- discusses the application of mathematical modelling in engineering problem solving.
- discusses the conservation laws and how numerical methods are useful in engineering problem solving.

1.5 Exercises

1. An investor invested ₹$10,000/-$ at 10% simple interest per year. With the return from the investment, he wants to buy a T.V. that costs ₹ $20,000/-$. After how many years will he be able to buy the T.V.?

2. A car starts from a place A and travels at a speed of 30 km/hr towards another place B. At the same time another car starts from B and travels towards A at a speed of 20 km/hr. If the distance between A and B is 120 kms, after how much time will the cars meet?

3. A farmhouse uses atleast 1000 kg of special food daily. The special food is a mixture of corn and bean with the composition

Material	**Nutrient Present Per Kg**		**Cost Per Kg**
	Protein	**Fibre**	
Corn	0.09	0.02	₹ 10
Bean	0.6	0.06	₹ 20

The dietary requirements of the special food are atleast 30% protein and atmost 5% fibre. Formulate this problem to minimise the cost of the food.

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UNIT 1

2

APPROXIMATION & ROUND-OFF ERRORS

Unit Structure

2.0 Objectives

After reading this chapter, you should be able to:

1. Know the concept of significant digits.
2. Know the concept Accuracy & precision.
3. find the true and relative true error,
4. find the approximate and relative approximate error,
5. relate the absolute relative approximate error to the number of significant digits
6. Know other different errors including round-off error.

2.1 Introduction

Approximate numbers: There are two types of numbers exact and approximate. Exact numbers are 2, 4, 9, $\frac{7}{2}$, 6.45, ... etc. but there are numbers such that $\frac{4}{3}$ (=1.333…..) , $\sqrt{2}$ (= 1.414213 ...) and π (= 3.141592. …) which cannot be expressed by a finite number of digits. These may be approximated by numbers 1.3333,1.4141, and 3.1416, respectively. Such numbers, which represent the given numbers to a certain degree of accuracy, are called approximate numbers.

Rounding-off: There are numbers with many digits, e.g., $\frac{22}{7}$ = 3.142857143.

In practice, it is desirable to limit such numbers to a manageable number of digits, such as 3.14 or 3.143. This process of dropping unwanted digits is called rounding-off.

In this chapter information concerned with the quantification of error is discussed in the first sections. This is followed by a section on one of the two major forms of numerical error: round-off error. Round-off error is due to the fact that computers can represent only quantities with a finite number of digits.

2.2 Significant Figures

Whenever we use a number in a computation, we must have assurance that it can be used with confidence. For example, Fig. 2.1 depicts a speedometer and odometer from an automobile.

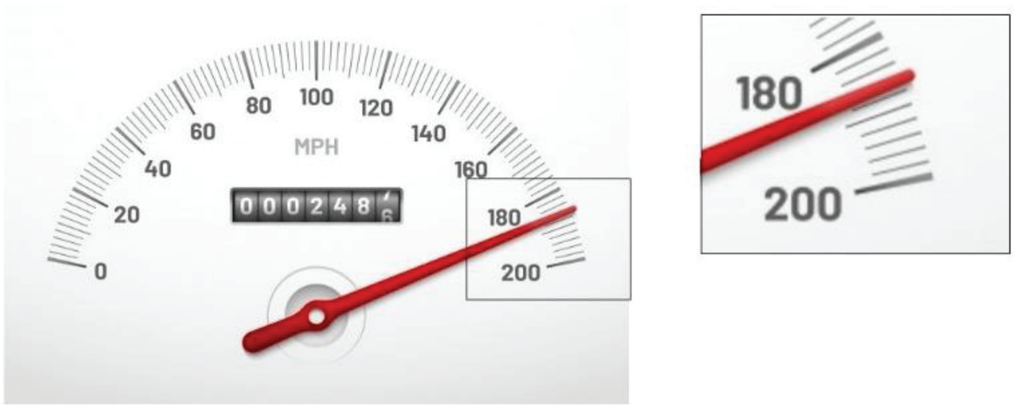


Fig. 2.1: An automobile speedometer and odometer illustrating the concept of a significant figure.

Visual inspection of the speedometer indicates the speed between 180 and 200 MPH. Because the indicator is lesser than the midpoint between the markers on the gauge, it can be said with assurance that the car is traveling at approximately 190 MPH. However, let us say that we insist that the speed be estimated to one decimal place. For this case, one person might say 188.8, whereas another might say 188.9 MPH. Therefore, because of the limits of this instrument, only the first three digits can be used with confidence. Estimates of the fourth digit (or higher) must be viewed as approximations. It would be nonsensical to claim, on the basis of this speedometer, that the automobile is traveling at 188.8642138 MPH. In contrast, the odometer provides up to six certain digits. From Fig. 2.1, we can conclude that the car has traveled slightly less than 248.5 km during its lifetime. In this case, the fifth digit (and higher) is uncertain.

Definition: *The number of significant figures or significant digits in the representation of a number is the number of digits that can be used with confidence*. In particular, for our purposes, the number of significant digits is equal to the number of digits that are known (or assumed) to be correct plus one estimated digit.

- The concept of a significant figure, or digit, has been developed to formally designate the reliability of a numerical value.
- The significant digits of a number are those that can be used with confidence.
- They correspond to the number of certain digits plus one estimated digit.

Significant digits. The digits used to express a number are called significant digits.

The digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are significant digits. ‘0’ is also a significant digit except when it is used to fix the decimal point or to fill the places of unknown or discarded digits.

For example, each of the numbers 7845, 3.589, and 0.4758 contains 4 significant figures while the numbers 0.00386, 0.000587, 0.0000296 contain only three significant figures (since zeros only help to fix the position of the decimal point).

Similarly, in the number 0.0003090, the first four ‘0’ s’ are not significant digits since they serve only to fix the position of the decimal point and indicate the place values of the other digits. The other two ‘0’ s’ are significant.

To be more clear, the number 3.0686 contains five significant digits.

The significant figure in a number in positional notation consists of

1) All non-zero digits
2) Zero digits which
   - lie between significant digits;
   - lie to the right of decimal point and at the same time to the right of a non-zero digit;

are specifically indicated to be significant.

The significant figure in a number written in scientific notation

(e.g., $M \times 10^k$) consists of all the digits explicitly in M.

Example 1

Give some examples of showing the number of significant digits.

Solution

a) 0.0459 has three significant digits
b) 4.590 has four significant digits
c) 4008 has four significant digits
d) 4008.0 has five significant digits
e) 1.079×10^3 has four significant digits
f) 1.0790×10^3 has five significant digits
g) 1.07900×10^3 has six significant digits

Significant digits are counted from left to right starting with the non- zero digit on the left.

A list is provided to help students understand how to calculate significant digits in a given number:

Number	Significant digits	Number of significant digits
3969	3, 9, 6, 9	04
3060	3, 0, 6	03
3900	3, 9	02
39.69	3, 9, 6, 9	04
0.3969	3, 9, 6, 9	04

Number	Significant digits	Number of significant digits
39.00	3, 9, 0, 0	04
0.00039	3, 9	02
0.00390	3, 9, 0	03
3.0069	3, 0, 0, 6, 9	05
3.9×10^6	3, 9	02
3.909×10^5	3, 9, 0, 9	04
6×10^{-2}	6	01

The concept of significant figures has two important implications for study of numerical methods:

1. *Numerical methods yield approximate results*. We must, therefore, develop criteria to specify how confident we are in our approximate result. One way to do this is in terms of significant figures. For example, we might decide that our approximation is acceptable if it is correct to four significant figures.

2. *Although quantities such as π, e, or √7 represent specific quantities, they cannot be expressed exactly by a limited number of digits.*

For example,

$\pi = 3.141592653589793238462643...$

ad infinitum. Because computers retain only a finite number of significant figures, such numbers can never be represented exactly. The omission of the remaining significant figures is called round-off error.

The concept of significant figures will have relevance to our definition of accuracy and precision in the next section.

Applications:

The output from a physical measuring device or sensor is generally known to be correct up to a fixed number of digits.

For example, if the temperature is measured with a thermometer is calibrated between 85.6 and 85.7 degrees. the first three digits of the temperature are known i.e. 8, 5, and 6, but do not know the value of any subsequent digits.

The first unknown digit is sometimes estimated at half of the value of the calibration size, or 0.05 degrees in our case. If we did this, the temperature would be reported to four significant digits as 85.65 degrees.

Alternatively, we can choose to report the temperature to only three significant digits.

In this case, we could truncate or chop off the unknown digits to give a result of 48.6 degrees, or round off the result to the nearest tenth of a degree to give either 48.6 or 48.7 degrees depending on whether the actual reading was more or less than half-way between the two calibrations.

Round-off is generally the preferred procedure in this example, but without knowing which technique was adopted, we would really only be confident in the first two digits of the temperature.

Hence, if the temperature is reported as 48.6 degrees without any further explanation, we do not know whether the 6 is a correct digit or an estimated digit.

2.3 Accuracy and Precision

The errors associated with both calculations and measurements can be characterized with regard to their accuracy and precision.

Accuracy refers to how closely a computed or measured value agrees with the true value.

Accuracy tells us how close a measured value is to the actual value.

It is associated with the quality of data and numbers of errors present in the data set. Accuracy can be calculated using a single factor or measurement.

Precision tells us how close measured values are to each other.

It often results in round off errors. To calculate precision, multiple measurements are required.

Example:- The distance between point A and B is 7.15. On measuring with different devices the distance appears as:

Data set 1: 6.34, 6.31, 6.32

Data set 2 : 7.11, 7.19, 7.9

Data set 1 is more precise since they are close to each other and data set 2 is more accurate since they are close to the actual value.

To round-off a number to n significant digits to get accuracy, discard all digits to the right of the n^{th} digit and if this discarded number is

- less than 5 in $(n + 1)^{th}$ place, leave the n^{th} digit unaltered. *e.g.*, 7.893 to 7.89.
- greater than 5 in $(n + 1)^{th}$ place, increase the n^{th} digit by unity, *e.g.*, 6.3456 to 6.346.
- exactly 5 in $(n + 1)^{th}$ place, increase the n^{th} digit by unity if it is odd, otherwise leave it unchanged.

  e.g., $12.675 \simeq 12.68$

  $12.685 \simeq 12.68$

The number thus rounded-off is said to be correct to n significant figures. A list is provided for explanatory proposes:

Number	**Rounded-off to**		
	Three digits	**Four digits**	**Five digits**
00.543241	00.543	00.5432	00.54324
39.5255	39.5	39.52	39.526
69.4155	69.4	69.42	69.416
00.667676	00.668	00.6677	00.66768

Graphical Illustration of Precision & Accuracy

Precision & Accuracy is illustrated graphically using an analogy from target practice.

The bullet holes on each target in Fig. 2.2 can be thought of as the predictions of a numerical technique, whereas the bull's-eye represents the truth.

- *Inaccuracy* (also called *bias*) is defined as systematic deviation from the truth. Is illustrated on circles on right side
- *Imprecision* (also called *uncertainty*), on the other hand, refers to the magnitude of the scatter is represented on left side circles where the shots are tightly grouped.

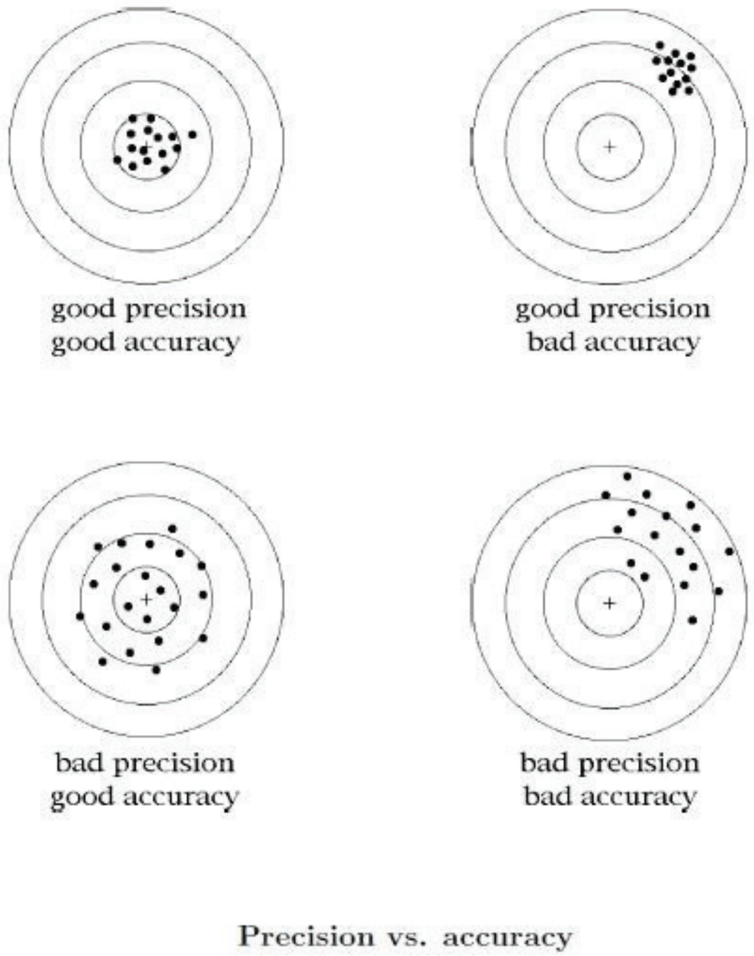


Fig 2.2 Graphical Illustration of Precision & Accuracy

2.4 Error Definitions

Machine epsilon

We know that a computer has a finite word length, so only a fixed number of digits is stored and used during computation. Hence, even in storing an exact decimal number in its converted form in the computer memory, an error is introduced. This error is machine dependant and is called machine epsilon.

Error = True value – Approximate

In any numerical computation, we come across the following types of errors:

Inherent errors. Errors which are already present in the statement of a problem before its solution are called inherent errors. Such errors arise either due to the fact that the given data is approximate or due to limitations of mathematical tables, calculators, or the digital computer.

Inherent errors can be minimized by taking better data or by using high precision computing aids. Accuracy refers to the number of significant digits in a value, for example, 53.965 is accurate to 5 significant digits.

Precision refers to the number of decimal positions or order of magnitude of the last digit in the value. For example, in 53.965, precision is 10^{-3}.

Example. *Which of the following numbers has the greatest precision?*

4.3201, 4.32, 4.320106.

Sol. In 4.3201, precision is 10^{-4}

In 4.32, precision is 10^{-2}

In 4.320106, precision is 10^{-6}.

Truncation errors

Truncation errors are caused by using approximate results or by replacing an infinite process with a finite one.

If we are using a decimal computer having a fixed word length of 4 digits, rounding-off of 13.658 gives 13.66, whereas truncation gives 13.65.

Therefore this difference in last significant digits causes error as result of difference in processing of rounding-off & truncating a number.

Numerical errors arise from the use of approximations to represent exact mathematical operations and quantities. These include truncation errors, which result when approximations are used to represent exact mathematical procedures, and round-off errors, which result when numbers having limited significant figures are used to represent exact numbers.

For both types, the relationship between the exact, or true, result and the approximation can be formulated as

$$\text{True value} = \text{approximation} + \text{error} \tag{2.1}$$

By rearranging Eq. (2.1), we find that the numerical error is equal to the discrepancy between the truth and the approximation, as in

$$E_t = \text{true value} - \text{approximation} \tag{2.2}$$

where E_t is used to designate the exact value of the error. The subscript t is included to designate that this is the “true” error. This is in contrast to other cases, as described shortly, where an “approximate” estimate of the error must be employed.

A shortcoming of this definition is that it takes no account of the order of magnitude of the value under examination. For example, an error of a centimeter is much more significant if we are measuring a rivet rather than a bridge. One way to account for the magnitudes of the quantities being evaluated is to normalize the error to the true value, as in

$$\text{True fractional relative error} = \frac{true\ error}{true\ value}$$

where, as specified by Eq. (2.2), error = true value − approximation.

The relative error can also be multiplied by 100 percent to express it as

$$\varepsilon_t = \frac{true\ error}{true\ value}\ X\ 100 \qquad (2.3)$$

where ε_t designates the true percent relative error.

2.4.1 Calculation of Errors

Example 2.4.1

Problem Statement. Suppose that you have the task of measuring the lengths of a bridge and a rivet and come up with 9999 and 9 cm, respectively. If the true values are 10,000 and 10 cm, respectively, compute (a) the true error and (b) the true percent relative error for each case.

Solution.

(a) The error for measuring the bridge is [Eq. (2.2)]

E_t = 10,000 − 9999 = 1 cm

and for the rivet it is

E_t = 10 − 9 = 1 cm

(b) The percent relative error for the bridge is [Eq. (2.3)]

$$\varepsilon_t = \frac{1}{10{,}000}\ X\ 100 = 0.01\%$$

and for the rivet it is

$$\varepsilon_t = \frac{1}{10}\ X\ 100 = 10\%$$

Thus, although both measurements have an error of 1 cm, the relative error for the rivet is much greater. We would conclude that we have done an adequate job of measuring the bridge, whereas our estimate for the rivet leaves something to be desired.

Notice that for Eqs. (2.2) and (2.3), E and ε are subscripted with a t to signify that the error is normalized to the true value.

In real-world applications, we will obviously not know the true answer a priori. For these situations, an alternative is to normalize the error using the best available estimate of the true value, that is, to the approximation itself, as

$$\varepsilon_a = \frac{\text{approximate error}}{\text{approximation}} X\,100 \qquad (2.4)$$

where the subscript a signifies that the error is normalized to an approximate value.

Note also that for real-world applications, Eq. (2.2) cannot be used to calculate the error term for Eq. (2.4). One of the challenges of numerical methods is to determine error estimates in the absence of knowledge regarding the true value.

For example, certain numerical methods use an iterative approach to compute answers. In such an approach, a present approximation is made on the basis of a previous approximation. This process is performed repeatedly, or iteratively, to successively compute (we hope) better and better approximations.

For such cases, the error is often estimated as the difference between previous and current approximations. Thus, percent relative error is determined according to

$$\varepsilon_a = \frac{\text{current approximation} - \text{previous approximation}}{\text{current approximation}} X\,100 \qquad (2.5)$$

The signs of Eqs. (2.2) through (2.5) may be either positive or negative. If the approximation is greater than the true value (or the previous approximation is greater than the current approximation), the error is negative; if the approximation is less than the true value, the error is positive. Also, for Eqs. (2.3) to (2.5), the denominator may be less than zero, which can also lead to a negative error. Often, when performing computations, we may not be concerned with the sign of the error, but we are interested in whether the percent absolute value is lower than a prespecified percent tolerance ε_s . Therefore, it is often useful to employ the absolute value of Eqs. (2.2) through (2.5). For such cases, the computation is repeated until

$$|\varepsilon_a| < \varepsilon_s \qquad (2.6)$$

If this relationship holds, our result is assumed to be within the prespecified acceptable level ε_s . Note that for the remainder of this text, we will almost exclusively employ absolute values when we use relative errors.

It is also convenient to relate these errors to the number of significant figures in the approximation.

It can be shown (Scarborough, 1966) that if the following criterion is met, we can be assured that the result is correct to at least n significant figures.

$$\varepsilon_s = (0.5 \times 102{-}n)\% \qquad (2.7)$$

2.4.2 Error Estimates for Iterative Methods

Problem Statement. In mathematics, functions can often be represented by infinite series. For example, the exponential function can be computed using

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \qquad (2.8)$$

Thus, as more terms are added in sequence, the approximation becomes a better and better estimate of the true value of e^x. Equation (Eq. 2.8) is called a *Maclaurin series expansion.*

Starting with the simplest version, $e^x = 1$, add terms one at a time to estimate $e^{0.5}$.

After each new term is added, compute the true and approximate percent relative errors with Eqs. (2.3) and (2.5), respectively. Note that the true value is $e^{0.5}$ = 1.648721 . . . Add terms until the absolute value of the approximate error estimate ε_a falls below a prespecified error criterion ε_s conforming to three significant figures.

Solution. First, Eq. (2.7) can be employed to determine the error criterion that ensures a result is correct to at least three significant figures:

$$\varepsilon_s = (0.5 \times 102{-}3)\% = 0.05\%$$

Thus, we will add terms to the series until ε_a falls below this level.

The first estimate is simply equal to Eq. (2.8) with a single term. Thus, the first estimate is equal to 1. The second estimate is then generated by adding the second term, as in

$e^x = 1 + x$

or for $x = 0.5$,

$e^{0.5} = 1 + 0.5 = 1.5$

This represents a true percent relative error of [Eq. (2.3)]

$$\varepsilon_t = \frac{1.648721 - 1.5}{1.648721} X\, 100 = 9.02\%$$

Equation (2.5) can be used to determine an approximate estimate of the error, as in

$$\varepsilon_a = \frac{1.5 - 1}{1.5} X\, 100 = 33\%$$

Because ε_a is not less than the required value of ε_s, we would continue the computation by adding another term, $x^2/2!$, and repeating the error calculations. The process is continued until $\varepsilon_a < \varepsilon_s$.

The entire computation can be summarized as

Terms	Result	$\varepsilon_t(\%)$	$\varepsilon_a(\%)$
1	1	39.3	
2	1.5	9.02	33.3
3	1.625	1.44	7.69
4	1.645833333	0.175	1.27
5	1.648437500	0.0172	0.158
6	1.648697917	0.00142	0.0158

Thus, after six terms are included, the approximate error falls below ε_s = 0.05% and the computation is terminated. However, notice that, rather than three significant figures, the result is accurate to five! This is because, for this case, both Eqs. (2.5) and (2.7) are conservative.

That is, they ensure that the result is at least as good as they specify.

2.5 Round-Off errors

Rounding errors. Rounding errors arise from the process of rounding- off numbers during the computation. They are also called procedual errors or numerical errors. Such errors are unavoidable in most of the calculations due to limitations of computing aids.

These errors can be reduced, however, by

I. changing the calculation procedure so as to avoid subtraction ofnearly equal numbers or division by a small number

II. retaining at least one more significant digit at each step and rounding-off at the last step. Rounding-off may be executed in two ways:

Chopping. In chopping, extra digits are dropped by truncation of number. Suppose we are using a computer with a fixed word length of four digits, then a number like 12.92364 will be stored as 12.92.

We can express the number 12.92364 in the floating print form as

True $\quad x = 12.92364$

$$= 0.1292364 \times 10^2 = (0.1292 + 0.0000364) \times 10^2$$

$$= 0.1292 \times 10^2 + 0.364 \times 10^{-4+2}$$

$$= f_x \, . \, 10^{E} + g_x \, . \, 10^{E-d}$$

$$= \text{Approximate } x + \text{Error}$$

$$\therefore \text{Error} = g_x \, . \, 10^{E-d}, \; 0 \le g_x \le d$$

Here, g_x is the mantissa, d is the length of mantissa and E is exponent

$$\text{Since} \quad 0 \le g_x < 1$$

$$\therefore \text{Absolute error} \le 10^{E-d}$$

Case I. If $g_x < 0.5$ then approximate $x = f_x \, . \, 10^{E}$

Case II. If $g_x \ge .5$ then approximate $x = f_x \, . \, 10^{E} + 10^{E-d}$

Error = True value – Approximate value

$$= f_x \, . \, 10^{E} + g_x \, . \, 10^{E-d} - f_x \, .10^{E} - 10^{E-d}$$

$$= (g_x - 1) \, . \, 10^{E-d}$$

$$\text{absolute error} \le 0.5.(10)^{E-d}.$$

Symmetric round-off. In symmetric round-off, the last retained significant digit is rounded up by unity if the first discarded digit is ≥ 5, otherwise the last retained digit is unchanged.

2.6 Problems on Error Definitions, Significant numbers & accuracy

Problem 1. Suppose 1.414 is used as an approximation to $\sqrt{2}$. Find the absolute and relative errors.

Sol. True value $= \sqrt{2} = 1.41421356$

Approximate value $= 1.414$

Error = True value – Approximate value

$$= \sqrt{2} - 1.414 = 1.41421356 - 1.414$$

$$= 0.00021356$$

Absolute error e_a $= |\text{ Error }|$

$$= |\,0.00021356\,| = 0.21356 \times 10^{-3}$$

Relative error e_r $= \dfrac{e_a}{\text{True value}} = \dfrac{0.21356 \times 10^{-3}}{\sqrt{2}}$

$$= 0.151 \times 10^{-3}.$$

Problem 2. If 0.333 is the approximate value of $\frac{1}{3}$, find the absolute, relative, and percentage errors.

Solution. True value (X) = $\frac{1}{3}$

Approximate value (X') = 0.333

$\therefore$ Absolute error $e_a = |\,X - X'|$

$$|\,\tfrac{1}{3} - 0.333| = |\,0.333333 - 0.333\,| = .000333$$

Relative error $e_r = \frac{e_a}{x} = \frac{0.000333}{0.333333} = 0.000999$

Percentage error $e_p = e_r \times 100 = .000999 \times 100 = .099\%$

Problem 3. An approximate value of π is given by 3.1428571 and its true value is 3.1415926. Find the absolute and relative errors.

Sol. True value = 3.1415926

Approximate value =3.1428571

Error = True value – Approximate value

$$= 3.1415926 - 3.1428571$$

$$= -0.0012645$$

Absolute error $e_a = |\text{ Error }| = 0.0012645$

Relative error $e_r = \frac{e_a}{True\ Value} = \frac{0.0012645}{3.1415926}$

$$= 0.000402502.$$

Problem 4. Three approximate values of the number $\frac{1}{3}$ are given as 0.30, 0.33,and 0.34. Which of these three is the best approximation?

Sol. The best approximation will be the one which has the least absolute error.

True value = $\frac{1}{3}$ = 0.33333

Case I. Approximate value = 0.30

Absolute error = | True value – Approximate value |
= | 0.33333 – 0.30 |
= 0.03333

Case II. Approximate value = 0.33

Absolute error = | True value – Approximate value |
= | 0.33333 – 0.33 |
= 0.00333.

Case III. Approximate value = 0.34

Absolute error = | True value – Approximate value |
= | 0.33333 – 0.34 |
= | – 0.00667 | = 0.00667

Since the absolute error is **least** in case II, 0.33 is the best approximation.

Problem 5. Find the relative error of the number 8.6 if both of its digits are correct.

Sol. Here, $e_a = 0.5$ $\quad\quad \because\ e_a = \frac{1}{2} X\ 10^{-1}$

$$\therefore e_r = \frac{0.5}{8.6} = .0058$$

Problem 6. Find the relative error if $\frac{2}{3}$ is approximated to 0.667.

Sol. True value $= \frac{2}{3} = 0.666666$

Approximate value = 0.667

Absolute error e_a = | True value – approximate value |

= | .666666 – .667 | = .000334

Relative error $\quad e_r = \frac{.000334}{.666666} = .0005$

Problem 7. Find the percentage error if 625.483 is approximated to three significant figures.

Sol. $e_a = |\ 625.483 - 625\ | = 0.483$

$$e_r = \frac{e_a}{625.483} = \frac{0.483}{625.483} = 0.000772$$

$$e_p = e_r \times 100$$

$$e_p = 0.000772 \times 100 = 0.77\%$$

Problem 8. Round-off the numbers 865250 and 37.46235 to four significant figures and compute e_a, e_r, e_p in each case.

Sol. (*i*) Number rounded-off to four significant digits = 865200

$$X = 865250$$

$$X' = 865200$$

$$\text{Error} = X - X' = 865250 - 865200 = 50$$

$$\text{Absolute error } e_a = |\text{ error }| = 50$$

$$\text{Relative error} = e_r = \frac{e_a}{X} = \frac{50}{865250} = 5.77 \times 10^{-5}$$

$$\text{Percentage error } e_p = e_r \times 100 = 5.77 \times 10^{-3}$$

(*ii*) Number rounded-off to four significant digits = 37.46

$$X = 37.46235$$

$$X' = 37.46$$

$$\text{Error} = X - X' = 0.00235$$

$$\text{Absolute error } e_a = |\text{ error }| = 0.00235$$

$$\text{Relative error} = e_r = \frac{e_a}{X} = \frac{0.00235}{37.46235} = 6.2729 \times 10^{-5}$$

$$\text{Percentage error } e_p = e_r \times 100 = 6.2729 \times 10^{-3}.$$

Problem 9. Round-off the number 75462 to four significant digits and then calculate the absolute error and percentage error.

Sol. Number rounded-off to four significant digits = 75460

$$\text{Absolute error } e_a = |\, 75462 - 75460 \,| = 2$$

$$\text{Relative error} = e_r = \frac{e_a}{75462} = \frac{2}{75462} = 0.0000265$$

$$\text{Percentage error } e_p = e_r \times 100 = 0.00265$$

Problem 10. Find the absolute, relative, and percentage errors if x is rounded-off to three decimal digits. Given x = 0.005998.

Sol. Number rounded-off to three decimal digits =.006

$$\text{Error} = .005998 - .006 = -.000002$$

$$\text{Absolute error } e_a = |\text{ error }| = .000002$$

$$\text{Relative error} = e_r = \frac{e_a}{0.005998} = \frac{.000002}{0.005998} = 0.0033344$$

$$\text{Percentage error } e_p = e_r \times 100 = 0.33344$$

2.7 Summary

The second chapter of this book introduces the learner with the concepts of approximations, rounding, significant digits and error which plays a very crucial role in solving a problem using numerical methods. Rules to round-off a given data and rules to determine significant digits are discussed which is useful in achieving the desired accuracy in the given problem.

2.8 References

Following books are recommended for further reading :-

1) Numerical Methods for Engineers Steven C Chapra & Raymond P Canale – 6^{th} Edition

2) Introductory Methods of Numerical Methods by S S Shastri

3) Computer based Numerical & Statistical Techniques – M. Goyal

4) Numerical & Statistical Methods by Bhupendra T Kesria, Himalaya Publishing House

2.9 Exercises

1. Round-off the following numbers correct to four significant digits:

   3.26425, 35.46735, 4985561, 0.70035, 0.00032217, 1.6583, 30.0567, 0.859378, 3.14159.

2. The height of an observation tower was estimated to be 47 m. whereas its actual height was 45 m. Calculate the percentage of relative error in the measurement.

3. If true value = $\frac{10}{3}$, approximate value = 3.33, find the absolute and relative errors.

4. Round-off the following numbers to two decimal

   48.21416, 2.3742, 52.275, 2.375, 2.385, 81.255.

5. If X = 2.536, find the absolute error and relative error when

   (i) X is rounded-off

   (ii) X is truncated to two decimal digits.

6. If $\pi = \frac{22}{7}$ is approximated as 3.14, find the absolute error, relative error, and percentage of relative error.

7. Given the solution of a problem as X' = 35.25 with the relative error in the solution atmost 2%, find, to four decimal digits, the range of values within which the exact value of the solution must lie.

8. Given that:

   $a = 10.00 \pm 0.05$, $b = 0.0356 \pm 0.0002$

   $c = 15300 \pm 100$, $d = 62000 \pm 500$

   Find the maximum value of the absolute error in
   (i) $a + b + c + d$ (ii) $a + 5c - d$ (iii) d3.

9. What do you understand by machine epsilon of a computer? Explain.

10. What do you mean by truncation error? Explain with examples.

❖❖❖

UNIT 1

3

TRUNCATION ERRORS & THE TAYLOR SERIES

Unit Structure

3.0 Objectives

After reading this chapter, you should be able to

1. Understand the basics of Taylor's theorem,
2. Write trigonometric functions as Taylor's polynomial,
3. Use Taylor's theorem to find the values of a function at any point, given the values of the function and all its derivatives at a particular point,
4. Calculate errors and error bounds of approximating a function by Taylor series
5. Revisit whenever Taylor's theorem is used to derive or explain numerical methods for various mathematical procedures.

3.1 Introduction

Truncation errors are those that result from using an approximation in place of an exact mathematical procedure. A truncation error was introduced into the numerical solution because the difference equation only approximates the true value of the derivative. In order to gain insight into the properties of such errors, we now turn to a mathematical formulation that is used widely in numerical methods to express functions in an approximate manner—the Taylor series.

In this chapter Understanding the basics of Taylor's theorem, Taylors series Truncation errors, remainder in taylor series, error propagation, Total numerical errors is covered

3.2 The Taylor's Series

Taylor's theorem and its associated formula, the Taylor series, is of great value in the study of numerical methods. In essence, the *Taylor series provides a means to predict a function value at one point in terms of the function value and its derivatives at another point*. In particular, the theorem states that any smooth function can be approximated as a polynomial.

A useful way to gain insight into the Taylor series is to build it term by term. For example, the first term in the series is

$$f(x_{i+1}) \cong f(x_i) \tag{3.1}$$

This relationship, called the *zero-order approximation*, indicates that the value of f at the new point is the same as its value at the old point. This result makes intuitive sense because if x_i and x_{i+1} are close to each other, it is likely that the new value is probably similar to the old value.

Equation (3.1) provides a perfect estimate if the function being approximated is, in fact, a constant. However, if the function changes at all over the interval, additional terms of the Taylor series are required to provide a better estimate.

For example, the *first-order approximation* is developed by adding another term to yield

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)\,(x_{i+1} - x_i) \tag{3.2}$$

The additional first-order term consists of a slope $f(x_i)$ multiplied by the distance between x_i and x_{i+1}. Thus, the expression is now in the form of a straight line and is capable of predicting an increase or decrease of the function between x_i and x_{i+1}.

Although Eq. (3.2) can predict a change, it is exact only for a straight-line, or linear,

trend. Therefore, a second-order term is added to the series to capture some of the curvature that the function might exhibit:

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 \tag{3.3}$$

In a similar manner, additional terms can be included to develop the complete Taylor series expansion:

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \frac{f'''(x_i)}{3!}(x_{i+1} - x_i)^3 + \cdots + \frac{f^n(x_i)}{n!}(x_{i+1} - x_i)^n + R_n \tag{3.4}$$

Note that because Eq. (3.4) is an infinite series, an equal sign replaces the approximate sign that was used in Eqs. (3.1) through (3.3). A remainder term is included to account for all terms from n + 1 to infinity:

$$R_n = \frac{f^{n+1}(\xi)}{(n+1)!}(x_{i+1} - x_i)^{n+1} \tag{3.5}$$

where the subscript *n* connotes that this is the remainder for the *n*th-order approximation and ξ is a value of *x* that lies somewhere between x_i and x_{i+1}.

ξ is a value that provides an exact determination of the error.

It is often convenient to simplify the Taylor series by defining a step size

$h = x_{i+1} - x_i$ and expressing Eq. (3.4) as

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(x_i)}{3!}h^3 + \cdots + \frac{f^n(x_i)}{n!}h^n + R_n \tag{3.6}$$

where the remainder term is now

$$R_n = \frac{f^{n+1}(\xi)}{(n+1)!}h^{n+1} \tag{3.7}$$

3.2.1 Taylor's Theorem

If the function *f* and its first n + 1 derivatives are continuous on an interval containing *a* and *x*, then the value of the function at *x* is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_n \tag{3.8}$$

where the remainder R_n is defined as

$$R_n = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt \qquad (3.9)$$

where t = a dummy variable. Equation (3.8) is called the Taylor series or Taylor's formula.

If the remainder is omitted, the right side of Eq. (3.8) is the Taylor polynomial approximation to $f(x)$. In essence, the theorem states that any smooth function can be approximated as a polynomial.

Equation (3.9) is but one way, called the integral form, by which the remainder can be expressed. An alternative formulation can be derived on the basis of the integral mean-value theorem.

3.2.2.Taylor Series Approximation of a Polynomial

Problem Statement. Use zero- through fourth-order Taylor series expansions to approximate the function

$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$

from $x_i = 0$ with $h = 1$. That is, predict the function's value at $x_{i+1} = 1$.

Solution. Because we are dealing with a known function, we can compute values for

$f(x)$ between 0 and 1.

Therefore $f(x)$ by substituting x=0 becomes

$$f(0) = -0.1(0)^4 - 0.15(0)^3 - 0.5(0)^2 - 0.25(0) + 1.2$$

$$= -0 - 0 - 0 - 0 + 1.2$$

$$f(0) = 1.2$$

and

$f(x)$ by substituting x=1 becomes

$$f(1) = -0.1(1)^4 - 0.15(1)^3 - 0.5(1)^2 - 0.25(1) + 1.2$$

$$= -0.1 - 0.15 - 0.5 - 0.25 + 1.2$$

$$f(1) = 0.2$$

The results (Fig. 3.1) indicate that the function starts at $f(0)=1.2$ and then curves downward to $f(1) = 0.2$

Thus, the true value that we are trying to predict is 0.2.

The Taylor series approximation with n = 0 is (Eq. 3.1)

$$f(x_{i+1}) \simeq 1.2$$

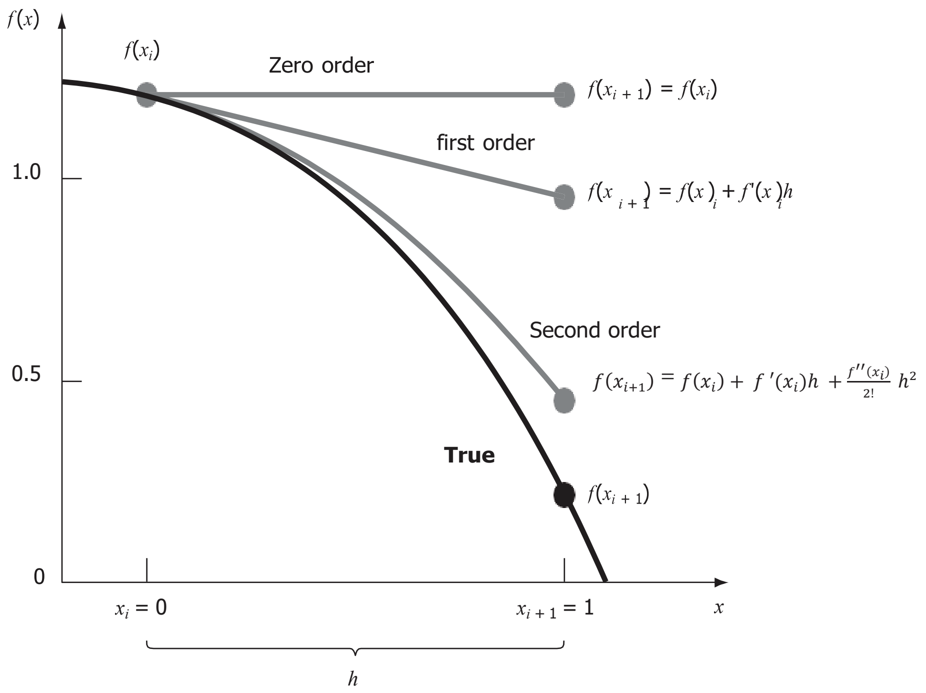


Fig 3.1

The approximation of $f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$ at $x = 1$ by zero-order, first-order, and second-order Taylor series expansions.

Thus, as in Fig. 3.1, the zero-order approximation is a constant. Using this formulation results in a truncation error E_t

E_t = true value – approximation [Refer Eq. (2.2) in prev. chapter]

$E_t = 0.2 - 1.2 = -1.0$

at $x = 1$.

For $n = 1$, the first derivative must be determined and evaluated at $x = 0$:

$f'(x) = -0.1 \times 4x^3 - 0.15 \times 3x^2 - 0.5 \times 2x - 0.25 \times 1 = -0.25$

$f'(0) = -0.4(0.0)^3 - 0.45(0.0)^2 - 1.0(0.0) - 0.25 = -0.25$

Therefore, the first-order approximation is [Eq. (3.2)]

$f(x_{i+1}) = \quad 1.2 - 0.25h$

which can be used to compute $f(1) = 0.95$. Consequently, the approximation begins to capture the downward trajectory of the function in the form of a sloping straight line(Fig. 3.1). This results in a reduction of the truncation error to

$E_t = 0.2 - 0.95 = -0.75$

For $n = 2$, the second derivative is evaluated at $x = 0$:

$f''(0) = -1.2(0.0)^2 - 0.9(0.0) - 1.0 = -1.0$

Therefore, according to Eq. (3.3),

$f(x_{i+1}) \quad 1.2 - 0.25h - 0.5h^2$

and substituting $h = 1, f(1)$ 0.45. The inclusion of the second derivative now adds some downward curvature resulting in an improved estimate, as seen in Fig. 3.1. The truncationerror is reduced further to $0.2 - 0.45 = -0.25$.

Additional terms would improve the approximation even more. In fact, the inclusion of the third and the fourth derivatives results in exactly the same equation we started with:

$f(x) = 1.2 - 0.25h - 0.5h^2 - 0.15h^3 - 0.1h^4$

where the remainder term is

$$R_4 = \frac{f^{(5)}(\xi)}{5!} h^5 = 0$$

because the fifth derivative of a fourth-order polynomial is zero. Consequently, the Taylor series expansion to the fourth derivative yields an exact estimate at $x_{i+1} = 1$:

$f(1) = 1.2 - 0.25(1) - 0.5(1)^2 - 0.15(1)^3 - 0.1(1)^4 = 0.2$

Eq. (3.7) is useful for gaining insight into truncation errors.

This is because we *do* have control over the term h in the equation. In other words, we can choose how far away from x we want to evaluate $f(x)$, and we can control the number of terms we include in the expansion. Consequently, Eq. (3.9) is usually expressed as

$Rn = O(h^{n+1})$

where the nomenclature $O(h^{n+1})$means that the truncation error is of the order of h^{n+1}.

3.2.3 The Remainder for the Taylor Series Expansion

Before demonstrating how the Taylor series is actually used to estimate numerical errors, we must explain why we included the argument ξ in Eq. (3.7). A mathematical derivation is presented in point 3.2.1. We will now develop an alternative exposition based on a somewhat more visual interpretation. Then we can extend this specific case to the more general formulation.

Suppose that we truncated the Taylor series expansion [Eq. (3.6)] after the zero-order term to yield

$f(x_{i+1}) \simeq f(x_i)$

A visual depiction of this zero-order prediction is shown in Fig. 3.2. The remainder, or error, of this prediction, which is also shown in the illustration, consists of the infinite series of terms that were truncated:

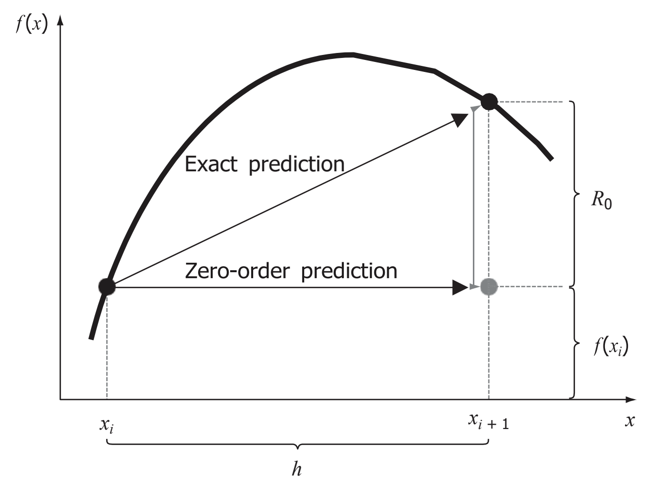
$$R_0 = f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(x_i)}{3!}h^3 + \cdots$$

It is obviously inconvenient to deal with the remainder in this infinite series format.

One simplification might be to truncate the remainder itself, as in

$$R_0 \cong f'(x_i)h$$

Fig 3.2 Graphical depiction of a zero-order Taylor series prediction and remainder.



The use of Taylor series exists in so many aspects of numerical methods. For example, you must have come across expressions such as

$$(1)\ \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$(2)\ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$(3)\ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

All the above expressions are actually a special case of Taylor series called the Maclaurin series. Why are these applications of Taylor's theorem important for numerical methods? Expressions such as given in Equations (1), (2) and (3) give you a way to find the approximate values of these functions by using the basic arithmetic operations of addition, subtraction, division, and multiplication.

Example 1

Find the value of $e^{0.25}$ using the first five terms of the Maclaurin series.

Solution

The first five terms of the Maclaurin series for e^x is

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{0.25} \approx 1 + 0.25 + \frac{0.25^2}{2!} + \frac{0.25^3}{3!} + \frac{0.25^4}{4!}$$

$$= 1.2840$$

The exact value of $e^{0.25}$ up to 5 significant digits is also 1.2840.

But the above discussion and example do not answer our question of what a Taylor series is.

Here it is, for a function $f(x)$

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots \quad (4)$$

Provided all derivatives of $f(x)$ exist and are continuous between x and $x+h$.

As Archimedes would have said (*without the fine print*), "*Give me the value of the function at a single point, and the value of all (first, second, and so on) its derivatives, and I can give you the value of the function at any other point*".

It is very important to note that the Taylor series is not asking for the expression of the function and its derivatives, just the value of the function and its derivatives at a single point.

Example 2

Take $f(x)=\sin(x)$, we all know the value of $\sin\left(\frac{\pi}{2}\right)=1$. We also know the $f'(x)=\cos(x)$ and $\cos\left(\frac{\pi}{2}\right)=0$. Similarly $f''(x)=-\sin(x)$ and $\sin\left(\frac{\pi}{2}\right)=1$. In a way, we know the value of $\sin(x)$ and all its derivatives at $x=\frac{\pi}{2}$. We do not need to use any calculators, just plain differential calculus and trigonometry would do. Can you use Taylor series and this information to find the value of $\sin(2)$?

Solution

$$x=\frac{\pi}{2}$$

$$x+h=2$$

$$h=2-x$$

$$=2-\frac{\pi}{2}$$

$$=0.42920$$

So

$$f(x+h)=f(x)+f'(x)h+f''(x)\frac{h^2}{2!}+f'''(x)\frac{h^3}{3!}+f''''(x)\frac{h^4}{4!}+\cdots$$

$$x=\frac{\pi}{2}$$

$$h=0.42920$$

$$f(x)=\sin(x),\ f\left(\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)=1$$

$$f'(x) = \cos(x),\ f'\left(\frac{\pi}{2}\right) = 0$$

$$f''(x) = -\sin(x),\ f''\left(\frac{\pi}{2}\right) = -1$$

$$f'''(x) = -\cos(x),\ f'''\left(\frac{\pi}{2}\right) = 0$$

$$f''''(x) = \sin(x),\ f''''\left(\frac{\pi}{2}\right) = 1$$

Hence

$$f\left(\frac{\pi}{2} + h\right) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)h + f''\left(\frac{\pi}{2}\right)\frac{h^2}{2!} + f'''\left(\frac{\pi}{2}\right)\frac{h^3}{3!} + f''''\left(\frac{\pi}{2}\right)\frac{h^4}{4!} + \cdots$$

$$f\left(\frac{\pi}{2} + 0.42920\right) = 1 + 0(0.42920) - 1\frac{(0.42920)^2}{2!} + 0\frac{(0.42920)^3}{3!} + 1\frac{(0.42920)^4}{4!} + \cdots$$

$$= 1 + 0 - 0.092106 + 0 + 0.00141393 + \cdots$$

$$\cong 0.90931$$

The value of $\sin(2)$ I get from my calculator is 0.90930 which is very close to the value I just obtained. Now you can get a better value by using more terms of the series. In addition, you can now use the value calculated for $\sin(2)$ coupled with the value of $\cos(2)$ (which can be calculated by Taylor series just like this example or by using the $\sin^2 x + \cos^2 x \equiv 1$ identity) to find value of $\sin(x)$ at some other point. In this way, we can find the value of $\sin(x)$ for any value from $x = 0$ to 2π and then can use the periodicity of $\sin(x)$, that is $\sin(x) = \sin(x + 2n\pi), n = 1,2,\ldots$ to calculate the value of $\sin(x)$ at any other point.

Example 3

Derive the Maclaurin series of $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Solution

In the previous example, we wrote the Taylor series for $\sin(x)$ around the point $x = \frac{\pi}{2}$. Maclaurin series is simply a Taylor series for the point $x = 0$.

$$f(x)=\sin(x),\ f(0)=0$$

$$f'(x)=\cos(x), f'(0)=1$$

$$f''(x)=-\sin(x), f''(0)=0$$

$$f'''(x)=-\cos(x), f'''(0)=-1$$

$$f''''(x)=\sin(x), f''''(0)=0$$

$$f'''''(x)=\cos(x)\, , f'''''(0)=1$$

Using the Taylor series now,

$$f(0+h)=f(0)+f'(0)h+f''(0)\frac{h^2}{2!}+f'''(0)\frac{h^3}{3!}+f''''(0)\frac{h^4}{4}+f'''''(0)\frac{h^5}{5}+\cdots$$

$$f(h)=f(0)+f'(0)h+f''(0)\frac{h^2}{2!}+f'''(0)\frac{h^3}{3!}+f''''(0)\frac{h^4}{4}+f'''''(0)\frac{h^5}{5}+\cdots$$

$$f(x+h)=f(x)+f'(x)h+f''(x)\frac{h^2}{2!}+f'''(x)\frac{h^3}{3!}+f''''(x)\frac{h^4}{4}+f'''''(x)\frac{h^5}{5}+\cdots$$

$$=0+1(h)-0\frac{h^2}{2!}-1\frac{h^3}{3!}+0\frac{h^4}{4}+1\frac{h^5}{5}+\cdots$$

$$=h-\frac{h^3}{3!}+\frac{h^5}{5!}+\cdots$$

So

$$f(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$$

$$\sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$$

Example 4

Find the value of $f(6)$ given that $f(4)=125$, $f'(4)=74$, $f''(4)=30$, $f'''(4)=6$ and all other higher derivatives of $f(x)$ at $x=4$ are zero.

Solution

$$f(x+h)=f(x)+f'(x)h+f''(x)\frac{h^2}{2!}+f'''(x)\frac{h^3}{3!}+\cdots$$

$x=4$

$h=6-4\ =2$

Since fourth and higher derivatives of $f(x)$ are zero at $x=4$.

$$f(4+2) = f(4) + f'(4)2 + f''(4)\frac{2^2}{2!} + f'''(4)\frac{2^3}{3!}$$

$$f(6) = 125 + 74(2) + 30\left(\frac{2^2}{2!}\right) + 6\left(\frac{2^3}{3!}\right)$$

$$= 125 + 148 + 60 + 8$$

$$= 341$$

Note that to find $f(6)$ exactly, we only needed the value of the function and all its derivatives at some other point, in this case, $x = 4$. We did not need the expression for the function and all its derivatives. Taylor series application would be redundant if we needed to know the expression for the function, as we could just substitute $x = 6$ in it to get the value of $f(6)$.

Actually the problem posed above was obtained from a known function $f(x) = x^3 + 3x^2 + 2x + 5$ where $f(4) = 125$, $f'(4) = 74$, $f''(4) = 30$, $f'''(4) = 6$, and all other higher derivatives are zero.

3.2.4 Error in Taylor Series

As you have noticed, the Taylor series has infinite terms. Only in special cases such as a finite polynomial does it have a finite number of terms. So whenever you are using a Taylor series to calculate the value of a function, it is being calculated approximately.

The Taylor polynomial of order n of a function $f(x)$ with $(n+1)$ continuous derivatives in the domain $[x, x+h]$ is given by

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + \cdots + f^{(n)}(x)\frac{h^n}{n!} + R_n(x+h)$$

where the remainder is given by

$$R_n(x+h) = \frac{(h)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

where

$$x < c < x+h$$

that is, c is some point in the domain $(x, x+h)$.

Example 5

The Taylor series for e^x at point $x = 0$ is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

a) What is the truncation (true) error in the representation of e^1 if only four terms of the series are used?

b) Use the remainder theorem to find the bounds of the truncation error.

Solution

a) If only four terms of the series are used, then

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^1 \approx 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!}$$

$$= 2.66667$$

The truncation (true) error would be the unused terms of the Taylor series, which then are

$$E_t = \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$= \frac{1^4}{4!} + \frac{1^5}{5!} + \cdots$$

$$\cong 0.0516152$$

b) But is there any way to know the bounds of this error other than calculating it directly? Yes,

$$f(x+h) = f(x) + f'(x)h + \cdots + f^{(n)}(x)\frac{h^n}{n!} + R_n(x+h)$$

where

$$R_n(x+h) = \frac{(h)^{n+1}}{(n+1)!} f^{(n+1)}(c),\ x < c < x+h, \text{ and}$$

c is some point in the domain $(x, x+h)$.

So in this case, if we are using four terms of the Taylor series, the remainder is given by $(x = 0, n = 3)$

$$R_3(0+1) = \frac{(1)^{3+1}}{(3+1)!} f^{(3+1)}(c)$$

$$= \frac{1}{4!} f^{(4)}(c)$$

$$= \frac{e^c}{24}$$

Since

$$x < c < x + h$$

$$0 < c < 0 + 1$$

$$0 < c < 1$$

The error is bound between

$$\frac{e^0}{24} < R_3(1) < \frac{e^1}{24}$$

$$\frac{1}{24} < R_3(1) < \frac{e}{24}$$

$$0.041667 < R_3(1) < 0.113261$$

So the bound of the error is less than 0.113261 which does concur with the calculated error of 0.0516152 .

Example 6

The Taylor series for e^x at point $x = 0$ is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

As you can see in the previous example that by taking more terms, the error bounds decrease and hence you have a better estimate of e^1. How many terms it would require to get an approximation of e^1 within a magnitude of true error of less than 10^{-6}?

Solution

Using $(n+1)$ terms of the Taylor series gives an error bound of

$$R_n(x+h)=\frac{(h)^{n+1}}{(n+1)!}f^{(n+1)}(c)$$

$$x=0, h=1, f(x)=e^x$$

$$R_n(1)=\frac{(1)^{n+1}}{(n+1)!}f^{(n+1)}(c)$$

$$=\frac{(1)^{n+1}}{(n+1)!}e^c$$

Since

$$x<c<x+h$$

$$0<c<0+1$$

$$0<c<1$$

$$\frac{1}{(n+1)!}<|R_n(1)|<\frac{e}{(n+1)!}$$

So if we want to find out how many terms it would require to get an approximation of e^1 within a magnitude of true error of less than 10^{-6},

$$\frac{e}{(n+1)!}<10^{-6}$$

$$(n+1)!>10^6 e$$

$(n+1)!>10^6\times 3$ (as we do not know the value of e but it is less than 3).

$$n\geq 9$$

So 9 terms or more will get e^1 within an error of 10^{-6} in its value.

We can do calculations such as the ones given above only for simple functions. To do a similar analysis of how many terms of the series are needed for a specified accuracy for any general function, we can do that based on the concept of absolute relative approximate errors discussed in Chapter 01.02 as follows.

We use the concept of absolute relative approximate error (see Chapter 02 for details), which is calculated after each term in the series is added. The maximum value of m, for which the absolute relative approximate error is less than $0.5\times 10^{2-m}\%$ is the least number of significant digits correct in the answer. It establishes the accuracy of the approximate value of a function without the knowledge of remainder of Taylor series or the true error.

3.3 Error Propagation

The purpose of this section is to study how errors in numbers can propagate through mathematical functions. For example, if we multiply two numbers that have errors, we would like to estimate the error in the product.

Numerical solutions involve a series of computational steps. Therefore, it is necessary to understand the way the error propagates with progressive computations.

Propagation of error (or propagation of uncertainty) is the effect of variables, uncertainties (or errors) on the uncertainty of a function based on them.

3.3.1 Functions of a Single Variable

Suppose that we have a function $f(x)$ that is dependent on a single independent variable x. Assume that x is an approximation of x. We, therefore, would like to assess the effect of the discrepancy between x and x on the value of the function. That is, we would like to estimate

$\Delta f(\bar{x}) = |f(x) - f(\bar{x})|$

The problem with evaluating $\Delta f(\bar{x})$ is that $f(x)$ is unknown because x is unknown. We can overcome this difficulty if $\dot{x}$ is close to x and $f(\bar{x})$ is continuous and differentiable. If these conditions hold, a Taylor series can be employed to compute $f(x)$ near $f(\bar{x})$ as in

$$f(x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{f''(\bar{x})}{2!}(x-\bar{x})^2 + \cdots$$

Dropping the second- and higher-order terms and rearranging yields

$f(x) - f(\bar{x}) \cong f'(\bar{x})(x-\bar{x})$

or

$$\Delta f(\bar{x}) = |f'(\bar{x})|\Delta\bar{x} \qquad (3.3.1)$$

Where $\Delta f(\bar{x}) = |f(x) - f(\bar{x})|$

represents an estimate of the error of the function and $\Delta \bar{x} = |x - \bar{x}|$

represents an estimate of the error of x. Equation (3.3.1) provides the capability to approximate the error in $f(x)$ given the derivative of a function and an estimate of the error in the independent variable. Figure 3.3.1 is a graphical illustration of the operation.

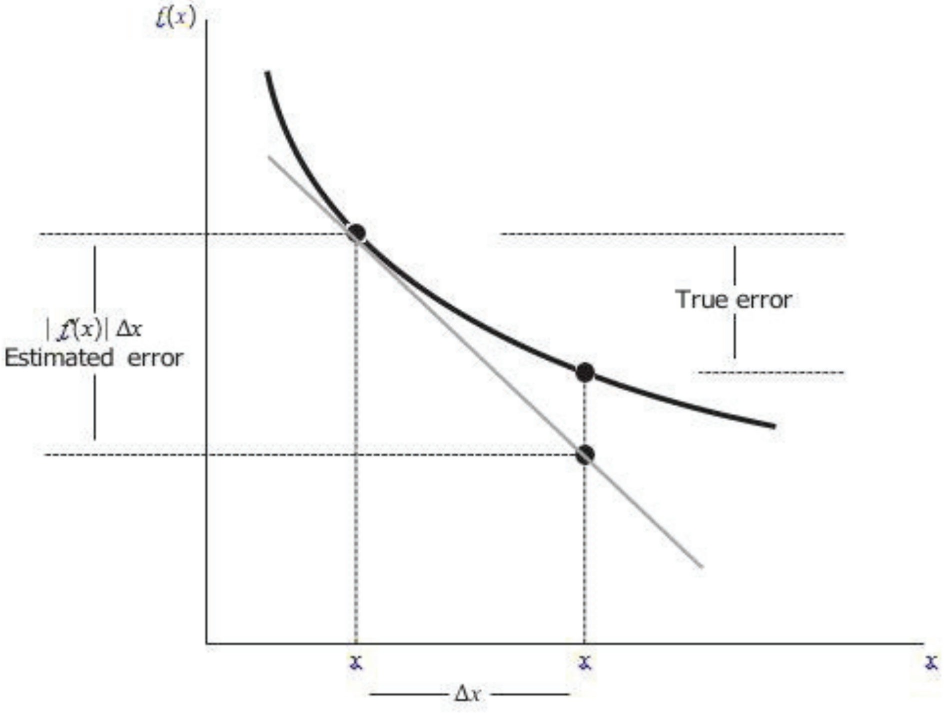


FIGURE 3.3.1 Graphical depiction of first order error propagation.

3.3.2 Error Propagation in a Function of a Single Variable

Problem Statement

Given a value of $\bar{x} = 2.5$ with an error of $\Delta \bar{x} = 0.01$, estimate the resulting error in the function, $f(x) = x^3$.

Solution. Using Eq. (4.25),

$\Delta f(\bar{x}) \cong 3(2.5)^2(0.01) = 0.1875$

Because f (2.5) = 15.625, we predict that

$f(2.5) = 15.625 \pm 0.1875$

or that the true value lies between 15.4375 and 15.8125. In fact, if x were actually 2.49, the function could be evaluated as 15.4382, and if x were 2.51, it would be 15.8132. For this case, the first-order error analysis provides a fairly close estimate of the true error.

3.4 Total Numerical Error

The total numerical error is the summation of the truncation and round-off errors.

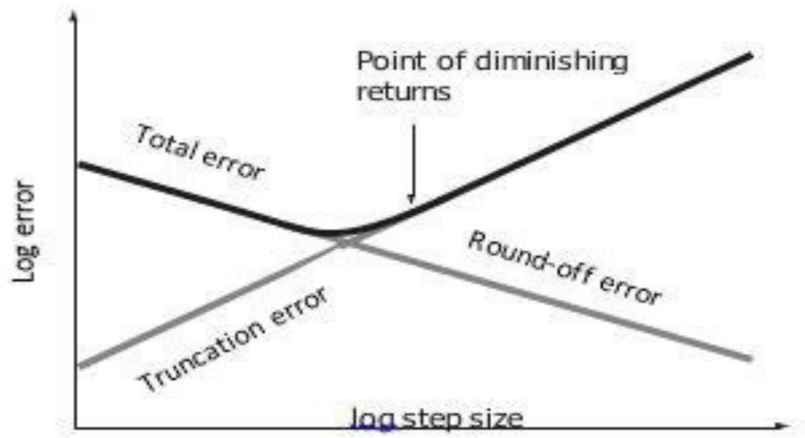


FIGURE 3.4.1

A graphical depiction of the trade-off between round-off and truncation error that sometimes comes into play in the course of a numerical method. The point of diminishing returns is shown, where round-off error begins to negate the benefits of step-size reduction.

In general, the only way to minimize round-off errors is to increase the number of significant figures of the computer. Further, we have noted that round-off error will increase due to subtractive cancellation or due to an increase in the number of computations in an analysis. The truncation error can be reduced by decreasing the step size. Because a decrease in step size can lead to subtractive cancellation or to an increase in computations, the truncation errors are decreased as the round-off errors are increased. Therefore, we are faced by the following dilemma: The strategy for decreasing one component of the total error leads to an increase of the other component. In a computation, we could conceivably decrease the step size to minimize truncation errors only to discover that in doing so, the round-off error begins to dominate the solution and the total error grows! Thus, our remedy becomes our problem (Fig. 3.4.1). One challenge that we face is to determine an appropriate step size for a particular computation. We would like to choose a large step size in order to decrease the amount of calculations and round-off errors without incurring the penalty of a large truncation error. If the total error is as shown in Fig. 3.4.1, the challenge is to identify the point of diminishing returns where round-off error begins to negate the benefits of step-size reduction.

In actual cases, however, such situations are relatively uncommon because most computers carry enough significant figures that round-off errors do not predominate. Nevertheless, they sometimes do occur and suggest a sort of "numerical uncertainty principle" that places an absolute limit on the accuracy that may be obtained using certain computerized numerical methods.

3.5 Formulation Error and Data Uncertainty

Formulation Errors

Formulation, or model, are the errors resulting from incomplete mathematical models when some latent effects are not taken into account or ignored.

Data Uncertainty

Errors resulting from the accuracy and/or precision of the data

- When with biased (underestimation/overestimation) or imprecise instruments
- We can use descriptive statistics (viz. mean and variance) to provide a measure of the bias and imprecision.

3.6 Summary

The third chapter of this book discusses the truncation error and Taylor series expansion which is one of the most widely used example to illustrate truncation error and truncation error bound. Formulation error and data uncertainty is also discussed in this chapter.

3.7 References

Following books are recommended for further reading:-

1) Introductory Methods of Numerical Methods by S S Shastri

2) Numerical Methods for Engineers Steven C Chapra & Raymond P Canale – 6th Edition

3) Numerical & Statistical Methods by Bhupendra T Kesria, Himalaya Publishing House

4) Computer based Numerical & Statistical Techniques – M. Goyal

3.8 Exercise

1. Finding Taylor polynomials

Find the Taylor polynomials of orders 0,1 and 2 generated by *f* at a

a) $f(x) = \frac{1}{x}$, a = 2

b) $f(x) = \frac{1}{(x+2)}$, a = 0

c) $f(x) = \sin x$, $a = \frac{\pi}{4}$

d) $f(x) = \cos x$, $a = \frac{\pi}{4}$

2. Find Maclaurin series for the following functions

a) e^{-x}

b) $e^{\frac{x}{2}}$

c) $\frac{1}{(1+x)}$

d) $\frac{1}{(1-x)}$

e) $sin3x$

f) $\sin\frac{x}{2}$

3. Finding Taylor Series

Find the Taylor series generated by *f* at x=a.

a) $f(x) = x^3-2x+4$, a = 2

b) $f(x) = 2x^3+x^2+3x-8$, a = 1

c) $f(x) = x^4+x^2+1$, a = -2

d) $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2$, a = -1

e) $f(x) = \frac{1}{x^2}$, a = 1

f) $f(x) = \frac{1}{(1-x)}$, a = 0

g) $f(x) = e^x$, a = 2

h) $f(x) = 2^x$, a = 1

4. Use the taylor series generated by e^x at x=a to show that

$$e^x = e^a \left[1 + (x-a) + \frac{(x-a)^2}{2!} + \cdots\right]$$

5. $(\boldsymbol{continuation\ of\ above\ problem})$ Find the Taylor series generated by e^x at x=1 compare your answer with the formula in example 4

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UNIT 2

4

SOLUTIONS OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

Unit Structure

4.0 Objectives

This chapter will enable the learner to:

- understand the concepts of simple root, multiple roots, algebraic equations, transcendental equations, direct methods, iterative methods.

- find roots of an equation using Bisection method, Newton-Raphson method, Regula-Falsi method, Secant method.
- understand the geometrical interpretation of these methods and derive the rate of convergence.

4.1 Introduction

The solution of an equation of the form $f(x) = 0$ is the set of all values which when substituted for unknowns, make an equation true. Finding roots of an equation is a problem of great importance in the fields of mathematics and engineering. In this chapter we see different methods to solve a given equation.

4.1.1 Simple and Multiple Roots

Definition 4.1.1.1 (Root of an Equation). A number ζ is said to be a root or a zero of an equation $f(x) = 0$ if $f(\zeta) = 0$.

Definition 4.1.1.2 (Simple Root). A number ζ is said to be a simple root of an equation $f(x) = 0$ if $f(\zeta) = 0$ and $f^{'}(\zeta) \neq 0$. In this case we can write $f(x)$ as

$$f(x) = (x - \zeta)g(x), \text{ where } g(\zeta) \neq 0.$$

Definition 4.1.1.3 (Multiple Root). A number ζ is said to be a multiple root of multiplicity m of an equation $f(x) = 0$ if $f(\zeta) = 0, f^{'}(\zeta) = 0,\ldots, f^{(m-1)}(\zeta) = 0$ and $f^{(m)}(\zeta) \neq 0$. In this case we can write $f(x)$ as

$$f(x) = (x - \zeta)^m g(x), \text{ where } g(\zeta) \neq 0.$$

4.1.2 Algebraic and Transcendental Equations

Definition 4.1.2.1 (Algebraic Equation). A polynomial equation of the form $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n = 0,\ a_n \neq 0$ where $a_i \in \mathbb{C}$ for $0 \leq i \leq n$ is called an algebraic equation of degree n.

Definition 4.1.2.2 (Transcendental Equation). An equation which contains exponential functions, logarithmic functions, trigonometric functions etc. is called a transcendental function.

4.1.3 Direct Methods and Iterative Methods

Definition 4.1.2.1 (Direct Methods). A method which gives an exact root in a finite number of steps is called direct method.

Definition 4.1.2.2 (Iterative Methods). A method based on successive approximations, that is starting with one or more initial approximations to the root, to obtain a sequence of approximations or iterates which converge to the root is called an iterative method.

4.1.4 Intermediate Value Theorem

Iterative methods are based on successive approximations to the root starting with one or more initial approximations. Choosing an initial approximation for an iterative method plays an important role in solving the given equation in a smaller number of iterates. Initial approximation to the root can be taken from the physical considerations of the given problem or by graphical methods.

As an example of finding initial approximation using physical considerations of the given problem, consider $f(x) = x^3 - 28$. Then to find the root of $f(x) = 0$, one of the best initial approximation is $x = 3$ as cube of $x = 3$ is close to the given value 28

To find initial approximation graphically, consider an example of

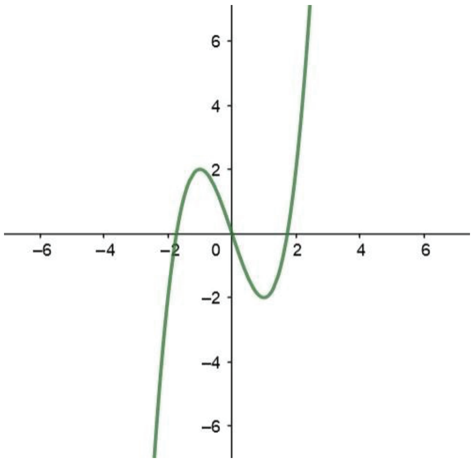

Figure 1: $f(x) = x^3 - 3x$

$f(x) = x^3 - 3x$. The value of x at which the graph of $f(x)$ intersects x- axis gives the root of $f(x) = 0$. From Figure 1, it is clear that the roots of $f(x) = 0$ lies close to 2 and −2. Hence the best initial approximations will be 2 and −2.

Intermediate Value Theorem is another commonly used method to obtain the initial approximations to the root of a given equation.

Theorem 4.1.4.1 (Intermediate Value Theorem). If $f(x)$ is a continuous function on some interval $[a,b]$ and $f(a)f(b) < 0$ then the equation $f(x) = 0$ has at least one real root or an odd number of real roots in the interval (a,b).

4.1.5 Rate of Convergence

An iterative method is said to have a rate of convergence α, if α is the largest positive real number for which there exists a finite constant $C \neq 0$ such that

$$|\epsilon_{k+1}| \leq C|\epsilon_k|^{\alpha}$$

where $\epsilon_i = x_i - \zeta$ is the error in the i^t iterate for $i \in \mathbb{N} \cup \{0\}$. The constant C is called the asymptotic error constant.

4.2 Bisection Method

Bisection method is based on the repeated application of intermediate value theorem. Let $I_0 = (a_0, b_0)$ contain the root of the equation $f(x) = 0$. We find $m_1 = \frac{a_0+b_0}{2}$ by bisecting the interval I_0. Let I_1 be the interval (a_0, m_1), if $f(a_0)f(m_1) < 0$ or the interval (m_1, b_0), if $f(m_1)f(b_0) < 0$. Thus interval I_1 also contains the root of $f(x) = 0$. We bisect the interval I_1 and take I_2 as the subinterval at whose end points the function $f(x)$ takes the values of opposite signs and hence I_2 also contains the root.

Repeating the process of bisecting intervals, we get a sequence of nested subintervals $I_0 \supset I_1 \supset I_2 \supset \cdots$ such that each subinterval contains the root. The midpoint of the last subinterval is taken as the desired approximate root.

Example 4.2.1. Find the smallest positive root of $f(x) = x^3 - 5x + 1 = 0$ by performing three iterations of Bisection Method.

Solution: Here, $f(0) = 1$ and $f(1) = -3$. That is $f(0)f(1) < 0$ and hence the smallest positive root lies in the interval $(0, 1)$. Taking $a_0 = 0$ and $b_0 = 1$, we get

(First Iteration)

$$m_1 = \frac{a_0 + b_0}{2} = \frac{0+1}{2} = 0.5$$

Since, $f(m_1) = -1.375$ and $f(a_0)f(m_1) < 0$, the root lies in the interval $(0, 0.5)$.

Taking $a_1 = 0$ and $b_1 = 0.5$, we get

(Second Iteration)

$$m_2 = \frac{a_1 + b_1}{2} = \frac{0+0.5}{2} = 0.25$$

Since, $f(m_2) = -0.234375$ and $f(a_1)f(m_2) < 0$, the root lies in the interval $(0, 0.25)$.

Taking $a_2 = 0$ and $b_2 = 0.25$, we get

(Third Iteration)

$$m_3 = \frac{a_2 + b_2}{2} = \frac{0 + 0.25}{2} = 0.125$$

Since, $f(m_3) = 0.37695$ and $f(m_3)f(b_2) < 0$, the approximate root lies in the interval $(0.125, 0.25)$.

Since we have to perform three iterations, we take the approximate root as midpoint of the interval obtained in the third iteration, that is $(0.125, 0.25)$. Hence the approximate root is 0.1875.

4.3 Newton-Raphson Method

Newton-Raphson method is based on approximating the given equation $f(x) = 0$ with a first degree equation in x. Thus, we write

$$f(x) = a_0 x + a_1 = 0$$

whose root is given by $x = -\frac{a_1}{a_0}$ such that the parameters a_1 and a_0 are to be determined. Let x_k be the k^{th} approximation to the root. Then

$$f(x_k) = a_0 x_k + a_1 \quad (4.3.1)$$

and

$$f'(x_k) = a_0 \quad (4.3.2)$$

Substituting the value of a_0 in 4.3.1 we get $a_1 = f(x_k) - f'(x_k)x_k$. Hence,

$$x = -\frac{a_1}{a_0} = -\frac{f(x_k) - f'(x_k)x_k}{f'(x_k)}.$$

Representing the approximate value of x by x_{k+1} we get

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad k = 0, 1, \cdots (4.3.3)$$

This method is called the Newton-Raphson method to find the roots of $f(x) = 0$.

Alternative:

Let x_k be the k^{th} approximation to the root of the equation $f(x) = 0$ and h be an increment such that $x_k + h$ is an exact root. Then

$$f(x_k + h) = 0.$$

Using Taylor series expansion on $f(x_k + h)$ we get,

$$0 = f(x_k + h) = f(x_k) + hf'(x_k) + \frac{1}{2!}h^2 f''(x_k) + \cdots$$

Neglecting the second and higher powers of h we get,

$$f(x_k) + h f'(x_k) = 0.$$

Thus,

$$h = -\frac{f(x_k)}{f'(x_k)}$$

We put $x_{k+1} = x_k + h$ and obtain the iteration method as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad k = 0, 1, \cdots$$

Newton-Raphson method requires two function evaluations f_k and f_k' for each iteration.

4.3.1 Geometrical Interpretation

We approximate $f(x)$ by a line taken as a tangent to the curve at x_k which gives the next approximation x_{k+1} as the x-intercept as in Figure 2.

Example 4.3.1. Find the approximate root correct upto two decimal places for $f(x) = x^4 - x - 10$ using Newton-Raphson Method with initial approximation $x_0 = 1$.

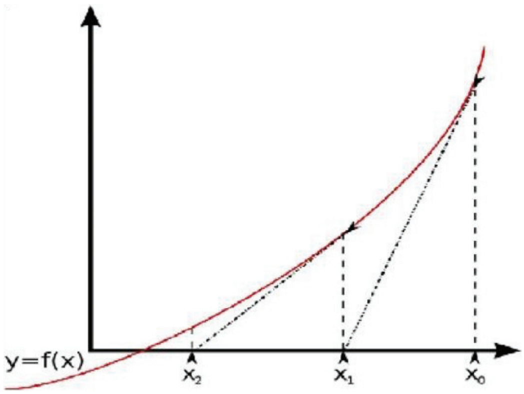

Figure 2: Newton-Raphson Method

Solution: Here $f(x) = x^4 - x - 10 = 0$ implies $f'(x) = 4x^3 - 1$. For $x_0 = 1$, $f(x_0) = -10$ and $f'(x_0) = 3$. By Newton-Raphson iteration Formula,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Thus,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4.3333$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.2908.$$

Similarly, $x_3 = 2.5562$, $x_4 = 2.0982$, $x_5 = 1.8956$, $x_6 = 1.8568$, $x_7 = 1.8555$.

4.3.2 Rate of Convergence

Let $f(x)$ be a continuous function. Then the Newton-Raphson method to find the approximate root of $f(x) = 0$ is given by equation 4.3.3.

Let ζ be the exact root of $f(x)$. Then we write x_{k+1}, x_k in terms of ζ as

$x_{k+1} = \zeta + \epsilon_{k+1}$ and $x_k = \zeta + \epsilon_k$, where I is the error in the i^{th} iteration.

Let

$$f_k = f(x_k) = f(\zeta + \epsilon_k).$$

Thus, equation 4.3.3 becomes

$$\zeta + \epsilon_{k+1} = \zeta + \epsilon_k - \frac{f(\zeta + \epsilon_k)}{f'(\zeta + \epsilon_k)}.$$

That is

$$\epsilon_{k+1} = \epsilon_k - \frac{f(\zeta + \epsilon_k)}{f'(\zeta + \epsilon_k)}. \qquad (4.3.4)$$

Using Taylor series expansion on $f(\zeta + \epsilon_k)$ and $f'(\zeta + \epsilon_k)$, we get

$$f(\zeta + \epsilon_k) = f(\zeta) + \epsilon_k f'(\zeta) + \frac{1}{2!}\epsilon_k^2 f''(\zeta) + \cdots$$

and

$$f'(\zeta + \epsilon_k) = f'(\zeta) + \epsilon_k f''(\zeta) + \frac{1}{2!}\epsilon_k^2 f'''(\zeta) + \cdots$$

Thus, equation 4.3.4 becomes

$$\begin{aligned}\epsilon_{k+1} &= \epsilon_k - \frac{\left[f(\zeta)+\epsilon_k f'(\zeta)+\frac{1}{2!}\epsilon_k^2 f''(\zeta)+\cdots\right]}{\left[f'(\zeta)+\epsilon_k f''(\zeta)+\frac{1}{2!}\epsilon_k^2 f'''(\zeta)+\cdots\right]} \\ &= \epsilon_k - \left[\epsilon_k + \frac{1}{2!}\frac{f''(\zeta)}{f'(\zeta)}\epsilon_k^2+\cdots\right]\left[1+\frac{f''(\zeta)}{f'(\zeta)}\epsilon_k+\frac{1}{2!}\frac{f'''(\zeta)}{f'(\zeta)}\epsilon_k^2+\cdots\right]^{-1} \\ &= \epsilon_k - \left[\epsilon_k + \frac{1}{2!}\frac{f''(\zeta)}{f'(\zeta)}\epsilon_k^2+\cdots\right]\left[1-\frac{f''(\zeta)}{f'(\zeta)}\epsilon_k-\frac{1}{2!}\frac{f'''(\zeta)}{f'(\zeta)}\epsilon_k^2+\cdots\right]\end{aligned}$$

Neglecting the third and higher powers of ϵ_k, we get

$$\begin{aligned}\epsilon_{k+1} &= \frac{1}{2}\frac{f''(\zeta)}{f'(\zeta)}\epsilon_k^2 \\ &= C\epsilon_k^2\end{aligned}$$

where $C = \frac{1}{2}\frac{f''(\zeta)}{f'(\zeta)}$. Thus, Newton-Raphson method has a second order rate of convergence.

4.4 Regula-Falsi Method

Given a continuous function $f(x)$, we approximate it by a first-degree equation of the form $a_0x + a_1$, such that $a_0 \neq 0$ in the neighbourhood of the root. Thus, we write

$$f(x) = a_0x + a_1 = 0 \tag{4.4.1}$$

Then

$$f(x_k) = a_0x_k + a_1 \text{ and} \tag{4.4.2}$$

$$f(x_{k-1}) = a_0x_{k-1} + a_1 \tag{4.4.3}$$

On solving equation 4.4.2 and 4.4.3, we get

$$a_0 = \frac{f_k - f_{k-1}}{x_k - x_{k-1}}$$

and

$$a_1 = \frac{x_k f_{k-1} - x_{k-1} f_k}{x_k - x_{k-1}}$$

From 4.4.1, we have $x = -\frac{a_1}{a_0}$. Hence substituting the values of a_0 and a_1 and writing x as x_{k+1} we get

$$x_{k+1} = -\frac{\left(\frac{x_k f_{k-1} - x_{k-1} f_k}{x_k - x_{k-1}}\right)}{\left(\frac{f_k - f_{k-1}}{x_k - x_{k-1}}\right)}$$

That is (4.4.4)

$$x_{k+1} = \frac{f_k x_{k-1} - f_{k-1} x_k}{f_k - f_{k-1}}$$

which can be expressed as

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k, \qquad k = 1, 2, \cdots \quad (4.4.5)$$

Here, we take the approximations x_k and x_{k-1} such that $f(x_k)f(x_{k-1}) < 0$. This method is known as Regula Falsi Method or False Position Method. This method requires only one function evaluation per iteration.

Example 4.4.1. Perform four iterations of Regula Falsi method for $f(x) = x^3 - 5x + 1$ such that the root lies in the interval $(0, 1)$.

Solution: Since the root lies in the interval $(0, 1)$, we take $x_0 = 0$ and $x_1 = 1$. Then $f(x_0) = f(0) = 1$ and $f(x_1) = f(1) = -3$. By Regula Falsi Method,

$$x_2 = x_1 - \frac{x_1 - x_0}{f_1 - f_0} f_1 = 0.25$$

and $f(0.25) = -0.234375$

As $f(x_0)f(x_2) < 0$ and $f(x_1)f(x_2) > 0$, by Intermediate Value property, the root lies in the interval (x_0, x_2). Hence

$$x_3 = x_2 - \frac{x_2 - x_0}{f_2 - f_0} f_2 = 0.202532$$

and $f(x_3) = -0.004352297$.

Similarly, we get $x_4 = 0.201654$ and $x_5 = 0.20164$. Hence, we get the approximate root as 0.20164.

4.4.2 Rate of Convergence

Let $f(x)$ be a continuous function. Then the Regula Falsi method to find the approximate root of $f(x) = 0$ is given by equation 4.4.5. Let ζ be the exact root of $f(x)$. Then we write x_{k+1}, x_k, x_{k-1} in terms of ζ as

$$x_{k+1} = \zeta + \epsilon_{k+1}, x_k = \zeta + \epsilon_k, x_{k-1} = \zeta + \epsilon_{k-1}$$

where ϵ_i is the error in the i^{th} iteration.

Hence, $f_k = f(x_k) = f(\zeta+\epsilon_k)$, $f_{k-1} = f(x_{k-1}) = f(\zeta+\epsilon_{k-1})$.

Thus, equation 4.4.5 becomes

$$\zeta+\epsilon_{k+1} = \zeta+\epsilon_k - \frac{\zeta+\epsilon_k-(\zeta+\epsilon_{k-1})}{f(\zeta+\epsilon_k)-f(\zeta+\epsilon_{k-1})}f(\zeta+\epsilon_k)$$

$$\epsilon_{k+1} = \epsilon_k - \frac{\zeta+\epsilon_k-(\zeta+\epsilon_{k-1})}{f(\zeta+\epsilon_k)-f(\zeta+\epsilon_{k-1})}f(\zeta+\epsilon_k)$$

Applying Taylor expansion on $f(\zeta+\epsilon_k)$ and $f(\zeta+\epsilon_{k-1})$ we get

$$\epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k-\epsilon_{k-1})\left[f(\zeta)+\epsilon_k f'(\zeta)+\frac{1}{2!}\epsilon_k^2 f''(\zeta)+\cdots\right]}{\left[f(\zeta)+\epsilon_k f'(\zeta)+\frac{1}{2!}\epsilon_k^2 f''(\zeta)+\cdots\right]-\left[f(\zeta)+\epsilon_{k-1}f'(\zeta)+\frac{1}{2!}\epsilon_{k-1}^2 f''(\zeta)\right.}$$

Since ζ is an exact root of $f(x) = 0$ we get

$$\begin{aligned}
\epsilon_{k+1} &= \epsilon_k - \frac{(\epsilon_k-\epsilon_{k-1})\left[\epsilon_k f'(\zeta)+\frac{1}{2!}\epsilon_k^2 f''(\zeta)+\cdots\right]}{\left[\epsilon_k f'(\zeta)+\frac{1}{2!}\epsilon_k^2 f''(\zeta)+\cdots\right]-\left[\epsilon_{k-1}f'(\zeta)+\frac{1}{2!}\epsilon_{k-1}^2 f''(\zeta)+\cdots\right]}\\
&= \epsilon_k - \frac{(\epsilon_k-\epsilon_{k-1})\left[\epsilon_k f'(\zeta)+\frac{1}{2!}\epsilon_k^2 f''(\zeta)+\cdots\right]}{(\epsilon_k-\epsilon_{k-1})f'(\zeta)+\frac{1}{2!}(\epsilon_k^2-\epsilon_{k-1}^2)f''(\zeta)+\cdots}\\
&= \epsilon_k - \left[\frac{\epsilon_k f'(\zeta)+\frac{1}{2!}\epsilon_k^2 f''(\zeta)+\cdots}{f'(\zeta)+\frac{1}{2!}(\epsilon_k+\epsilon_{k-1})f''(\zeta)+\cdots}\right]\\
&= \epsilon_k - \left[\frac{\epsilon_k+\frac{1}{2!}\frac{\epsilon_k^2 f''(\zeta)}{f'(\zeta)}+\cdots}{1+\frac{(\epsilon_k+\epsilon_{k-1})f''(\zeta)}{2!f'(\zeta)}+\cdots}\right]\\
&= \epsilon_k - \left[\epsilon_k+\frac{1}{2!}\frac{\epsilon_k^2 f''(\zeta)}{f'(\zeta)}+\cdots\right]\left[1+\frac{(\epsilon_k+\epsilon_{k-1})f''(\zeta)}{2!f'(\zeta)}+\cdots\right]^{-1}\\
&= \epsilon_k - \left[\epsilon_k+\frac{1}{2!}\frac{\epsilon_k^2 f''(\zeta)}{f'(\zeta)}+\cdots\right]\left[1-\frac{(\epsilon_k+\epsilon_{k-1})f''(\zeta)}{2!f'(\zeta)}+\cdots\right]
\end{aligned}$$

Neglecting the higher powers of ϵ_k and ϵ_{k-1}, we get

$$\epsilon_{k+1} = \frac{\epsilon_k\epsilon_{k-1}f''(\zeta)}{2!f'(\zeta)}$$

Hence

$$\epsilon_{k+1} = C\epsilon_k\epsilon_{k-1} \tag{4.4.6}$$

where $C = \frac{f''(\zeta)}{2!f'(\zeta)}$.

Since in Regula Falsi method, one of the x_0 or x_1 is fixed, equation 4.4.6 becomes

$\epsilon_{k+1} = C\epsilon_0\epsilon_k = C'\epsilon_k$, if x_0 is fixed and $\epsilon_{k+1} = C\epsilon_1\epsilon_k = C'\epsilon_k$, if x_1 is fixed.

In both the cases, the rate of convergence is 1. Thus, Regula Falsi method has a linear rate of convergence.

4.5 Secant Method

Given a continuous function $f(x)$, we approximate it by a first-degree equation of the form $a_0x + a_1$, such that $a_0 \neq 0$ in the neighbourhood of the root. Thus, we write

$$f(x) = a_0x + a_1 = 0 \qquad (4.5.1)$$

Then

$$f(x_k) = a_0x_k + a_1 \qquad (4.5.2)$$

and

$$f(x_{k-1}) = a_0x_{k-1} + a_1 \qquad (4.5.3)$$

On solving equation 4.5.2 and 4.5.3, we get

$$a_0 = \frac{f_k - f_{k-1}}{x_k - x_{k-1}}$$

and

$$a_1 = \frac{x_k f_{k-1} - x_{k-1} f_k}{x_k - x_{k-1}}$$

From 4.5.1, we have $x = -\frac{a_1}{a_0}$. Hence substituting the values of a_0 and a_1 and writing x as x_{k+1} we get

$$x_{k+1} = -\frac{\left(\frac{x_k f_{k-1} - x_{k-1} f_k}{x_k - x_{k-1}}\right)}{\left(\frac{f_k - f_{k-1}}{x_k - x_{k-1}}\right)}$$

That is

$$x_{k+1} = \frac{f_k x_{k-1} - f_{k-1} x_k}{f_k - f_{k-1}} \qquad (4.5.4)$$

which can be expressed as

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k, \qquad k = 1, 2, \cdots \qquad (4.5.5)$$

This method is known as Secant Method or Chord Method.

Example 4.5.1. Using Secant Method, find the root of $f(x) = cosx - xe^x = 0$ taking the initial approximations as 0 and 1.

Solution: Here $x_0 = 0$, $f(x_0) = 1$ and $x_1 = 1$, $f(x_1) = -2.17798$. Using Secant formula

$$x_{k+1} = x_k - \frac{x_k - x_{k+1}}{f_k - f_{k+1}} f_k$$

we get

$$x_2 = x_1 - \frac{x_1 - x_0}{f_1 - f_0} f_1 = 0.314665$$

and $f(x_2) = 0.519872$. Then

$$x_3 = x_2 - \frac{x_2 - x_1}{f_2 - f_1} f_2 = 0.44674$$

and $f(x_3) = 0.2036$. Similarly, we get $x_4 = 0.531706$ and $x_5 = 0.51691$. Hence the approximate root is 0.51691.

4.5.2 Rate of Convergence

Let $f(x)$ be a continuous function. Then the Secant method to find the approximate root of $f(x) = 0$ is given by equation 4.5.5. Let ζ be the exact root of $f(x)$. Then we write x_{k+1}, x_k, x_{k-1} in terms of ζ as

$$x_{k+1} = \zeta + \epsilon_{k+1}, x_k = \zeta + \epsilon_k, x_{k-1} = \zeta + \epsilon_{k-1}$$

where ϵ_i is the error in the i^{th} iteration.

Hence, $f_k = f(x_k) = f(\zeta + \epsilon_k), f_{k-1} = f(x_{k-1}) = f(\zeta + \epsilon_{k-1})$. Thus, equation 4.5.5 becomes

$$\zeta + \epsilon_{k+1} = \zeta + \epsilon_k - \frac{\zeta + \epsilon_k - (\zeta + \epsilon_{k-1})}{f(\zeta + \epsilon_k) - f(\zeta + \epsilon_{k-1})} f(\zeta + \epsilon_k)$$

$$\epsilon_{k+1} = \epsilon_k - \frac{\zeta + \epsilon_k - (\zeta + \epsilon_{k-1})}{f(\zeta + \epsilon_k) - f(\zeta + \epsilon_{k-1})} f(\zeta + \epsilon_k)$$

Applying Taylor expansion on $f(\zeta + \epsilon_k)$ and $f(\zeta + \epsilon_{k-1})$ we get

$$\epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1}) \left[f(\zeta) + \epsilon_k f'(\zeta) + \frac{1}{2!}\epsilon_k^2 f''(\zeta) + \cdots \right]}{\left[f(\zeta) + \epsilon_k f'(\zeta) + \frac{1}{2!}\epsilon_k^2 f''(\zeta) + \cdots \right] - \left[f(\zeta) + \epsilon_{k-1} f'(\zeta) + \frac{1}{2!}\epsilon_{k-1}^2 f''(\zeta) + \cdots \right]}$$

Since ζ is an exact root of $f(x) = 0$ we get

$$\begin{aligned}
\epsilon_{k+1} &= \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1})\left[\epsilon_k f'(\zeta) + \frac{1}{2!}\epsilon_k^2 f''(\zeta) + \cdots\right]}{\left[\epsilon_k f'(\zeta) + \frac{1}{2!}\epsilon_k^2 f''(\zeta) + \cdots\right] - \left[\epsilon_{k-1} f'(\zeta) + \frac{1}{2!}\epsilon_{k-1}^2 f''(\zeta) + \cdots\right]} \\
&= \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1})\left[\epsilon_k f'(\zeta) + \frac{1}{2!}\epsilon_k^2 f''(\zeta) + \cdots\right]}{(\epsilon_k - \epsilon_{k-1}) f'(\zeta) + \frac{1}{2!}(\epsilon_k^2 - \epsilon_{k-1}^2) f''(\zeta) + \cdots} \\
&= \epsilon_k - \left[\frac{\epsilon_k f'(\zeta) + \frac{1}{2!}\epsilon_k^2 f''(\zeta) + \cdots}{f'(\zeta) + \frac{1}{2!}(\epsilon_k + \epsilon_{k-1}) f''(\zeta) + \cdots}\right] \\
&= \epsilon_k - \left[\frac{\epsilon_k + \frac{1}{2!}\frac{\epsilon_k^2 f''(\zeta)}{f'(\zeta)} + \cdots}{1 + \frac{(\epsilon_k + \epsilon_{k-1}) f''(\zeta)}{2! f'(\zeta)} + \cdots}\right] \\
&= \epsilon_k - \left[\epsilon_k + \frac{1}{2!}\frac{\epsilon_k^2 f''(\zeta)}{f'(\zeta)} + \cdots\right]\left[1 + \frac{(\epsilon_k + \epsilon_{k-1}) f''(\zeta)}{2! f'(\zeta)} + \cdots\right]^{-1} \\
&= \epsilon_k - \left[\epsilon_k + \frac{1}{2!}\frac{\epsilon_k^2 f''(\zeta)}{f'(\zeta)} + \cdots\right]\left[1 - \frac{(\epsilon_k + \epsilon_{k-1}) f''(\zeta)}{2! f'(\zeta)} + \cdots\right]
\end{aligned}$$

Neglecting the higher powers of $_k$ andϵ_{k-1}, we get

$$\epsilon_{k+1} = \frac{\epsilon_k \epsilon_{k-1} f''(\zeta)}{2! f'(\zeta)}$$

$$\text{Hence } \epsilon_{k+1} = C\epsilon_k \epsilon_{k-1} \qquad (4.5.6)$$

where $C = \frac{f''(\zeta)}{2! f'(\zeta)}$.

Considering the general equation of rate of convergence, we have

$$\epsilon_{k+1} = A\epsilon_k^p \qquad (4.5.7)$$

which implies

$$\epsilon_k = A\epsilon_{k-1}^p.$$

Then

$$\epsilon_{k-1} = A^{\frac{-1}{p}} \epsilon_k^{\frac{1}{p}}$$

Substituting the value of ϵ_{k-1} in 4.5.6 we get

$$\epsilon_{k+1} = C\epsilon_k A^{\frac{-1}{p}} \epsilon_k^{\frac{1}{p}}$$

From equation 4.5.7

$$\epsilon_{k+1} = A\epsilon_k^p = C\epsilon_k^{1+\frac{1}{p}} A^{\frac{-1}{p}} \qquad (4.5.8)$$

Comparing the powers of ϵ_k we get

$$p = 1 + \frac{1}{p}$$

which implies

$$p = \frac{1 \pm \sqrt{5}}{2}$$

Neglecting the negative value of p, we get the rate of convergence of Secant method as $\frac{1+\sqrt{5}}{2} \approx 1.618$. On comparing the constants of 4.5.8 we get,

$$A = C^{\frac{p}{1+p}}.$$

Thus, $\epsilon_{k+1} = C^{\frac{p}{1+p}} \epsilon_k^p$ where $p = \frac{1+\sqrt{5}}{2}$ and $C = \frac{f''(\zeta)}{2!f'(\zeta)}$.

4.6 Geometrical Interpretation of Secant and Regula Falsi Method

Geometrically, in Secant and Regula Falsi method we replace the function $f(x)$ by a chord passing through (x_k, f_k) and (x_{k-1}, f_{k-1}). We take the next root approximation as the point of intersection of the chord with the x- axis.

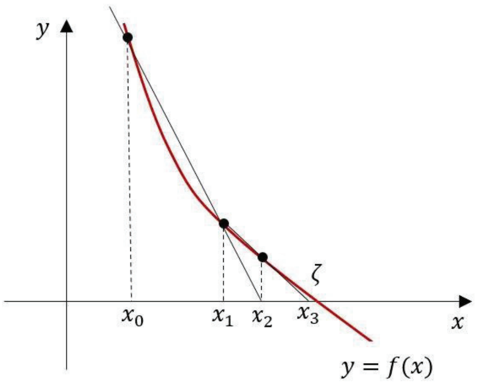

Figure 3: Secant Method

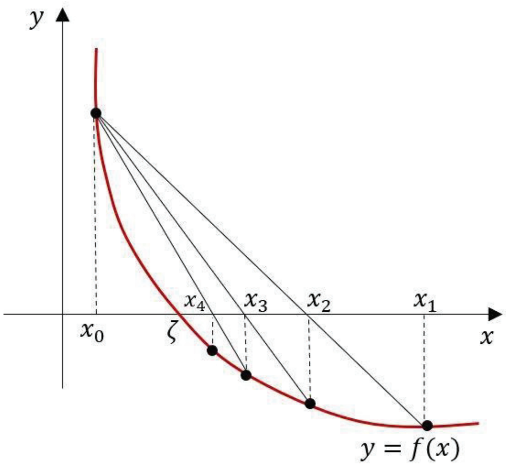

Figure 4: Regula Falsi Method

4.7 Summary

In this chapter, iteration methods to find the approximate roots of an equation is discussed.

The concepts of simple roots, multiple roots, algebraic and transcendental equations are discussed.

Intermediate value property to find the initial approximations to the root is discussed. Bisection method, Newton-Raphson method, Regula Falsi method and Secant method to find the approximate root of an equation is discussed.

Geometrical interpretation and rate of convergence of each method are discussed.

4.8 Exercise

1. Perform four iterations of bisection method to obtain a root of $f(x) = cosx - xe^x$.

2. Determine the initial approximation to find the smallest positive root for $f(x) = x^4 - 3x^2 + x - 10$ and find the root correct to five decimal places by Newton Raphson Method.

3. Perform four iterations of Newton Raphson method to obtain the approximate value of $17^{\frac{1}{3}}$ starting with the initial approximation $x_0 = 2$.

4. Using Regula Falsi Method, find the root of $f(x) = cosx - xe^x = 0$ taking the initial approximations as 0 and 1.

5. Perform three iterations of Secant Method for $f(x) = x^3 - 5x + 1$ such that the root lies in the interval $(0, 1)$.

6. For $f(x) = x^4 - x - 10$ determine the initial approximations to find the smallest positive root correct up to three decimal places using Newton Raphson method, Secant Method and Regula Falsi Method.

7. Show that the Newton-Raphson method leads to the recurrence

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right)$$

to find the square-root of a.

8. For $f(x) = x - e^{-x} = 0$ determine the initial approximation to find the smallest positive root. Find the root correct to three decimal places using Regula Falsi and Secant method.

9. Find the approximate root correct upto three decimal places for $f(x) = cosx - xe^{x}$ using Newton- Raphson Method with initial approximation $x_0 = 1$.

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UNIT 2

5

INTERPOLATION

Unit Structure

5.0 Objectives

This chapter will enable the learner to:

- Understand the concepts of interpolation and interpolating polynomial. Prove the existence and uniqueness of an interpolating polynomial.
- To find interpolating polynomial using Lagrange method and Newton Divided Difference method.

- To understand the concepts of finite difference operators and to relate between difference operators and divided differences.
- To find interpolating polynomial using finite differences using Newton Forward Difference and Backward Difference Interpolation.

5.1 Introduction

If we have a set of values of a function $y = f(x)$ as follows:

x	x_0	x_1	x_2	$\cdots$	x_n
y	y_0	y_1	y_2	$\cdots$	y_n

then the process of finding the value of $f(x)$ corresponding to any value of $x = x_i$ between x_0 and x_n is called interpolation. If the function $f(x)$ is explicitly known, then the value of $f(x)$ corresponding to any value of $x = x_i$ can be found easily. But, if the explicit form of the function is not known then finding the value of $f(x)$ is not easy. In this case, we approximate the function by simpler functions like polynomials which assumes the same values as those of $f(x)$ at the given points x_0, x_1, $x_2, \cdots, x_n$.

Definition 5.1.1 (Interpolating Polynomial). A polynomial $P(x)$ is said to be an interpolating polynomial of a function $f(x)$ if the values of $P(x)$ and/or its certain order derivatives coincides with those of $f(x)$ for given values of x.

If we know the values of $f(x)$ at $n + 1$ distinct points say $x_0 < x_1 < x_2 < \cdots < x_n$, then interpolation is the process of finding a polynomial $P(x)$ such that

(a) $P(x_i) = f(x_i), \quad i = 0, 1, 2, \cdots, n$

or

(a) $P(x_i) = f(x_i), \qquad i = 0, 1, 2, \cdots, n$

(b) $P'(x_i) = f'(x_i), \quad i = 0, 1, 2, \cdots, n$

5.1.1 Existence and Uniqueness of Interpolating Polynomial

Theorem 5.1.1.1. Let $f(x)$ be a continuous function on $[a,b]$ and let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be the $n + 1$ distinct points such that the value of $f(x)$ is known at these points. Then there exist a unique polynomial $P(x)$ such that $P(x_i) = f(x_i)$, for $i = 0,1,2,\cdots,n$.

Proof. We intend to find a polynomial $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ such that $P(x_i) = f(x_i)$, for $i = 0,1,2,\cdots,n$. That is

$$\begin{aligned}
a_0 + a_1x_0 + a_2x_0^2 + \cdots + a_nx_0^n &= f(x_0)\\
a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_nx_1^n &= f(x_1)\\
a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_nx_2^n &= f(x_2)\\
\vdots \qquad\qquad & \quad \vdots\\
a_0 + a_1x_n + a_2x_n^2 + \cdots + a_nx_n^n &= f(x_n)
\end{aligned} \tag{5.1.1.1}$$

Then the polynomial $P(x)$ exists only if the system of equations 5.1.1.1 has a unique solution. That is, the polynomial $P(x)$ exists only if the Vandermonde's determinant is non-zero (in other words, $P(x)$ exists only if the determinant of the co-efficient matrix is non-zero). Let

$$V(x_0, x_1, \cdots, x_{n-1}, x_n) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix}$$

By the properties of determinants, we get

$$V(x_0, x_1, \cdots, x_{n-1}, x_n) = \prod_{\substack{i,j=0\\ i>j}}^{n} (x_i - x_j)$$

Since each x_i's are distinct, we get $\prod_{\substack{i,j=0\\ i>j}}^{n} (x_i - x_j) \neq 0$ and hence the system of equations 5.1.1.1 has a unique solution.

To prove the uniqueness of $P(x)$, let there exists another polynomial $Q(x)$ such that $Q(x_i) = f(x_i)$, for $i = 0, 1, 2, \cdots, n$.

Let

$$R(x) = P(x) - Q(x).$$

As $P(x)$ and $Q(x)$ are polynomials of degree $\leq n$, we get that $R(x)$ is also a polynomial of degree $\leq n$. Also, $R(x_i) = 0$, for $i = 0, 1, 2, \cdots, n$.

That is $R(x)$ is a polynomial of degree $\leq n$ with $n + 1$ distinct roots $x_0, x_1, x_2, \cdots, x_n$ which implies that $R(x) = 0$. Hence $P(x) = Q(x)$. $\square$

5.2 Lagrange Interpolation

5.2.1 Linear Interpolation

For linear interpolation, $n = 1$ and we find a 1-degree polynomial

$$P_1(x) = a_1x + a_0$$

such that

$$f(x_0) = P_1(x_0) = a_1x_0 + a_0$$

and

$$f(x_1) = P_1(x_1) = a_1x_1 + a_0.$$

We eliminate a_0 and a_1 to obtain $P_1(x)$ as follows:

$$\begin{vmatrix} P_1(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0$$

On simplifying, we get

$P_1(x)(x_0 - x_1) - f(x_0)(x - x_1) + f(x_1)(x - x_0) = 0$ and hence

$$\begin{aligned} P_1(x) =& \frac{(x-x_1)}{(x_0-x_1)}f(x_0) + \frac{(x-x_0)}{(x_1-x_0)}f(x_1) \\ =& l_0(x)f(x_0) + l_1(x)f(x_1) \end{aligned} \quad (5.2.1.2)$$

where $l_0(x) = \frac{(x-x_1)}{(x_0-x_1)}$ and $l_1(x) = \frac{(x-x_0)}{(x_1-x_0)}$ are called the Lagrange Fundamental Polynomials which satisfies the condition $l_0(x) + l_1(x) = 1$.

Example 5.2.1. Find the unique polynomial $P(x)$ such that $P(2) = 1.5$, $P(5) = 4$ using Lagrange interpolation.

Solution: By Lagrange interpolation formula,

$$\begin{aligned} P_1(x) =& \frac{(x-x_1)}{(x_0-x_1)}f(x_0) + \frac{(x-x_0)}{(x_1-x_0)}f(x_1) \\ =& \frac{(x-5)}{(2-5)}1.5 + \frac{(x-2)}{(5-2)}4 \\ =& \frac{5x-1}{6} \end{aligned}$$

5.2.2 Quadratic Interpolation

For quadratic interpolation, $n = 2$ and we find a 2-degree polynomial

$$P_2(x) = a_2x^2 + a_1x + a_0$$

such that

$$f(x_0) = P_2(x_0) = a_2x_0^2 + a_1x_0 + a_0$$

$$f(x_1) = P2(x_1) = a_2x_1^2 + a_1x_1 + a_0.$$

$$f(x_2) = P_2(x_2) = a_2x_2^2 + a_1x_2 + a_0$$

We eliminate a_0, a_1 and a_2 to obtain $P_2(x)$ as follows:

$$\begin{vmatrix} P_2(x) & 1 & x & x^2 \\ f(x_0) & 1 & x_0 & x_0^2 \\ f(x_1) & 1 & x_1 & x_1^2 \\ f(x_2) & 1 & x_2 & x_2^2 \end{vmatrix} = 0$$

On simplifying, we get

$$\begin{aligned} P_2(x) =& \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) \\ &+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2) \\ =& l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2) \end{aligned}$$

where $l_0(x) + l_1(x) + l_2(x) = 1$.

Example 5.2.2. Find the unique polynomial $P(x)$ such that $P(3) = 1$, $P(4) = 2$ and $P(5) = 4$ using Lagrange interpolation.

Solution: By Lagrange interpolation formula,

$$\begin{aligned} P_2(x) =& \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) \\ &+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2) \\ =& \frac{(x-4)(x-5)}{(3-4)(3-5)}(1) + \frac{(x-3)(x-5)}{(4-3)(4-5)}(2) + \frac{(x-3)(x-4)}{(5-3)(5-4)}(4) \\ =& \frac{x^2-5x+8}{2} \end{aligned}$$

5.2.3 Higher Order Interpolation

The Lagrange Interpolating polynomial $P(x)$ of degree n for given $n + 1$ distinct points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ is given by

$$P_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i) \quad (5.2.3.1)$$

where,

$$l_i(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}$$

for $i = 0, 1, \cdots, n$.

5.3 Newton Divided Difference Interpolation

5.3.1 Linear Interpolation

For linear interpolation, $n = 1$ and we find a 1-degree polynomial

$$P_1(x) = a_1x + a_0$$

such that

$$f(x_0) = P_1(x_0) = a_1x_0 + a_0$$

and

$$f(x_1) = P_1(x_1) = a_1x_1 + a_0.$$

We eliminate a_0 and a_1 to obtain $P_1(x)$ as follows:

$$\begin{vmatrix} P_1(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0$$

On simplifying, we get

$$P_1(x) = f(x_0) + (x - x_0)\frac{f(x_1) - f(x_0)}{(x_1 - x_0)}$$

Let $f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}$, then

$$P_1(x) = f(x_0) + (x - x_0)\, f[x_0, x_1] \qquad (5.3.1.1)$$

The ratio $f[x_0, x_1]$ is called the first divided difference of $f(x)$ relative to x_0 and x_1. The equation 5.3.1.1 is called the linear Newton interpolating polynomial with divided differences.

Example 5.3.1. Find the unique polynomial $P(x)$ such that $P(2) = 1.5$, $P(5) = 4$ using Newton divided difference interpolation.

Solution: Here

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{5}{6}$$

Hence, by Newton divided difference interpolation,

$$\begin{aligned} P_1(x) =& f(x_0) + (x - x_0) f[x_0, x_1] \\ =& 1.5 + \frac{5}{6}(x - 2) \\ =& \frac{5x - 1}{6} \end{aligned}$$

5.3.2 Higher Order Interpolation

We define higher order divided differences as

$$\begin{aligned} f[x_0, x_1, x_2] =& \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\ =& \frac{1}{x_2 - x_0}\left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}\right] \\ =& \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} \end{aligned}$$

Hence, in general we have

$$f[x_0, x_1, x_2, \cdots, x_{k-1}, x_k] = \frac{f[x_1, x_2, \cdots, x_k] - f[x_0, x_1, x_2, \cdots, x_{k-1}]}{x_k - x_0}$$

for $k = 3, 4, \cdots, n$ and in terms of function values, we have

$$f[x_0, x_1, x_2, \cdots, x_{k-1}, x_k] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod\limits_{\substack{j=0 \\ i \neq j}}^{n} (x_i - x_j)}$$

Then the Newton's Divided Difference interpolating polynomial is given by

$$\begin{aligned} P_n(x) =& f[x_0] + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \cdots \\ &+ (x - x_0)(x - x_1) \cdots (x - x_{n-1}) f[x_0, x_1, \cdots, x_n] \qquad (5.3.2.1) \end{aligned}$$

Example 5.3.2. Construct the divided difference table for the given data and hence find the interpolating polynomial.

x	0.5	1.5	3.0	5.0	6.5	8.0
$f(x)$	1.625	5.875	31.000	131.000	282.125	521.000

Solution: The divided difference table will be as follows:

x	$f(x)$	I order d.d.	II order d.d.	III order d.d.	IV order d.d.
0.5	1.625				
		4.25			
1.5	5.875		5		
		16.75		1	
3.0	31.000		9.5		0
		50.00		1	
5.0	131.000		14.5		0
		100.75		1	
6.5	282.125		19.5		
		159.25			
8.0	521.000				

From the table, as the fourth divided differences are zero, the interpolating polynomial will be given as

$$P_3(x) = f[x_0] + (x - x_0)f[x_0,x_1] + (x - x_0)(x - x_1)f[x_0,x_1,x_2]$$

$$+ (x - x_0)(x - x_1)(x - x_2)f[x_0,x_1,x_2,x_3]$$

$$= 1.625 + (x - 0.5)(4.25) + 5(x - 0.5)(x - 1.5)$$

$$+ 1(x - 0.5)(x - 1.5)(x - 3)$$

$$= x^3 + x + 1$$

5.4 Finite Difference Operators

Consider $n+1$ equally spaced points $x_0, x_1, \cdots, x_n$ such that $x_i = x_0 + ih, i = 0, 1, \cdots, n$. We define the following operators:

1. Shift Operator (E): $E(f(x_i)) = f(x_i + h)$.

2. Forward Difference Operator (Δ): $\Delta(f(x_i)) = f(x_i + h) - f(x_i)$.

3. Backward Difference Operator (∇): $\nabla(f(x_i)) = f(x_i) - f(x_i - h)$.

4. Central Difference Operator (δ $\delta(f(x_i)) = f(x_i + \frac{h}{2}) - f(x_i - \frac{h}{2})$.

5. Average Operator (μ $\mu(f(x_i)) = \frac{1}{2}\left[f(x_i + \frac{h}{2}) + f(x_i - \frac{h}{2})\right]$.

Some properties of these operators:

$\Delta f_i = \nabla f_{i+1} = \delta f_{i+\frac{1}{2}}$.

$\Delta = E - I$

$\nabla = I - E^{-1}$

$\delta = E^{1/2} - E^{-1/2}$

$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

Now we write the Newton's divided differences in terms of forward and backward difference operators. Consider

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}$$

As $x_i = x_0 + ih,\ i = 0, 1, \cdots, n$, we get $x_1 - x_0 = h$ and hence

$$f[x_0, x_1] = \frac{1}{h}\Delta f_0$$

Now we consider

$$\begin{aligned} f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\ &= \frac{\frac{1}{h}\Delta f_1 - \frac{1}{h}\Delta f_0}{2h} \\ &= \frac{1}{2!h^2}\Delta^2 f_0 \end{aligned}$$

Thus, by induction we have

$$f[x_0, x_1, \cdots, x_n] = \frac{1}{n!h^n}\Delta^n f_0.$$

Similarly, for backward difference operator, consider

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = \frac{1}{h}\nabla f_1$$

$$\begin{aligned} f[x_0, x_1, x_2] =& \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\ =& \frac{\frac{1}{h}\nabla f_2 - \frac{1}{h}\nabla f_1}{2h} \\ =& \frac{1}{2!h^2}\nabla^2 f_2 \end{aligned}$$

Thus, by induction we have

$$f[x_0, x_1, \cdots, x_n] = \frac{1}{n!h^n}\nabla^n f_n.$$

5.5 Interpolating polynomials using finite differences operators

5.5.1 Newton Forward Difference Interpolation

Newton's forward difference interpolating polynomial is obtained by substituting the divided differences in 5.3.2.1 with the forward differences. That is

$$\begin{aligned} P_n(x) =& f[x_0] + (x - x_0)f[x_0, x_1] + \cdots \\ &+ (x - x_0)(x - x_1)\cdots(x - x_{n-1})f[x_0, x_1, \cdots, x_n] \\ =& f(x_0) + \frac{(x - x_0)}{h}\Delta f_0 + \frac{(x - x_0)(x - x_1)}{2!h^2}\Delta^2 f_0 + \cdots \\ &+ \frac{(x - x_0)(x - x_1)\cdots(x - x_{n-1})}{n!h^n}\Delta^n f_0 \end{aligned} \qquad (5.5.1.1)$$

Let$u = \frac{(x-x_0)}{h}$, then 5.5.1.1 can be written as

$$\begin{aligned} P_n(x) =& P_n(x_0 + hu) \\ =& f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \cdots \\ &+ \frac{u(u-1)\cdots(u-n+1)}{n!}\Delta^n f(x_0) \end{aligned} \qquad (5.5.1.2)$$

5.5.2 Newton Backward Difference Interpolation

Newton interpolation with divided differences can be expressed in terms of backward differences by evaluating the differences at the end point x_n. Hence, we, write

$$f(x) = f(x_n + \frac{x - x_n}{h}h) = f(x_n + hu) = E^u f(x_n) = (1 - \nabla)^{-u} f(x_n)$$

where$u = \frac{x-x_n}{h}$.

Expanding $(1-\nabla)^{-u}$ in binomial series, we get

$$\begin{aligned} f(x) =& f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \cdots \\ &+ \frac{u(u+1)\cdots(u+n-1)}{n!}\nabla^n f(x_n) + \cdots \end{aligned}$$

Neglecting the higher order differences, we get the interpolating polynomial as

$$\begin{aligned} P_n(x) =& P_n(x_n + hu) \\ =& f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \cdots \\ &+ \frac{u(u+1)\cdots(u+n-1)}{n!}\nabla^n f(x_n) \qquad (5.5.2.1) \end{aligned}$$

Example 5.5. Obtain the Newton's forward and backward difference interpolating polynomial for the given data

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.40	1.56	1.76	2.00	2.28

Solution: The difference table will be as follows:

x	$f(x)$	I order d.d.	II order d.d.	III order d.d.	IV order d.d.
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0	
0.3	1.76		0.04		0
		0.24		0	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

From the table, as the third differences onwards are zero, the interpolating polynomial using Newton's forward differences will be given as

$$\begin{aligned} P_2(x) =& 1.4 + \frac{0.16}{0.1}(x-0.1) + \frac{0.04}{0.01}\frac{(x-0.1)(x-0.2)}{2} \\ =& 2x^2 + x + 1.28 \end{aligned}$$

and the Newton's backward difference interpolating polynomial will be given as

$$\begin{aligned} P_2(x) =& 2.28 + \frac{0.28}{0.1}(x-0.5) + \frac{0.04}{0.01}\frac{(x-0.5)(x-0.4)}{2} \\ =& 2x^2 + x + 1.28 \end{aligned}$$

5.6 Summary

In this chapter, interpolation methods to approximate a function by a family of simpler functions like polynomials is discussed.

Lagrange Interpolation method and Newton Divided Difference Interpolation method are discussed.

Difference operators namely shift operator, forward, backward and central difference operators and average operator are discussed.

The relation between divided difference and the forward and backward difference operators are discussed and hence the Newton's divided difference interpolation is expressed in terms of forward and backward difference operators.

5.7 Exercises

1. Given $f(2) = 4$ and $f(2.5) = 5.5$, find the linear interpolating polynomial using Lagrange interpolation and Newton divided difference interpolation.

2. Using the data $\sin 0.1 = 0.09983$ and $\sin 0.2 = 0.19867$, find an approximate value of $\sin 0.15$ using Lagrange interpolation.

3. Using Newton divided difference interpolation, find the unique polynomial of degree 2 such that $f(0) = 1, f(1) = 3$ and $f(3) = 55$.

4. Calculate the n^{th} divided difference of $\frac{1}{x}$ for points $x_0, x_1, \cdots, x_n$.

5. Show that $\delta = \nabla(1-\nabla)^{-1/2}$ and $\mu = \left[1+\frac{\delta^2}{4}\right]^{1/2}$.

6. For the given data, find the Newton's forward and backward difference polynomials.

x	0	0.1	0.2	0.3	0.4	0.5
$f(x)$	−1.5	−1.27	−0.98	−0.63	−0.22	0.25

7. Calculate $f[x_1, x_2, x_3, x_4]$ $\quad f(x) = \frac{1}{x^2}$.

8. If $f(x) = e^{ax}$, show that $\Delta^n f(x) = (e^{ah} - 1)^n e^{ax}$.

9. Using the Newton's backward difference interpolation, construct the interpolating polynomial that fits data.

x	0.1	0.3	0.5	0.7	0.9	1.1
$f(x)$	−1.699	−1.073	−0.375	0.443	1.429	2.631

10. Find the interpolating polynomial that fits the data as follows:

x	−2	−1		0	1	3	4
$f(x)$	9	16		17	18	44	81

❖❖❖

UNIT 3

6

SOLUTION OF SIMULTANEOUS ALGEBRAIC EQUATIONS

Unit Structure

6.0 Objectives

Student will be able to understand the following from the Chapter:

Method to represent equations in Matrix Form.

Rules of Elementary Transformation.

Application of Dynamic Iteration Method.

6.1 Introduction

An equation is an expression having two equal sides, which are separated by an *equal* to sign. For Example: $9 + 4 = 13$

Here, a mathematical operation has been performed in which 4 and 9 has been added and the result 13 has been written alongside with an **equal** to sign.

Similarly, an **Algebraic** equation is a mathematical expression having two equal sides separated by an *equal to* sign, in which one side the expression is formulated by a set of variables and on the other side there is a constant value. For example $2x^2 + 5y^3 = 9$, Here, the set of variables x and y has been used to provide a unique pair of values whose sum will be equal to 9.

If the **powers** (degrees) of the variables used is 1, then these algebraic equations are known as Linear Algebraic equation. For example: $2x + 3y + 5z = 8$.

It may happen, that there are more than equations,where there will be at-lest one unique pair which will satisfy all the Algebraic equations. The procedure to find these unique pairs are known as Solutions of Simultaneous Algebraic Equations.

There are two types of methods:

1. Gauss-Jordan Method
2. Gauss-Seidel Method

6.2 Gauss-Jordan Method

Gauss Jordan Method is an algorithm used to find the solution of simultaneous equations. The algorithm uses **Matrix Approach** to determine the solution.

The method requires elementary transformation or elimination using ow operations. Hence, it is also known as **Gauss-Elimination Method**.

Steps of Algorithm:

i Represent the set of equation in the following format:

$$A \times X = B$$

where,

A: Coefficient Matrix

X: Variable Matrix B: Constant Matrix

Example:

Convert the following equations in Matrix format:

$3x + 5y = 12$

$2x + y = 1$

The Matrix representation is:

In the above set of equations, the coefficients are the values which are written along-with the variables.

The Constant matrix are the values which are written after the equal to sign.
Hence, the coefficient matrix is given as:

$$\begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix}$$

Hence, the variable matrix is given as:

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

And, the Constant matrix is:

$$\begin{bmatrix} 12 \\ 1 \end{bmatrix}$$

ii Temporarily Combine the Coefficient A and Constant B Matrices in the following format.

$$[A : B]$$

iii Perform row Transformation considering following Do's and Don't:

Allowed	**Not Allowed**
Swapping of Rows $Ra \leftrightarrow Rb$ where a 6= b	Swapping between Row and Column $Ra \leftrightarrow Cb$
Mathematical Operations between Rows allowed: *Addition, Subtraction.* $Ra \leftrightarrow Ra \pm Rb$ *Mathematical Operations with constant value allowed:* *Multiplication, Division.* $R_a \leftrightarrow R_a \times k$ $R_a \leftrightarrow \frac{R_a}{k}$	*Mathematical Operations between Rows not allowed:* *Multiplication, Division.* $R_a \leftrightarrow R_a \times R_b$ $R_a \leftrightarrow \frac{R_a}{R_b}$ *Mathematical Operations with constant value allowed:* *Addition, Subtraction.* $R_a \leftrightarrow R_a \pm k$

Row Transformation is done to convert the **Coefficient Matrix** A to **Unit Matrix** of same dimension as that of A.

6.2.1 Solved Examples:

i. Solve the system-

$6x + y + z = 20\ x + 4y - z = 6\ x - y + 5z = 7$

using Gauss-Jordan Method

Sol. Given:
$6x + y + z = 20\ x + 4y - z = 6\ x - y + 5z = 7$

The matrix representation is-

$$\begin{bmatrix} 6 & 1 & 1 \\ 1 & 4 & -1 \\ 1 & -1 & 5 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 6 \\ 7 \end{bmatrix}$$

Followed by the echolon form, formed by combining Coefficient Matrix and Constant Matrix.

$$\begin{bmatrix} 6 & 1 & 1 & : & 20 \\ 1 & 4 & -1 & : & 6 \\ 1 & -1 & 5 & : & 7 \end{bmatrix}$$

Perform elementary ow transformation in the above matrix to convert

matrix *A* to the following:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 & 1 & : & 20 \\ 1 & 4 & -1 & : & 6 \\ 1 & -1 & 5 & : & 7 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 4 & -1 & : & 6 \\ 6 & 1 & 1 & : & 20 \\ 1 & -1 & 5 & : & 7 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 - 6R_1$

$$\begin{bmatrix} 1 & 4 & -1 & : & 6 \\ 0 & -23 & 7 & : & -16 \\ 1 & -1 & 5 & : & 7 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 4 & -1 & : & 6 \\ 0 & -23 & 7 & : & -16 \\ 0 & -5 & 6 & : & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 4 & -1 & : & 6 \\ 0 & -23 & 7 & : & -16 \\ 0 & 18 & -1 & : & 17 \end{bmatrix}$$

$R_2 \leftrightarrow \dfrac{R_2}{-23}$

$$\begin{bmatrix} 1 & 4 & -1 & : & 6 \\ 0 & 1 & \frac{-7}{23} & : & \frac{16}{23} \\ 0 & -18 & 1 & : & -17 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - 18R_2$

$$\begin{bmatrix} 1 & 4 & -1 & : & 6 \\ 0 & 1 & \dfrac{-7}{23} & : & \dfrac{16}{23} \\ 0 & 0 & \dfrac{103}{23} & : & \dfrac{103}{23} \end{bmatrix}$$

$R_2 \leftrightarrow \dfrac{23}{103} \times R_3$

$$\begin{bmatrix} 1 & 4 & -1 & : & 6 \\ 0 & 1 & \dfrac{-7}{23} & : & \dfrac{16}{23} \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - 4R_2$

$$\begin{bmatrix} 1 & 0 & \dfrac{5}{23} & : & \dfrac{74}{23} \\ 0 & 1 & \dfrac{-7}{23} & : & \dfrac{16}{23} \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 + \dfrac{7}{23} \times R_2$

$$\begin{bmatrix} 1 & 0 & \dfrac{5}{23} & : & \dfrac{74}{23} \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - \dfrac{5}{23} \times R_3$

$$\begin{bmatrix} 1 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

The solution of the equations are: $x = 3$; $y = 1$ and $z = 1$.

ii. Solve the system-

$2x + y + z = 10$ $3x + 2y + 3z = 18$ $x + 4y + 9z = 16$

using Gauss-Jordan Method

Sol. Given:

$2x + y + z = 10$ $3x + 2y + 3z = 18$ $x + 4y + 9z = 16$

The matrix representation is-

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$$

Followed by the echolon form, formed by combining Coefficient Matrix and Constant Matrix.

$$\begin{bmatrix} 2 & 1 & 1 & : & 10 \\ 3 & 2 & 3 & : & 18 \\ 1 & 4 & 9 & : & 16 \end{bmatrix}$$

Perform elementary ow transformation in the above matrix to convert matrix A to the following:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & : & 10 \\ 3 & 2 & 3 & : & 18 \\ 1 & 4 & 9 & : & 16 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 4 & 9 & : & 16 \\ 3 & 2 & 3 & : & 18 \\ 2 & 1 & 1 & : & 10 \end{bmatrix}$$

$$R_2 \leftrightarrow R_2 - 3R_1$$

$$\begin{bmatrix} 1 & 4 & 9 & : & 16 \\ 0 & -10 & -24 & : & -30 \\ 2 & 1 & 1 & : & 10 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 4 & 9 & : & 16 \\ 0 & -10 & -24 & : & -30 \\ 0 & -7 & -17 & : & -22 \end{bmatrix}$$

$R_3 \leftrightarrow -R_3$

$$\begin{bmatrix} 1 & 4 & 9 & : & 16 \\ 0 & -10 & -24 & : & -30 \\ 0 & 7 & 17 & : & 22 \end{bmatrix}$$

$R_2 \leftrightarrow -R_2$

$$\begin{bmatrix} 1 & 4 & 9 & : & 16 \\ 0 & 10 & 24 & : & 30 \\ 0 & 7 & 17 & : & 22 \end{bmatrix}$$

$R_2 \leftrightarrow \frac{R_2}{10}$

$$\begin{bmatrix} 1 & 4 & 9 & : & 16 \\ 0 & 1 & \frac{24}{10} & : & 3 \\ 0 & 7 & 17 & : & 22 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - 7R_2$

$$\begin{bmatrix} 1 & 4 & 9 & : & 16 \\ 0 & 1 & \frac{24}{10} & : & 3 \\ 0 & 0 & \frac{1}{5} & : & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - 4R_2$

$$\begin{bmatrix} 1 & 0 & \frac{-3}{5} & : & 4 \\ 0 & 1 & \frac{24}{10} & : & 3 \\ 0 & 0 & \frac{1}{5} & : & 1 \end{bmatrix}$$

$R_3 \leftrightarrow 5R_3$

$$\begin{bmatrix} 1 & 0 & \frac{-3}{5} & : & 4 \\ 0 & 1 & \frac{24}{10} & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$R_2 \leftrightarrow R_2 - \frac{24}{10} R_3$$

$$\begin{bmatrix} 1 & 0 & \frac{-3}{5} & : & 4 \\ 0 & 1 & 0 & : & -9 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$R_1 \leftrightarrow R_1 + \frac{3}{5} R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & : & 7 \\ 0 & 1 & 0 & : & -9 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

The solution of the equations are: $x = 7$; $y = -9$ and $z = 5$.

6.3 Gauss-Seidel Method

Gauss Seidel Method uses Iterative methods to find the unique solution of Linear Algebraic equations. In this method, the present value of a variable depends on the past and present value of the other variables. This type of Iteration is known as *Dynamic Iteration Method*.

To achieve convergence of the values it is important to have **Diagonal Dominance**. In Diagonal Dominance, the first equation should have the highest coefficient among the set of x coefficients as well as it should be the highest coefficient within the same equation. Similarly, the second equation should have highest coefficient of y as well as among its own equation.

After ensuring Diagonal Dominance, the variable of each equation is represented as the function of other variables.

6.3.1 Solved Examples:

i. Solve the equation: $x + 4y - z = 6$ $6x + y + z = 20$ $x - y + 5z = 7$

by using Gauss-Seidel Method

Sol. Given:

$x + 4y - z = 6$ $6x + y + z = 20$ $x - y + 5z = 7$

On comparing the coefficient of x among the given set of equations. The maximum value is 6 which is present in the second equation. Considering the second equation, the maximum coefficient present among the variable is also 6 (Coefficient of x)

Hence, the first equation is: **6x+y+z=20 —(i)**

Similarly, among First and Third equation, the maximum value of the Coefficient of *y* is 4.

Hence, the second equation is: **x+4y-z=6—(ii)**

And, the third equation is: **x-y+5z=7—(iii)**

Now represent each variable as a function of other two variables like: Using equation 1:

$$x = \frac{20 - y - z}{6}$$

Using equation 2:

$$y = \frac{6 - x + z}{4}$$

Using equation 3:

$$z = \frac{7 + y - x}{5}$$

Consider the initial values of *x*, *y* and *z* as 0.

Now,

To implement the iteration the equations are re-written as:

$$x_n = \frac{20 - y_{n-1} - z_{n-1}}{6}$$
$$y_n = \frac{6 - x_n + z_{n-1}}{4}$$
$$z_n = \frac{7 + y_n - x_n}{5}$$

Where, x_n, y_n and z_n are the **Present** values of *x*, *y* and *z* respectively. x_{n-1}, y_{n-1} and z_{n-1} are the **Past** values of *x*, *y* and *z* respectively.

Means,

To calculate the Present value of *x*, we require Past values of *y* and *z*.

To calculate the Present value of *y*, we require Present value of *x* and Past value of *z*.

To calculate the Present value of *z*, we require Present values of *x* and *y*.

Hence the values of x, y and z will be:

i	x_n	y_n	z_n
0	0	0	0
1	3.33	0.6675	0.8675
2	3.0775	0.9475	0.974
3	3.0131	0.97412	0.992
4	3.0056	0.99665	0.9982
5	3.0035	0.99815	0.99893
6	3.0005	0.9996	0.99982
7	3.00009	0.99993	0.999668

On Approximation:

$x = 3$; $y = 1$ and $z = 1$ ii. Solve the equation:

$x_1 + 10x_2 + 4x_3 = 6$ $2x_1 + 10x_2 + 4x_3 = -15$

$9x_1 + 2x_2 + 4x_3 = 20$

by using Gauss-Seidel Method

Sol. Given:

$x_1 + 10x_2 + 4x_3 = 6$

$2x_1 - 4x_2 + 10x_3 = -15$

$9x_1 + 2x_2 + 4x_3 = 20$

On re-arranging to achieve the Diagonal Dominance:

$9x_1 + 2x_2 + 4x_3 = 20$ — (1) $x_1 + 10x_2 + 4x_3 = 6$ —(2)

$2x_1 - 4x_2 + 10x_3 = -15$ —(3)

Therefore,

$x_{1_n} = \dfrac{20 - 2x_{2_{n-1}} - 4x_{3_{n-1}}}{9}$ (From equation (1))

$x_2 = \dfrac{6 - x_{1_n} - 4x_{3_{n-1}}}{10}$ (From equation (2))

$x_3 = \dfrac{-15 - 2x_{1_n} + 4x_{2_n}}{10}$ (From equation (3)) Hence the values of x, y and z will be:

i	x_n	y_n	z_n
0	0	0	0
1	2.2222	0.3778	-1.7933
2	2.9353	1.0238	-1.6775
3	2.7403	0.99697	-1.6493
4	2.7337	0.98635	-1.6522
5	2.7373	0.98715	-1.6526
6	2.7373	0.98731	-1.6525
7	2.7373	0.9873	-1.6525

On Approximation:

$x = 2.7373$; $y = 0.9873$ and $z = -1.6525$

6.4 Summary

Linear Algebraic equations can be solved by using two methods.

- Gauss Seidel Method
- Gauss Jordan Method

Gauss Seidel Method uses iterative approach, following Diagonal Dominance principle.

Gauss Jordan Method uses Matrix approach of the form: $A \times X = B$, following Elementary Transformation principle.

6.5 References

(a) S. S. Shastry ”Introductory Methods of Numerical Methods”. (Chp 3)

(b) Steven C. Chapra, Raymond P. Canale ”Numerical Methods for Engineers”.

6.6 Unit End Exercise

Find the solution of the following set of equation by using Gauss Jordan method.

(a) $3x + 2y + 4z = 7$ $2x + y + z = 7$ $x + 3y + 5z = 2$

(b) $10x + y + z = 12$ $2x + 10y + z = 13$ $x + y + 3z = 5$

(c) $4x + 3y - z = 6$ $3x + 5y + 3z = 4$ $x + y + z = 1$

(d) $2x + y - z = -1$ $x - 2y + 3z = 9$

$3x - y + 5z = 14$

Find the solution of the following set of equation by using Gauss Seidel method.

(a) $10x + y + z = 12$ $2x + 10y + z = 13$ $x + y + 3z = 5$

(b) $28x + 4y - z = 32$ $2x + 17y + 4z = 35$ $x + 3y + 10z = 24$

(c) $7x_1 + 2x_2 - 3x_3 = -12$ $2x_1 + 5x_2 - 3x_3 = -20$ $x_1 - x_2 - 6x_3 = -26$

(d) $7x_1 + 2x_2 - 3x_3 = -12$ $2x_1 + 5x_2 - 3x_3 = -20$ $x_1 - x_2 - 6x_3 = -26$

(e) $10x + y + z = 12$ $x + 10y + z = 12$ $x + y + 10z = 12$

(Assume $x^{(0)} = 0.4$, $y^{(0)} = 0.6$, $z^{(0)} = 0.8$)

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UNIT 3

7

NUMERICAL DIFFERENTIATION AND INTEGRATION

Unit Structure

7.0 Objectives

Student will be able to understand the following from the Chapter:

Methods to compute value of Differentiation of a function at a particular value.

Methods to find the area covered by the curve within the the given interval.

Understand the importance of Interpolation Method.

7.1 Introduction

Differentiation is the method used to determine the slope of the curve at a particular point. Whereas, Integration is the method used to find the area between two values.

The solution of Differentiation and Integration at and between particular values can be easily determined by using some Standard rules. There are some complex function whose differentiation and Integration solution is very difficult to determine. Hence, there are some practical approaches which can be used to find the approximated value.

7.2 Numerical Differentiation

The value of differentiation or derivative of a function at a particular value can be determined by using Interpolation Methods.

(a) Newton's Difference Method (If the step size is constant.)

(b) La-grange's Interpolation Method (If step size is not constant.)

Newton's Difference Method is further divided into two types depending on the position of the sample present in the input data set.

Newton's **Forward** Difference Method: To be used when the input value is lying towards the **start** of the input data set.

Newton's **Backward** Difference Method: To be used when the input value is lying at the **end** of the input data set.

7.2.1 *Newton Forward Difference Method*

For a given set of values (x_i, y_i), $i = 0, 1, 2, \ldots n$ where x_i are at equal intervals and h is the interval of of the input values i.e. $x_i = x_0 + i \times h$ then,

$$y(x) = y_0 + p \triangle y_0 + \frac{p(p-1)}{2!} \triangle^2 y_0 + \frac{p(p-1)(p-2)}{3!} \triangle^3 y_0 + \cdots\cdots + \\ + \frac{p(p-1)(p-2) \cdot (p-n+1)}{n!} \triangle^n y_0$$

Since the above equation is in form of "p", therefore, chain rule is to be used for differentiation. Hence,

$$\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$$

We know that,

$$p = \frac{x - x_0}{h}$$

Where, x = Unknown Value x_0 = First Input value h = Step Size

Hence,

$$\frac{dp}{dx} = \frac{1}{h}$$

And,

$$\frac{dy}{dp} = \triangle y_0 + \frac{p-1+p}{2!}\triangle^2 y_0 + \frac{p(p-1)+p(p-2)+(p-1)(p-2)}{3!}\triangle^3 y_0 + \cdots\cdots$$

Hence due to simplification,

$$\frac{dy}{dx} = \frac{1}{h} \times \left(\triangle y_0 - \frac{\triangle^2 y_0}{2} + \frac{\triangle^3 y_0}{3} - \frac{\triangle^4 y_0}{4} + \frac{\triangle^5 y_0}{5} - \frac{\triangle^6 y_0}{6} + \cdots\right)$$

On differentiating second time, the expression becomes:

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \times \left(\triangle^2 y_0 - \triangle^3 y_0 + \frac{11\,\triangle^4\, y_0}{12} - \frac{5\,\triangle^5\, y_0}{6} + \frac{137\,\triangle^6\, y_0}{180} - \cdots\right)$$

7.2.2 *Newton Backward Difference Method*

For a given set of values (x_i, y_i), $i = 0, 1, 2, \ldots n$ where x_i are at equal intervals and h is the interval of of the input values i.e. $x_i = x_0 + i \times h$ then,

$$y(x) = y_n + p \nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \cdots\cdots + \cdots\cdots + \frac{p(p+1)(p+2)\cdot(p+n-1)}{n!} \nabla^n y_0$$

Since the above equation is in form of "p", therefore, chain rule is to be used for differentiation. Hence,

$$\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$$

We know that,

$$p = \frac{x - x_n}{h}$$

Where, x = Unknown Value x_n = Final Input value h = Step Size

Hence,

$$\frac{dp}{dx} = \frac{1}{h}$$

And,

$$\frac{dy}{dp} = \nabla y_n + \frac{p+1+p}{2!}\nabla^2 y_n + \frac{p(p+1)+p(p+2)+(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots\cdots$$

Hence due to simplification,

$$\frac{dy}{dx} = \frac{1}{h} \times \left(\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \frac{\nabla^5 y_n}{5} + \frac{\nabla^6 y_n}{6} + \cdots\right)$$

On differentiating second time, the expression becomes:

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \times \left(\nabla^2 y_n + \nabla^3 y_n + \frac{11\,\nabla^4\, y_n}{12} + \frac{5\,\nabla^5\, y_n}{6} + \frac{137\,\nabla^6\, y_n}{180} + \cdots\right)$$

7.2.3 *Solved Examples*

i. From the data table given below obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.2$

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Sol. The first step is to identify which method to be used.

Since in the question the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.2$ is to be determined.

Hence, **Forward Difference** to be used as the value lies at the start of the data set.

Therefore, Forward Difference Table is to be formed.

x	**y**	$4y$	4^2y	4^3y	4^4y	4^5y	4^6y
1	2.7183	0.6018	0.1333	0.0294	0.0067	0.0013	0.0001
1.2	3.3201	0.7351	0.1627	0.0361	0.0080	0.0014	
1.4	4.0552	0.8978	0.1988	0.0441	0.0094		
1.6	4.9530	1.0966	0.2429	0.0535			
1.8	6.0496	1.3395	0.2964				
2.0	7.3891	1.6359					
2.2	9.0250						

Now select the values corresponding to the row of $x = 1.2$. Since, value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ is to be determined at $x = 1.2$

Hence, the values are: $4y_0 = 0.7351$, $4^2y_0 = 0.1627$, $4^3y_0 = 0.0361$, $4^4y_0 = 0.0080$, $4^5y_0 = 0.0014$.

Substituting the values in the following formula:

$$\frac{dy}{dx} = \frac{1}{h} \times \left(\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \frac{\Delta^5 y_0}{5} \right)$$

$$\frac{dy}{dx} = 3.2031$$

Similarly, to find the value of $\frac{d^2y}{dx^2}$, Substitute the values in the following equation:

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \times \left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11\,\Delta^4\, y_0}{12} - \frac{5\,\Delta^5\, y_0}{6} \right)$$

where, $\Delta^2 y_0 = 0.1627$, $\Delta^3 y_0 = 0.0361$, $\Delta^4 y_0 = 0.0080$, $\Delta^5 y_0 = 0.0014$.

Therefore,

$$\frac{d^2y}{dx^2} = 3.3191$$

i. From the data table given below obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.8$

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Sol. The first step is to identify which method to be used.

Since in the question the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.8$ is to be determined.

Hence, **Backward Difference** to be used as the value lies at the end of the data set.

Therefore, Backward Difference Table is to be formed.

x	**y**	$5y$	5^2y	5^3y	5^4y	5^5y	5^6y
1	2.7183						
1.2	3.3201	0.6018					
1.4	4.0552	0.7351	0.1333				

x	y	5y	5^2y	5^3y	5^4y	5^5y	5^6y
1.6	4.9530	0.8978	0.1627	0.0294			
1.8	6.0496	1.0966	0.1988	0.0067			
2.0	7.3891	1.3395	0.2429	0.0441	0.0080	0.0013	
2.2	9.0250	1.6359	0.2964	0.0535	0.0094	0.0014	0.0001

Now select the values corresponding to the row of x = 1.2. Since,

value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ is to be determined at x = 1.8

Hence, the values are: $\nabla y_n = 1.0966,\ \nabla^2 y_n = 0.1988,\ \nabla^3 y_n = 0.0361,\ \nabla^4 y_n = 0.0067.$

Substituting the values in the following formula:

$$\frac{dy}{dx} = \frac{1}{h} \times \left(\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4}\right)$$

$$\frac{dy}{dx} = 6.04854.$$

Similarly, to find the value of $\frac{d^2y}{dx^2}$, Substitute the values in the following equation:

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \times \left(\nabla^2 y_n + \nabla^3 y_n + \frac{11\,\nabla^4\, y_n}{12}\right)$$

where, $\nabla^2 y_n = 0.1988,\ \nabla^3 y_n = 0.0361,\ \nabla^4 y_n = 0.0067.$

Therefore,

$$\frac{d^2y}{dx^2} = 3.3191$$

If **step size** of the input value is **not constant**, then the derivatives can be determined by simply differentiating the expression provided by the **Langrange's Polynomial**.

7.2.4 *Solved Examples*

i. Tabulate the following function: $y = x^3 - 10x + 6$ at $x_0 = 0.5$, $x_1 = 1$ and $x_2 = 2$. Compute its 1^{st} and 2^{nd} derivatives at x = 1.00 using Lagrange's interpolation method.

Sol. Given: $y = x^3 - 10x + 6$

At $x_0 = 0.5$ $y_0 = 0.5^3 - 10 \times 0.5 + 6 = 1.125$

At $x_1 = 1$ $y_1 = 1^3 - 10 \times 1 + 6 = -3$

At $x_2 = 2$ $y_2 = 2^3 - 10 \times 2 + 6 = -6$

Hence, the Lagange's Formula is:

$$y = \frac{(x-1)(x-2)}{(0.5-1)(0.5-2)} \times (1.125) + \frac{(x-0.5)(x-1)}{(2-0.5)(2-1)} \times (-6) + \frac{(x-0.5)(x-2)}{(1-0.5)(1-2)} \times (-3)$$

$$y = \frac{(x-1)(x-2)}{0.75} \times (1.125) + \frac{(x-0.5)(x-1)}{1.5} \times (-6) + \frac{(x-0.5)(x-2)}{0.5} \times (3)$$

Hence, $\frac{dy}{dx}$ will be obtained by differentiating both the sides

$$\frac{dy}{dx} = \frac{(x-1)+(x-2)}{0.75} \times (1.125) + \frac{(x-0.5)+(x-1)}{1.5} \times (-6) + \frac{(x-0.5)+(x-2)}{0.5} \times (3)$$

So therefore, at $x = 1$ the $\frac{dy}{dx}$ will be:

$$\frac{dy}{dx} = \frac{(1-1)+(1-2)}{0.75} \times (1.125) + \frac{(1-0.5)+(1-1)}{1.5} \times (-6) + \frac{(1-0.5)+(1-2)}{0.5} \times (3)$$

$$\frac{dy}{dx} = \frac{-1.125}{0.75} + \frac{-0.5}{1.5} \times (6) + (-3)$$

$$\frac{dy}{dx} = -6.5$$

7.3 Numerical Integration

Numerical Integration provides a set of methods to compute the Definite Integration of a Function between the given set of values.

There are three methods used to find the value of Integration:

- Trapezoidal Rule.
- Simpson's $\frac{1}{3^{rd}}$ Rule.
- Simpson's $\frac{3}{8^{th}}$ Rule.

7.3.1 Trapezoidal Rule

In this method the curve is divided into small trapeziums.

These trapeziums are then added to to find the complete area of the curve between two values. Hence, Let $y = f(x)$ then,

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\Big(y_0 + y_n + 2 \times (y_1 + y_2 + \ldots + y_{n-1})\Big)$$

Where,

$$h = \frac{x_n - x_0}{n}$$

x_n: Upper Limit x_0: Lower Limit $y_0, y_1, y_2, \cdots, y_n$ are the values of of y corresponding to $x_0, x_1, x_2, \cdots, x_n$

7.3.2 Simpson's $\frac{1}{3^{rd}}$ Rule

Let $y = f(x)$ then,

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\Big(y_0 + y_n + 4 \times \text{(Sum of Odd osition Terms)} + 2 \times$$

(Sum of Even osition Terms)

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\Big(y_0 + y_n + 4 \times (y_1 + y_3 + y_5 + \cdots) + 2 \times (y_2 + y_4 + y_6 + \cdots)\Big)$$

Where,

$$h = \frac{x_n - x_0}{n}$$

x_n: Upper Limit

x_0: Lower Limit

$y_0, y_1, y_2, \cdots, y_n$ are the values of of y corresponding to $x_0, x_1, x_2, \cdots, x_n$

7.3.2 Simpson's $\frac{3}{8^{th}}$ Rule

Let $y = f(x)$ then,

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\Big(y_0 + y_n + 2 \times \text{(Sum of Multiple of 3, position terms)} +$$

3 × (Sum of Remaining terms)

$$\int_{x_0}^{x_n} f(x)dx =$$

$$\frac{3h}{8}\Big(y_0 + y_n + 2 \times (y_3 + y_6 + y_9 + \cdots) + 3 \times (y_1 + y_2 + y_4 + y_5 + \cdots)\Big)$$

Where,

$$h = \frac{x_n - x_0}{n}$$

x_n: Upper Limit

x_0: Lower Limit

$y_0, y_1, y_2, \cdots, y_n$ are the values of of y corresponding to $x_0, x_1, x_2, \cdots, x_n$

7.3.3 *Solved Examples*

i. A solid of revolution is formed by rotating about the x-axis, the area between the x-axis,the lines $x = 0$ and $x = 1$ and the curve through the points below:

x	0.00	0.25	0.50	0.75	1.00
x	1.000	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed.

Sol. Let "V" be the volume of the solid formed by rotating the curve around x- axis, then

$$V = \pi \times \int_0^1 y^2 dx$$

Therefore, the tables is updated as:

x	0.00	0.25	0.50	0.75	1.00
x	1.000	0.9793	0.9195	0.8261	0.7081

Rewriting the same table to find the value of n.

x	$0.00 = x_0$	$0.25 = x_1$	$0.50 = x_2$	$0.75 = x_3$	$1.00 = x_4$
x	1.000	0.9793	0.9195	0.8261	0.7081

As the extreme or the last value is x_4. Hence, $n = 4$

Therefore,

$$h = \frac{1-0}{4}$$
$$h = \frac{1}{4}$$
$$h = 0.25$$

(a) <u>TRAPEZOIDAL RULE</u>

$$\pi \times \int_0^1 y^2 dx = \pi \times \frac{h}{2}\Big(y_0 + y_4 + 2 \times (y_1 + y_2 + y_3)\Big)$$

where, $h = 0.25, y_0 = 1.000, y_1 = 0.9793, y_2 = 0.9195, y_3 = 0.8261, y_4 = 0.7081$

On substituting the values we get:

$$\pi \times \int_0^1 y^2 dx = 3.36704$$

(b) SIMPSON'S $\frac{1}{3^{rd}}$ RULE

$$\pi \times \int_0^1 y^2 dx = \pi \times \frac{h}{3}\Big(y_0 + y_4 + 4 \times (y_1 + y_3) + 2 \times (y_2)\Big)$$

where, $h = 0.25, y_0 = 1.000, y_1 = 0.9793, y_2 = 0.9195, y_3 = 0.8261, y_4 = 0.7081$

On substituting the values we get:

$$\pi \times \int_0^1 y^2 dx = 2.81923$$

(c) SIMPSON'S $\frac{3}{8^{th}}$ RULE

$$\pi \times \int_0^1 y^2 dx = \pi \times \frac{3h}{8}(y_0 + y_4 + 2 \times (y_3) + 3 \times (y_1 + y_2))$$

where, $h = 0.25, y_0 = 1.000, y_1 = 0.9793, y_2 = 0.9195, y_3 = 0.8261, y_4 = 0.7081$

On substituting the values we get:

$$\pi \times \int_0^1 y^2 dx = 2.66741$$

7.4 Summary

Numerical Differentiation uses the methods to find the value of First ans Second Order Derivatives at a particular value of x or the input variable.

Numerical Integration provides methods to find the Definite Integration or the area covered by the curve between two points.

7.5 References

(a) S. S. Shastry ”Introductory Methods of Numerical Methods”.

(b) Steven C. Chapra, Raymond P. Canale ”Numerical Methods for Engineers”.

7.6 Unit End Exercise

(a) Find $\frac{d(J_0(x))}{dx}$ and $\frac{d^2(J_0(x))}{dx^2}$ at $x = 0.1,\ x = 0.2$ and $x = 0.4$ from the following table:

(0,1.0); (0.1,0.9975); (0.2,0.9900); (0.3,0.9776); (0.4,0.9604)

(b) The following table gives angular displacement a at different time t (time). (0,0.052); (0.02,0.105); (0.04,0.168); (0.06,0.242); (0.08,0.327) ; (0.10,0.408) ;

Calculate angular velocity and acceleration at t=0.04, 0.06, and 0.1

Angular Velocity :$\frac{d\theta}{dt}$; Angular Acceleration :$\frac{d^2\theta}{dt^2}$

(c) A cubic function $y = f(x)$ satisfies the following data.

x	0	1	3	4
y	1	4	40	85

Determine $f(x)$ and hence find $f'(2)$ and $f''(2)$

(d) Use Trapezoidal Rule to evaluate integral $\int_0^2 e^{-x}dx$ width of sub-interval (h) = 0.5.

(e) Using Simpson's rules. Evaluate $\int_0^6 \left(\frac{1}{x^4+1}\right) dx$ take $n = 6$.

(f) Using Simpson's rules. Evaluate $\int_{-3}^{3} (x^4)\, dx$ take $n = 6$.

(g) Using Simpson's rules. Evaluate $\int_0^6 \left(\frac{1}{x+1}\right) dx$ take $n = 6$.

(h) Use Trapezoidal Rule to evaluate integral $\int_0^2 x \times e^{-x}dx$ width of sub-interval (h) = 0.5.

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UNIT 3

8

NUMERICAL DIFFERENTIATION EQUATION

Unit Structure

8.0 Objectives

Student will be able to understand the following from the Chapter:

Methods to compute value of Differential Equation of a function at a particular value.

Practical or Software Implemented method to find the solution of Differential equation.

8.1 Introduction

Differential equation is defined as an expression which contains derivative terms. For example: $\frac{dy}{dx} = 2x + 3y$. Here the term $\frac{dy}{dx}$ is indicating that the occurrence of a Differential Equation.

The differential equations can be solved analytically using various methods like: *Variable separable*, *Substitution*, *Linear Differential Equation*, *Solution to Homogeneous Equation*, etc. but in this chapter, various practically approachable methods will be discussed to find the solution of a given differential equation at a particular value.

A Differential equation is defined on the basic of two terminologies:

Order: Number of times a variable is getting differentiated by an another variable.

Degree: Power of the highest order derivative in a differential equation.

For Example: Consider the following equations: A. $\frac{dy}{dx} = 4x + 5y$

B. $\frac{d^2y}{dx^2} = 5x^2 + 6xy$C. $\left(\frac{dy}{dx}\right)^2 = 5x^3 + 6y^2$

D. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = x - y$

In the above equations, eqn. *A* has **Order** = 1, because *y* is differentiated only 1 time w.r.t. *x*. The **degree** = 1 because the power of the only differential equation is 1.

In eqn. *B* has **Order** = 2, because *y* is differentiated 2 times w.r.t. *x*. The **degree** = 1 because the power of the only differential equation is 1.

In eqn. *C* has **Order** = 1, because *y* is differentiated only 1 time w.r.t. *x*. The **degree** = 2 because the power of the only differential equation is 2.

In eqn. *D*, there are two derivatives present. Hence, the term with maximum differentiation present is to be selected. Therefore, the equation has **Order** = 2, because *y* is differentiated only 2 time w.r.t. *x*. The **degree** = 1 because the power of the derivative of maximum order is 1.

8.2 Euler's Method

Euler's Method is a Practically implemented method used to find the solution of **first order** Differential equation.

Suppose the given differential equation is $\frac{dy}{dx} = f(x, y)$ with initial conditions $y(x_0) = y_0$ (The value of y at $x = x_0$ is y_0.) Then by Euler's method:

$$y_{n+1} = y_n + h \times f(x_n, y_n) \text{ for } n = 0,1,2,...$$

Where,

y_{n+1} is *Future Value of* y. y_n is *Present Value of* y.

h is *Common Difference or Step Size*.

8.2.1 Solved Examples

Find the value of y when $x = 0.1$, given that $y(0) = 1$ and $y^0 = x^2 + y$ by using Euler's Method.

Sol. Given: $y^0 = x^2 + y$ with initial condition $y(0) = 1$

$y' = \frac{dy}{dx} = x^2 + y$ and the meaning of $y(0) = 1$ is the value of y at $x = 0$ is 1. Hence,

$x_0 = 0$ and $y_0 = 1$

To find the value of y at $x = 0.1$, value of h is required.

Hence, $h = 0.05$.

According to Euler's Method, $y_{n+1} = y_n + h \times f(x_n, y_n)$.

Iteration 1:

$y_1 = y_0 + h \times f(x_0, y_0)$ $y_0 = 1$; $x_0 = 0$; $h = 0.05$ $\therefore f(x_n, y_n) = x^2 + y$ $y_1 = 1 + 0.05(0^2 + 1)$

$y_1 = 1.05$ (The value of y at $x = x_0 + h = 0 + 0.05 = 0.05$

Iteration 2:

$y_2 = y_1 + h \times f(x_1, y_1)$ $y_2 = 1.05$; $x_1 = x_0 + h = 0.05$ $y_2 = 1.05 + 0.05(0.05^2 + 1.05)$ $y_1 = 1.0265$ (The value of y at $x = x_1 + h = 0.05 + 0.05 = 0.10$

8.3 Euler's Modified Method

The values of y determined at every iteration may have some error depending on the value of selected Common Difference or Step Size h. Hence, to find the accurate value of y at a particular x, **Euler's Modified Method** is used.

In this method the value of the corresponding iteration is ensured by providing a process called *Iteration within Iteration method*.

This method is used to minimize the error. The iterative formula is given as;

$$y_{n+1}^{(m+1)} = y_n + \frac{h}{2}\left[f\left(x_m, y_m\right) + f \quad x_{n+1}, y_{n+1}^{m}\right)\Big]$$

Where,

x_m, y_m are the values to be used in the basic iteration and $y(m+1)^n$ is the value obtained while saturating the given "y" value in the same iteration.

The steps to use Euler's Modified Method is:

(a) Apply Euler's Method at every iteration to find the approximate value at new value of x using **Euler's Method**.

(b) Apply **Euler's Modified Method** in a particular iteration to saturate the value of y.

8.3.1 Solved Examples

Find the value of y when $x = 0.1$, given that $y(0) = 1$ and $y^0 = x^2 + y$ by using Euler's Modified Method.

Sol. Given: $y^0 = x^2 + y$ with initial condition $y(0) = 1$

$y' = \frac{dy}{dx} = x^2 + y$ and the meaning of $y(0) = 1$ is the value of y at $x = 0$

is 1. Hence,

$x_0 = 0$ and $y_0 = 1$

To find the value of y at $x = 0.1$, value of h is required.

Hence, $h = 0.05$.

Iteration 1:

Using Euler's Method $y_1 = y_0 + h \times f(x_0, y_0)$ $y_0 = 1$; $x_0 = 0$; $h = 0.05$ $\therefore f(x_n, y_n) = x^2 + y$ $y_1 = 1 + 0.05(0^2 + 1)$ $y_1 = 1.05$

Iteration 1 a :

Using Euler's Modified Method,

$$y_1^1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^0)\right]$$

where, $x_1 = x_0 + h = 0 + 0.05 = 0.05$ and y_1^0 is the initial value obtained by using Euler's Method.

$$\therefore y_1^1 = 1 + \frac{0.05}{2}\left[f(0^2 + 1) + f(0.05^2 + 1.05)\right]$$
$$y_1^1 = 1.0513$$

Since, $y_1^0 = 1.0500$ and $y_1^1 = 1.0513$ are not equal. Hence, apply 2^{nd} iteration.

Iteration 1 b :

Using Euler's Modified Method,

$$y_1^2 = y_0 + \frac{h}{2}\Big[f(x_0, y_0) + f(x_1, y_1^1)\Big]$$

where, $x_1 = x_0 + h = 0 + 0.05 = 0.05$ and $y_1^1 = 1.0513$ is the value obtained in *Iteration 1(a)*.

$$\therefore y_1^2 = 1 + \frac{0.05}{2}\Big[f(0^2 + 1) + f(0.05^2 + 1.0513)\Big]$$
$$y_1^2 = 1.0513$$

Since, values of y_1^2 and y_1^1 are equal. Hence, the value of y at $x = 0.05$ is 1.0513

Iteration 2:

Using Euler's Method $y_2 = y_1 + h \times f(x_1, y_1)$ $y_1 = 1.0513$; $x_1 = x_0 + h = 0.05$; $h = 0.05$

$\therefore f(x_n, y_n) = x^2 + y$

$y_2 = 1.0513 + 0.05(0.05^2 + 1.0513)$ $y_2 = 1.1044$

Iteration 2 a :

Using Euler's Modified Method,

$$y_2^1 = y_1 + \frac{h}{2}\Big[f(x_1, y_1) + f(x_2, y_2^0)\Big]$$

where, $x_2 = x_1 + h = 0.05 + 0.05 = 0.10$ and y_2^0 is the initial value obtained by using Euler's Method.

$$\therefore y_2^1 = 1.1044 + \frac{0.05}{2}\Big[f(0.05^2 + 1.0513) + f(0.1^2 + 1.1044)\Big]$$
$$y_2^1 = 1.1055$$

Since, $y_2^0 = 1.1044$ and $y_2^1 = 1.1055$ are not equal. Hence, apply 2^{nd} iteration.

Iteration 1 b :

Using Euler's Modified Method,

$$y_2^2 = y_1 + \frac{h}{2}\Big[f(x_1, y_1) + f(x_2, y_2^1)\Big]$$

where, $x_2 = x_1 + h = 0.05 + 0.05 = 0.10$ and y_2^1 is the value obtained at

Iteration 2 a.

$$\therefore y_2^2 = 1.1044 + \frac{0.05}{2}\left[f\left(0.05^2 + 1.1055\right) + f\left(0.1^2 + 1.1055\right)\right]$$
$$y_2^1 = 1.1055$$

Since, values of y_2^2 and y_2^1 are equal. Hence,

the value of y at $x = 0.1$ is 1.1055

8.4 Range Kutta Method

Range Kutta Method is an another method to find the solution of a First Order Differential Equations. It is mainly divided into two methods depending on the number of parameters used in a method.

Range Kutta 2^{nd} Order Method.

Range Kutta 4^{th} Order Method.

Range Kutta 2^{nd} order Method uses **Two** parameters k_1 and k_2 to find the value of y_{n+1}. The expression is given as:

$$y_{n+1} = y_n + \frac{1}{2} \times [k_1 + k_2]$$

where

$k_1 = h \times f(x_n, y_n)$ $k_2 = h \times f(x_n + h, y_n + k_1)$

Range Kutta 4^{th} order Method uses **Four** parameters k_1, k_2, k_3 and k_4 to find the value of y_{n+1}. The expression is given as:

$$y_{n+1} = y_n + \frac{1}{6} \times [k_1 + 2 \times k_2 + 2 \times k_3 + k_4]$$

where,

$$k_1 = h \times f\left(x_n, y_n\right)$$
$$k_2 = h \times f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$
$$k_3 = h \times f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$
$$k_2 = h \times f\left(x_n + h, y_n + k_3\right)$$

8.4.1 Solved Examples

Given; $\frac{dy}{dx} = y - x$, where $y_0 = 2$, Find $y(0.1)$ and $y(0.2)$, correct upto 4 decimal places.

Sol. Given: $\frac{dy}{dx} = y - x = f(x, y)$ $y(0) = 2$

$\therefore y_0 = 2$ and $x_0 = 0$

Range Kutta 2^{nd} Order Method:

$h = 0.1$

Iteration 1:

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$
$$k_1 = h \times f(x_0, y_0)$$
$$\therefore k_1 = 0.1(2 - 0)$$
$$\therefore k_1 = 0.2$$
$$k_2 = h \times f(x_0 + h, y_0 + k_1)$$
$$\therefore k_2 = 0.1 \times ((2 + 0.2) - (0 + 0.1))$$
$$\therefore k_2 = 0.21$$

So, $y_1 = 2 + \frac{1}{2}(0.2 + 0.21)$

$\therefore y_1 = 2.2050$ at $x = 0.1$ **Iteration 2:**

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$$

$k_1 = h \times f(x_1, y_1)$

$\therefore k_1 = 0.1(2.205 - 0.1)$ $\therefore k_1 = 0.2105$ $k_2 = h \times f(x_1 + h, y_1 + k_1)$

$\therefore k_2 = 0.1 \times ((2.2050 + 0.2105) - (0.1 + 0.1))$

$\therefore k_2 = 0.22155$

So, $y_1 = 2.205 + \frac{1}{2}(0.2105 + 0.22155)$

$\therefore y_1 = 2.421025$ at $x = 0.2$ **Runge Kutta 4^{th} Order:**

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2 \times k_2 + 2 \times k_3 + k_4)$$

Where,

$$k_1 = h \times f\left(x_n, y_n\right)$$
$$k_2 = h \times f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$
$$k_3 = h \times f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$
$$k_2 = h \times f\left(x_n + h, y_n + k_3\right)$$

Let h=0.1,

Iteration 1:

$$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2 \times k_2 + 2 \times k_3 + k_4\right]$$
$$k_1 = h \times f\left(x_0, y_0\right)$$
$$\therefore k_1 = h \times f(x_0, y_0)$$
$$\therefore k_1 = 0.1(2 - 0)$$
$$\therefore k_1 = 0.2$$
$$k_2 = h \times f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$\therefore k_2 = 0.1 \times \left(\left(2 + \frac{0.2}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right)$$
$$\therefore k_2 = 0.215$$
$$k_3 = h \times f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$\therefore k_3 = 0.1 \times \left(\left(2 + \frac{0.215}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right)$$
$$\therefore k_3 = 0.20575$$

$$k_4 = h \times f\left(x_0 + h, y_0 + k_3\right)$$
$$\therefore k_4 = 0.1 \times ((2 + 0.20575) - (0 + 0.1))$$
$$\therefore k_4 = 0.210575$$
$$\therefore y_1 = y_0 + \frac{1}{6}\left[k_1 + 2 \times k_2 + 2 \times k_3 + k_4\right]$$
$$\therefore y_1 = 2 + \frac{1}{6}\left[0.2 + 2 \times 0.215 + 2 \times 0.20575 + 0.210575\right]$$
$$\therefore y_1 = 2.2087 \text{ at } x_1 = 0.1$$

Iteration 2:

$$y_2 = y_1 + \frac{1}{6}\left[k_1 + 2 \times k_2 + 2 \times k_3 + k_4\right]$$
$$k_1 = h \times f\left(x_1, y_1\right)$$
$$\therefore k_1 = h \times f(x_1, y_1)$$
$$\therefore k_1 = 0.1(2.2087 - 0.1)$$
$$\therefore k_1 = 0.21087$$
$$k_2 = h \times f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$\therefore k_2 = 0.1 \times \left(\left(2.2087 + \frac{0.21087}{2}\right) - \left(0.1 + \frac{0.1}{2}\right)\right)$$
$$\therefore k_2 = 0.2164$$
$$k_3 = h \times f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$\therefore k_3 = 0.1 \times \left(\left(2.2087 + \frac{0.2164}{2}\right) - \left(0.1 + \frac{0.1}{2}\right)\right)$$
$$\therefore k_3 = 0.21669$$
$$k_4 = h \times f\left(x_1 + h, y_1 + k_3\right)$$
$$\therefore k_4 = 0.1 \times ((2.2087 + 0.21669) - (0.1 + 0.1))$$
$$\therefore k_2 = 0.222539$$
$$\therefore y_2 = y_1 + \frac{1}{6}\left[k_1 + 2 \times k_2 + 2 \times k_3 + k_4\right]$$
$$\therefore y_1 = 2.2087 + \frac{1}{6}\left[0.21087 + 2 \times 0.2164 + 2 \times 0.21669 + 0.222539\right]$$
$$\therefore y_1 = 2.4253$$

8.5 Taylor Series

The methods discussed in the previous sections, are applicable only for **First** order Differential Equation. But, Taylor Series method can be used to find the solution for higher order differential equations.

Taylor Series is a method used to represent a Function as a sum of infinite series represented in terms of derivatives derived at a particular point. Taylor series is mathematically expressed as:

$$f(x) = f(x_0) + \frac{xf'(x_0)}{1!} + \frac{x^2 f''(x_0)}{2!} + \frac{x^3 f'''(x_0)}{3!} \cdots$$

Similarly, to get an approximate value of y at $x = x_0 + h$ is given by following expression:

$$y(x_0 + h) = y_0 + hy_0' + \frac{h^2 y_0''}{2!} + \frac{h^3 y_0'''}{3!} + \cdots$$

8.5.1 Solved Examples

Example 12. Given the differential equation $y'' - xy' - y = 0$, with the condition $y(0) = 1$ and $y'(0) = 0$, Use Taylor series to determine the value of $y(0.1)$.

Solution: $y'' - xy' - y = 0$

Now, $y(0) = 1$ and $y'(0) = 0$ and also $x_0 = 0$

According to Taylor Series;

$$y(x) = y_0 + (x - x_0)\, y_0' + \frac{(x - x_0)^2}{2!} \times y_0'' + \frac{(x - x_0)^3}{3!} \times y_0''' + \frac{(x - x_0)^4}{4!} \times y_0^{iv} + \frac{(x - x_0)^5}{5!} \times y_0^{v}$$

Since, $y'' - xy' - y = 0$

$\therefore y'' = xy' + y \ldots\ldots (1)$

$\therefore y''(0) = x_0 y_0' + y_0$

$\therefore y''(0) = 0 \times 0 + 1$

$\therefore y''(0) = 1$

Differentiating Equation (1)

$y''' = y' + xy'' + y' \ldots\ldots (2)$

$\therefore y'''(0) = y_0' + x_0 y_0'' + y_0'$

$\therefore y'''(0) = 0 + 0 \times 1 + 0$

$\therefore y'''(0) = 0$

Differentiating Equation (2)

$y^{iv} = y'' + y'' + xy''' + y''$

$y^{iv} = 3y'' + xy''' \ldots\ldots (3)$

$\therefore y^{iv}(0) = 3y_0'' + x_0 y_0'''$

$\therefore y^{iv}(0) = 3 \times 1 + 0 \times 1 + 0$

$\therefore y^{iv}(0) = 3$

Differentiating Equation (3)

$y^{v} = 3y''' + y''' + xy^{iv}$

$y^{v} = 4y''' + xy^{iv} \ldots\ldots (4)$

$\therefore y^{v}(0) = 4y_0''' + x_0 y_0^{iv}$

$\therefore y^{v}(0) = 4 \times 0 + 0 \times 3$

$\therefore y^{v}(0) = 0$

Since, $y(x) = y_0 + (x)\, y_0' + \frac{(x)^2}{2!} \times y_0'' + \frac{(x)^3}{3!} \times y_0''' + \frac{(x)^4}{4!} \times y_0^{iv} + \frac{(x)^5}{5!} \times y_0^{v}$

Because $x_0 = 0$

Where, $y_0 = 1, y_0' = 0,\ y_0'' = 1,\ y_0''' = 0,\ y_0^{iv} = 3$ and $y_0^{v} = 0$

$$\therefore y(0.1) = 1 + 0.1 \times 0 + \frac{(0.1)^2}{2!} \times 1 + \frac{(0.1)^3}{3!} \times 0 + \frac{(0.1)^4}{4!} \times 3 + \frac{(0.1)^5}{5!} \times 0$$

$$y(0.1) = 1 + \frac{(0.1)^2}{2!} \times 1 + \frac{(0.1)^4}{4!} \times 3$$

$\therefore y(0.1) = 1.00501$

8.6 Summary

Euler's Method, Euler's Modified and Range Kutta Method are applicable only in First Order Differential Equation.

The First order differential equation should be of the form $\frac{dy}{dx} = f(x, y)$

Solution of Higher Order Differential Equation can be done by using Taylor's Method.

8.7 References

(a) S. S. Shastry "Introductory Methods of Numerical Methods".

(b) Steven C. Chapra, Raymond P. Canale "Numerical Methods for Engineers".

8.7 Unit End Exercise

(a) Use Euler's method to estimate $y(0.5)$ of the following equation with $h = 0.25$ and $\frac{dy}{dx} = x + y + xy$, $y(0) = 1$.

(b) Apply Euler's method to solve $\frac{dy}{dx} = x + 3y$ with $y(0) = 1$. Hence, find $y(1)$. Take $h = 0.2$

(c) Using Euler's method, find $y(2)$ where $\frac{dy}{dx} = 2y + x$ and $y(1) = 1$, take $h = 0.2$.

(d) Using Euler's method, find $y(2)$ where $\frac{dy}{dx} = 2 + \sqrt{xy}$ and $y(1) = 1$, take $h = 0.2$.

(e) Solve $y^0 = 1 - y$, $y(0) = 0$ by Euler's Modified Method and obtain y at $x = 0.1$ and $x = 0.2$

(f) Apply Euler's Modified method to find $y(1.2)$. Given $\frac{dy}{dx} = \frac{y + xy}{x}$, $h = 0.1$ and $y(1) = 2.718$. Correct upto three decimal places.

(g) Solve $\frac{dy}{dx} = ln(x + y)$, $y(1) = 2$. Compute y for $x = 1.2$ and $x = 1.4$ using

Euler's Modified Method.

(h) Using Range-Kutta Method of second order to find $y(0.2)$. Given $\frac{dy}{dx} = x + y$, $y(0) = 2$, $h = 0.1$

(i) Use Range-Kutta Method of fourth order to find $y(0.1)$, $y(0.2)$. Given $\frac{dy}{dx} = \frac{y}{2} - 3x,\ y(0) = 1$. (Take $h = 0.1$)

(j) Use Range-Kutta Method of fourth order to find $y(0.1)$. Given $\frac{dy}{dx} = \frac{y^2 + x^2}{10}$, $y(0) = 1$. (Take $h = 0.1$)

(k) Use Range-Kutta Method of second order to find $y(1.2)$. Given $\frac{dy}{dx} = y^2 + x^2$, $y(1) = 0$. (Take $h = 0.1$)

(l) Solve $\frac{dy}{dx} = \frac{y - x}{y + x}$, where $y(0) = 1$, to find $y(0.1)$ using Range-Kutta method.

(m) By using Runge-Kutta method of order 4 to evaluate $y(2.4)$ from the following differential equation $y^0 = f(x,y)$ where $f(x,y) = (x+1)y$. Initial condition $y(2) = 1$, $h = 0.2$ correct upto 4 decimal places.

(n) Use Taylor series method, for the equation, $\frac{dy}{dx} = y - xy$ and $y(0) = 2$ to find the value of y at $x = 1$.

(o) Use Taylor series method, for the equation, $\frac{dy}{dx} = xy + y^2$ and $y(0) = 1$ to find the value of y at $x = 0.1, 0.2, 0.3$.

(p) Use Taylor series method, for the equation, $\frac{dy}{dx} = \frac{1}{y + x^2}$ and $y(4) = 4$ to find the value of y at $x = 4.1, 4.2$.

(q) Use Taylor's series method, for the equation, $y^0 = x^2 - y$ and $y(0) = 1$ to find $y(0.1)$

❖❖❖

Unit 4

9

LEAST-SQUARES REGRESSION

Unit Structure

9.0 Objectives

This chapter would make you understand the following concepts:

- Correlation.
- Scatter Diagram .
- Linear Regression.
- Polynomial Regression
- Non- liner Regression.
- Multiple Linear Regression.

9.1 Introduction

The functional relationship between two or more variables is called 'Regression'. here in regression, the variables may be considered as independent or dependent. For example imagine the person walking on the straight road from point A to point B he took 10min, the distance from A to B is 100m, now to reach point C from B, he took another 10min while the distance from B to C is 100m, then how many minutes will he take to reach point E from C which is 200m long from Point C ? or if he walks continuously 20min & straight like this from point C where will he reach ? also what if the walking speed is not uniform ? or road is not straight ?

9.2 Basic Concepts of Correlation

Definition:

Two variables are said to be correlated if the change in the value of one variable causes corresponding change in the value of other variable.

9.2.1 Types of Correlation

1) **Positive & Negative Correlation :** If changes in the value of one variable causes change in value of other variable in the same direction then the variables are said to be positively correlated.

   eg. The more petrol you put in your car, the farther it can go.

   If changes in the value of one variable causes change in value of other variable in opposite direction then the variable are said to be negatively correlated eg. As a biker's speed increases, his time to get to the finish line decreases.

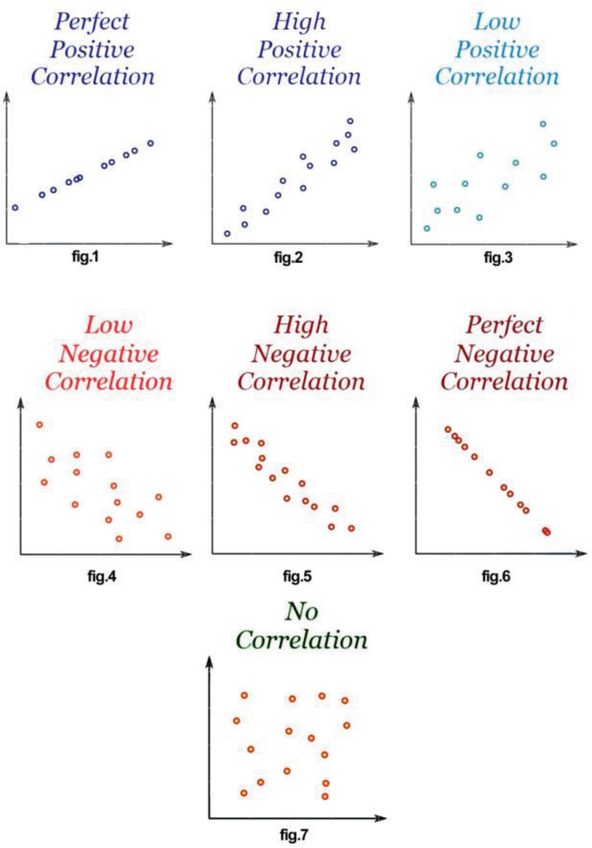
2) **Simple, Multiple & Partial Correlation :** The correlation between two variables is simple correlation. The Correlation between three or more variables is called multivariate correlation, For example the relationship between supply, price and profit of a commodity. in partial correlation though more than two factors are involved but correlation is studied only between two factors and the other factors are assumed to be constant.

3) **Linear and Non-linear Correlation :** if the nature of the graph is straight line the correlation is called linear and if the graph is not a straight line but curve is called non-linear correlation

9.2.2 Scatter Diagram

This is a graphical method to study correlation. In this method each pair of observations is represented by a point in a plane. The diagram formed by plotting all points is called scatter diagram. in this following Scatter diagram each dot coordinate is like $(x_1, y_1), (x_2, y_2), \ldots\ldots (x_n, y_n)$ on XY- plane

The Following are types of scatter diagram:



Merits & Demerits of Scatter Diagram

Merits:

i) Scatter diagram is easy to draw.

ii) It is not influenced by extreme items.

iii) It is easy to understand and non-mathematical method of studying correlation between two variables

Demerits:

It does not give the exact degree of correlation between two variables. it gives only a rough idea.

9.2.3 Karl Pearson's Coefficient of correlation (Product moment Coefficient of correlation)

Karl Pearson's Coefficient of Correlation is an extensively used mathematical method in which the numerical representation is applied to measure the level of relation between linear related variables. The coefficient of correlation is expressed by “r”.

Steps involved to calculate "r":

Step 1: Calculate the actual mean of x and actual mean of y

Step 2: Take deviation from the actual mean of x series, It gives column $X = (x - \bar{x})$.

Step 3: Take deviation from the actual mean of y series, It gives column $Y = (y - \bar{y})$.

Step 4: Calculate summation of deviation product of X and Y i.e. $\sum XY$.

Step 5 : Square the deviations of X and Y and calculate the summation i.e. $\sum X^2$ and $\sum Y^2$.

Step 6 : Use the following formula to calculate r

$$r = \frac{\sum XY}{\sqrt{\sum X^2 \sum Y^2}}$$

Interpretation of the values of :

Correlation Coefficient Value (r)	Direction and Strength of Correlation
-1	Perfectly negative
-0.8	Strongly negative
-0.5	Moderately negative
-0.2	Weakly negative
0	No association
0.2	Weakly positive
0.5	Moderately positive
0.8	Strongly positive
1	Perfectly positive

Example 9.2.3.1 :

Find the Coefficient of correlation for the following data and comment on its value.

x	6	13	9	10	6	4
y	2	15	17	13	7	6

Solution : Here $n = 6,\ \overline{x} = \frac{\sum x_n}{n} = \frac{48}{6} = 8$ and $\overline{y} = \frac{\sum y_n}{n} = \frac{60}{6} = 10$

Consider the following table for calculation

x	y	$X = (x - \overline{x})$	$Y = (y - \overline{y})$	XY	X^2	Y^2
6	2	-2	-8	16	4	64
13	15	5	5	25	25	25
9	17	1	7	7	1	49
10	13	2	3	6	4	9
6	7	-2	-3	6	4	9
4	6	-4	-4	16	16	16
48	**60**			**76**	**54**	**172**

$$r = \frac{\sum XY}{\sqrt{\sum X^2 \sum Y^2}} = \frac{76}{\sqrt{54 \times 172}} = 0.79$$

$r = 0.79$

Hence, **Strongly positive correlation between x and y**

Similarly we can calculate r value without deviation form means (i.e. Direct method) where $\boldsymbol{r} = \dfrac{\sum \boldsymbol{xy} - \frac{\sum x \sum y}{n}}{\sqrt{\sum \boldsymbol{x^2} - \frac{(\sum x)^2}{n}} \times \sqrt{\sum \boldsymbol{y^2} - \frac{(\sum y)^2}{n}}}$, where n =number of observations.

Also we can calculate r value by using assumed means of x and y

where $= \dfrac{\sum \boldsymbol{uv} - \frac{\sum u \sum v}{n}}{\sqrt{\sum \boldsymbol{u^2} - \frac{(\sum u)^2}{n}} \times \sqrt{\sum \boldsymbol{v^2} - \frac{(\sum v)^2}{n}}}$,

where n =number of observations,

$u = x - A$, here A is assumed mean of x series,

$v = y - B$, here B is assumed mean of y series.

9.3 Linear Regression

Regression is the method of estimating the value of one variable when other variable is known and when the variables are correlated. when the points of scatter diagram concentrate around straight line, the regression is called linear and this straight line is known as the line of regression

9.3.1 Regression Equations

Regression equations are algebraic forms of the regression lines. Since there are two regression lines, there are two regression equations the regression of x on y (y is independent and x is dependent) is used to estimate the values of x for the given changes in y and the regression equation of y on x (x is independent and y is dependent) is used to estimate the values of y for the given changes in x.

a) Regression equation of y on x:

The regression equation of y on x is expressed as $\boldsymbol{y = a + bx}$ ---(1)

where x = independent variable, y = dependent variable,
$a = y$ intercept, b = slope of the said line & n = number of observations.

The values of $'a'$and $'b'$ can be obtained by the Method of least squares. in this method the following two algebraic Normal equations are solved to determine the values of a and b :

$$\sum \boldsymbol{y} = \boldsymbol{na} + \boldsymbol{b} \sum \boldsymbol{x} \qquad \text{---(2)}$$

$$\sum \boldsymbol{xy} = \boldsymbol{a} \sum \boldsymbol{x} + \boldsymbol{b} \sum \boldsymbol{x^2} \qquad \text{---(3)}$$

b) Regression equation of x on y:

The regression equation of x on y is expressed as $x = c + dy$ ---(1')

where y = independent variable, x = dependent variable,
$c = y$ intercept, d = slope of the said line & n = number of observations.

The values of $'c'$ and $'d'$ can be obtained by the Method of least squares. in this method the following two algebraic Normal equations are solved to determine the values of c and d :

$$\boldsymbol{\sum x = nc + d\sum y} \quad \text{---(2')}$$

$$\boldsymbol{\sum yx = c\sum y + d\sum y^2} \quad \text{---(3')}$$

Regression equation using regression coefficients:

Direct method:

a) Regression equation of y on x :

When the values of x and y are large. In such case,
the equation $y = a + bx$ is changed to

$$\boldsymbol{y - \bar{y} = b_{yx}(x - \bar{x})} \quad \text{---(4)}$$

where $\bar{y}$ and $\bar{x}$ are arithmetic mean of y and x respectively

Dividing both side of equation (2) by n, we get $\bar{y} = a + b\bar{x}$
so that $a = \bar{y} + b\bar{x}$

Substituting this in equation (1), we get

$$y - \bar{y} = b(x - \bar{x})$$

Writing b with the usual subscript we get the equation (4)

again, multiplying equation (2) by $\sum x$ and equation (3) by n we have

$$\sum x \sum y = na \sum x + b(\sum x)^2$$

$$n\sum xy = na\sum x + nb(\sum x)^2$$

subtracting the first from second, we get

$$n\sum xy - \sum x \sum y = na(\sum x)^2 - b(\sum x)^2$$

$$b = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum x^2 - \frac{(\sum x)^2}{n}}$$

writing b with the usual subscript we get

Regression coefficient of y on x is $\boldsymbol{b_{yx}} = \frac{\sum \boldsymbol{xy} - \frac{\sum \boldsymbol{x} \sum \boldsymbol{y}}{\boldsymbol{n}}}{\sum \boldsymbol{x^2} - \frac{(\sum \boldsymbol{x})^2}{\boldsymbol{n}}}$

b) Regression equation of x on y

Processing the same way as before, the regression equation of x on y is

$$\boldsymbol{x - \overline{x} = b_{xy}(y - \overline{y})}$$

Where Regression coefficient of x on y is $\boldsymbol{b_{xy}} = \frac{\sum \boldsymbol{yx} - \frac{\sum \boldsymbol{y} \sum \boldsymbol{x}}{\boldsymbol{n}}}{\sum \boldsymbol{y^2} - \frac{(\sum \boldsymbol{y})^2}{\boldsymbol{n}}}$

Example 9.3.1.1:

Find the two-regression equation for the following data:

$\boldsymbol{x}$	16	18	20	23	26	27
$\boldsymbol{y}$	11	12	14	15	17	16

Solution :

Calculation for regression equations

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{xy}$	$\boldsymbol{x^2}$	$\boldsymbol{y^2}$
16	11	176	256	121
18	12	216	324	144
20	14	280	400	196
23	15	345	529	225
26	17	442	676	289
27	16	432	729	256
$\sum \boldsymbol{x}$ **=130**	$\sum \boldsymbol{y} = \mathbf{85}$	$\sum \boldsymbol{xy}$ **=1891**	$\sum \boldsymbol{x^2} = \mathbf{2914}$	$\sum \boldsymbol{y^2} = \mathbf{1231}$

$$\boldsymbol{\overline{x} = \frac{\sum x_n}{n} = \frac{130}{6} = 21.67, \quad \overline{y} = \frac{y_n}{n} = \frac{85}{6} = 14.17}$$

i) Regression Equation of y on x is $y - \bar{y} = b_{yx}(x - \bar{x})$

Regression coefficient of y on x is $\boldsymbol{b_{yx}} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum x^2 - \frac{(\sum x)^2}{n}}$

$$\boldsymbol{b_{yx}} = \frac{1891 - \frac{130*85}{6}}{2914 - \frac{130^2}{6}} = \mathbf{0.50}$$

Regression Equation of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$y - 14.17 = 0.50(x - 21.67)$

$\boldsymbol{y = 0.50x + 3.33}$

ii) Regression Equation of x on y is $x - \bar{x} = b_{xy}(y - \bar{y})$

Regression coefficient of x on y is $\boldsymbol{b_{xy}} = \frac{\sum yx - \frac{\sum y \sum x}{n}}{\sum y^2 - \frac{(\sum y)^2}{n}}$

$$\boldsymbol{b_{xy}} = \frac{1891 - \frac{85*130}{6}}{1231 - \frac{85^2}{6}} = \mathbf{1.84}$$

Regression Equation of x on y is

$x - \bar{x} = b_{xy}(y - \bar{y})$

$x - 21.67 = 1.84(y - 14.17)$

$\boldsymbol{x = 1.84x - 4.4}$

Deviations taken from arithmetic mean of x and y

a) Regression equation of y on x:

Another way of expressing regression equation of y on x is by taking deviation of x and y series from their respective actual means. in this case also, the equation $y = a + bx$ is changed to $\boldsymbol{y - \bar{y} = b_{yx}(x - \bar{x})}$

The value of b_{yx} can be easily obtained as follows:

$$\boldsymbol{b_{yx}} = \frac{\sum \chi\gamma}{\sum \chi^2}$$

Where $\chi = (x - \bar{x})$ and $\gamma = (y - \bar{y})$

The two normal equation which we had written earlier when changed in terms of χ and γ becomes

$$\sum\gamma = na + b\sum\chi \qquad \text{--- (5)}$$

$$\sum\chi\gamma = a\sum\chi + b\sum\chi^2 \qquad \text{--- (6)}$$

Since $\sum\chi = \sum\gamma = 0$ (deviation being taken from mean)

Equation (5) reduces to $na = 0 \therefore a = 0$

Equation (6) reduces to $\sum\chi\gamma = b\sum\chi^2$

$$\boldsymbol{b} \textbf{ or } \boldsymbol{b_{yx}} = \frac{\sum\chi\gamma}{\sum\chi^2}$$

After obtaining the value of b_{yx} the regression equation can easily be written in terms of x and y by substituting for $\chi = (x - \bar{x})$ and $\gamma = (y - \bar{y})$.

Deviations taken from arithmetic mean of x and y

The regression equation $x = c + dy$ is reduces to $\boldsymbol{x - \bar{x} = b_{xy}\,(y - \bar{y})}$

Where $\boldsymbol{b_{xy}} = \frac{\sum\chi\gamma}{\sum\gamma^2}$

Example 9.3.1.2 :

The following data related to advertising expenditure and sales

Advertising expenditure (In Lakhs)	5	6	7	8	9
Sales (In Lakhs)	20	30	40	60	50

i) Find the regression equations

ii) Estimate the likely sales when advertising expenditure is 12 lakhs Rs.

iii) What would be the advertising expenditure if the firm wants to attain sales target of 90 lakhs Rs.

Solution :

i) Let the advertising expenditure be denoted by x and sales by y

arithmetic mean of x is $\boldsymbol{\bar{x} = \frac{\sum x_n}{n} = \frac{35}{5} = 7}$

& arithmetic mean of y is $\boldsymbol{\bar{y} = \frac{\sum y_n}{n} = \frac{200}{5} = 40}$

Calculation for regression equations

x	y	$\chi = (x - \bar{x})$	$\gamma = (y - \bar{y})$	$\chi\gamma$	χ^2	γ^2
5	20	−2	−20	40	4	400
6	30	−1	−10	10	1	100
7	40	0	0	0	0	0
8	60	1	20	20	1	400
9	50	2	10	20	4	100
$\sum x = 35$	$\sum y = 200$	$\sum \chi = 0$	$\sum y = 0$	$\sum \chi\gamma = 90$	$\sum \chi^2 = 10$	$\sum \gamma^2 = 1000$

Regression equation of y on x

$b_{yx} = \frac{\sum \chi\gamma}{\sum \chi^2} = \frac{90}{10} = 9$

Regression equation of y on x is

$y - \bar{y} = b_{yx}(x - \bar{x})$

$y - 40 = 9(x - 7)$

$y - 40 = 9x - 63$

$\boldsymbol{y = 9x - 23}$

Regression equation of x on y

$b_{xy} = \frac{\sum \chi\gamma}{\sum \gamma^2} = \frac{90}{1000} = 0.09$

$x - \bar{x} = b_{xy}(y - \bar{y})$

$x - 7 = 0.09(y - 40)$

$x - 7 = 0.09y - 3.6$

$\boldsymbol{x = 0.09y + 3.4}$

ii) For estimate the likely sales when advertising expenditure is 12 lakhs

$\boldsymbol{y = 9x - 23} = 9(12) - 23 = \mathbf{85}$ **lakhs Rs.**

iii) The advertising expenditure if the firm wants to attain sales target of 90 lakhs Rs.

$\boldsymbol{x = 0.09y + 3.4} = 0.09(90) + 3.4 = \mathbf{11.5}$ **lakhs Rs.**

Deviation taken from the assumed mean:

a) Regression equation of y on x :

The equation $y = a + bx$ is changed to $y - \bar{y} = b_{yx}(x - \bar{x})$

When actual mean is not a whole number, but a fraction or when values of x and y are large

We use the formula: $\boldsymbol{b_{yx}} = \dfrac{\sum \boldsymbol{uv} - \frac{\sum \boldsymbol{u} \sum \boldsymbol{v}}{\boldsymbol{n}}}{\sum \boldsymbol{u^2} - \frac{(\sum \boldsymbol{u})^2}{\boldsymbol{n}}}$

Where $u = x - A$, $v = y - B$,

$A =$ Assumed mean for x , $B =$ Assumed mean for y

b) Regression equation of x on y :

The regression equation $x = c + dy$ is reduces to $x - \bar{x} = b_{xy}\,(y - \bar{y})$

Where $\boldsymbol{b_{xy}} = \dfrac{\sum \boldsymbol{uv} - \frac{\sum \boldsymbol{u} \sum \boldsymbol{v}}{\boldsymbol{n}}}{\sum \boldsymbol{v^2} - \frac{(\sum \boldsymbol{v})^2}{\boldsymbol{n}}}$

Where $u = x - A$, $v = y - B$,

$A =$ Assumed mean for x , $B =$ Assumed mean for y

Also the values of correlation coefficient, means and Standard deviations of two variables are given, in such a case we can find b_{xy} and b_{yx} as following:
Let $\bar{x}$, $\bar{y}$ be the means and σ_x , σ_y be the standard deviation of the x and y respectively and r be the correlation coefficient, then $\boldsymbol{b_{\chi Y}} = \boldsymbol{r}\frac{\sigma_\chi}{\sigma_Y}$ and $\boldsymbol{b_{Y\chi}} = \boldsymbol{r}\frac{\sigma_Y}{\sigma_\chi}$

Example 9.3.1.3 :

From the following data, calculate two lines of regression

$\boldsymbol{x}$	17	21	18	22	16
$\boldsymbol{y}$	60	70	68	70	66

i) Estimate value of y when x is 35

ii) Estimate value of x when y is 75

Solution :

Let 21 be the assumed mean for x i.e. $A = 21$ & 90 be the assumed mean for y, i.e. $B = 90$

Calculation for regression equations

x	y	$u = x - A$	$v = y - B$	uv	u^2	v^2
17	60	-4	-10	40	16	100
21	70	0	0	0	0	0
18	68	-3	-2	6	9	4
22	70	1	0	0	1	0
16	66	-5	-4	20	25	16
$\sum x = 94$	$\sum y = 334$	$\sum u = -11$	$\sum v = -16$	$\sum uv = 66$	$\sum u^2 = 51$	$\sum v^2 = 120$

Regression equation of y on x

$$b_{yx} = \frac{\sum uv - \frac{\sum u \sum v}{n}}{\sum u^2 - \frac{(\sum u)^2}{n}}, \qquad \text{Where } u = x - A, \qquad v = y - B,$$

$$b_{yx} = \frac{66 - \frac{(-11)\times(-16)}{5}}{51 - \frac{(-11)^2}{5}} = 1.1492$$

$\bar{x} = 18.8$ & $\bar{y} = 66.8$

Regression equation of y on x is $y - \bar{y} = b_{yx}(x - \bar{x})$

$y - 66.8 = 1.1492(x - 18.8)$

$y - 66.8 = 1.1492x - 21.6049$

$\boldsymbol{y = 1.1492x + 45.1951}$

i) When x is 35, y is

$y = 1.1492(35) + 45.1951$

$\boldsymbol{y = 85.4171}$

Regression equation of x on y

$$\boldsymbol{b_{xy}} = \frac{\sum \boldsymbol{uv} - \frac{\sum \boldsymbol{u} \sum \boldsymbol{v}}{\boldsymbol{n}}}{\sum \boldsymbol{v}^2 - \frac{(\sum \boldsymbol{v})^2}{\boldsymbol{n}}}, \qquad \text{Where } u = x - A, \qquad v = y - B,$$

$$\boldsymbol{b_{xy}} = \frac{\mathbf{66} - \frac{(\mathbf{-11})\times(\mathbf{-16})}{\mathbf{5}}}{\mathbf{120} - \frac{(\mathbf{-16})^2}{\mathbf{5}}} = 0.4476$$

$\bar{x} = 18.8$ & $\bar{y} = 66.8$

Regression equation of x on y is $x - \bar{x} = b_{xy}\,(y - \bar{y})$

$x - 18.8 = 0.4476\,(y - 66.8)$

$x - 18.8 = 0.4476y - 29.8996$

$\boldsymbol{x = 0.4476y - 11.0996}$

ii) When y is 75, x is

$x = 0.4476(75) - 11.0996$

$\boldsymbol{x = 22.47}$

9.3.2 Method of Least Squares

The famous German mathematician Carl Friedrich Gauss had investigated the method of least squares as early as 1794, but unfortunately he did not publish the method until 1809. In the meantime, the method was discovered and published in 1806 by the French mathematician Legendre, who quarrelled with Gauss about who had discovered the method first. The basic idea of the method of least squares is easy to understand. It may seem unusual that when several people measure the same quantity, they usually do not obtain the same results. In fact, if the same person measures the same quantity several times, the results will vary. What then is the best estimate for the true measurement? The method of least squares gives a way to find the best estimate, assuming that the errors (i.e. the differences from the true value) are random and unbiased, The method of least squares is a standard approach in regression analysis to approximate the solution of over determined systems by minimizing the sum of the squares of the residuals made in the results of every single equation. The most important application is in data fitting.

Steps Involved:

Step (i) Calculate the actual means of x and actual mean of y.

Step (ii) Calculate the summation of the x series and y series.

It'll gives $\sum x$ and $\sum y$ respectively.

Step (iii) Square the values of the series x and y and calculate its summation. it'll gives $\sum x^2$ and $\sum y^2$ respectively.

Step (iv) Multiply each value of the series x by the respective values of the series y and calculation its summation, it'll gives $\sum xy$.

Step (v) Solve the following Normal equations to determine the values of 'a' and 'b' for regression equation y on x.

$$\sum y = na + b \sum x$$

$$\sum xy = a \sum x + b \sum x^2$$

after determine the value of 'a' and 'b' put these values in the following equation.

$$y = a + bx$$

Step (v') Solve the following Normal equations to determine the values of 'c' and 'd' for regression equation y on x.

$$\sum x = nc + d \sum y$$

$$\sum yx = c \sum y + d \sum y^2$$

after determine the value of 'c' and 'd' put these values in the following equation.

$$x = c + dy.$$

Example 9.3.2.1 :

From the following data fit a regression line where y is dependent variable and x is independent variable (i.e. y on x) using the method of least square, also estimate the value of y when $x = 7.8$

x	2	4	6	8	10
y	15	14	8	7	2

Solution : Let $y = a + bx$ be the required equation of line (because y on x).

To determine the values of 'a' and 'b' for regression equation y on x

The two normal equations are $\sum y = na + b\sum x$,

$$\sum xy = a\sum x + b\sum x^2$$

where $n = 5$ (number of datasets)

x	y	xy	x^2
2	15	30	4
4	14	56	16
6	8	48	36
8	7	56	64
10	2	20	100
$\sum x = 30$	$\sum y = 46$	$\sum xy = 210$	$\sum x^2 = 220$

Substituting these calculated values into normal equation, we'll get

$46 = 5a + 30b$

$210 = 30a + 220b$

Solving the above equations, we get

$a = 19.1$ and $b = -1.65$

Hence, the required line $\boldsymbol{y = a + bx}$ is $\boldsymbol{y = 19.1 + (-1.65)x}$

or $\boldsymbol{y = 19.1 - 1.65x}$

When $= 7.8$, y is equal to $19.1 - 1.65(7.8) = 6.23$

Hence estimate value of $\boldsymbol{y}$ when $\boldsymbol{x = 7.8}$ is 6.23

Example 9.3.2.2 :

From the following data fit a regression line where x is dependent variable and y is independent variable (i.e. x on y) using the method of least square, also estimate the value of x when $y = 30$

x	7	8	11	12	14	16
y	20	12	15	19	8	25

Solution : Let $x = c + dy$ be the required equation of line (because x on y).

To determine the values of 'c' and 'd' for regression equation x on y

The two normal equations are $\sum x = nc + d\sum y$

$$\sum yx = c\sum y + d\sum y^2$$

where $n = 6$ (number of datasets)

x	y	yx	y^2
7	20	140	400
8	12	96	144
11	15	165	225
12	19	228	361
14	8	112	64
16	25	400	625
$\sum x = 68$	$\sum y = 99$	$\sum yx = 1141$	$\sum y^2 = 1819$

Substituting this calculated values into normal equation, we'll get

$$68 = 6c + 99d$$

$$1141 = 99c + 1819d$$

Solving the above equations, we get

$c = 9.64$ and $d = 0.10$

Hence, the required line $x = c + dy$ is $x = 9.64 + 0.10y$

When $= 30$, x is equal to $9.64 + 0.10(30) = 12.64$

Hence estimate value of x when $y = 30$ is 12.64

9.3.3 Properties of Regression Coefficients

i) Correlation coefficient is the geometric mean between the regression coefficients.

i.e. The coefficient of regression are $r\frac{\sigma_y}{\sigma_x}$ and $r\frac{\sigma_x}{\sigma_y}$

$\therefore$Geometric Mean between them $\sqrt{r\frac{\sigma_y}{\sigma_x} \times r\frac{\sigma_x}{\sigma_y}} = \sqrt{r^2} = \pm r$

Both the regression coefficients will have the same sign, i.e., they will be wither positive or negative. It is not possible for one to be positive and other to be negative.

If regression coefficients are positive, then r is positive and if regression coefficients are negative, r is negative .if $b_{xy} = 0.3$ and $b_{yx} = 0.6$

$$\frac{b_{xy} + b_{yx}}{2} = \frac{0.3 + 0.6}{2} = 0.45$$

then the value of $r = \sqrt{0.3 \times 0.6} = 0.42$ which is less than 0.45

The value of the correlation coefficient cannot exceed one, if one of the regression coefficient is greater than unity, the other must be less than unity.

if $b_{xy} = 1.4$ and $b_{yx} = 1.5$ the $r = \sqrt{1.4 \times 1.5} = 1.45$ which (greater than 1) is not possible.

The point $(\bar{x}, \bar{y})$ satisfies both the regression equations as it lies on both the lines so that it is the point of intersection of the two lines. This can be helpful to us whenever the regression equations are known and the mean values of x and y are to be obtained, In this case, the two regression equations can be solved simultaneously and the common solution represent $\bar{x}$ and $\bar{y}$

Example 9.3.3.1 :

From the given data calculate equations of two lines of regression

	Mean	**Standard deviation**
$\boldsymbol{x}$	20	3
$\boldsymbol{y}$	100	12

Coefficient of correlation is 0.8 i.e. $r = 0.8$

Solution :

We have $\bar{x} = 20,\ \sigma_x = 3,\ \bar{y} = 100,\ \sigma_y = 12$ and $r = 0.8$

i) To find the regression equation of y on x

$$b_{yx} = r \times \frac{\sigma_y}{\sigma_x} = 0.8 \times \frac{12}{3} = 3.2$$

Regression equation of y on x is $(y - \bar{y}) = b_{yx}(x - \bar{x})$

$$(y - 100) = 3.2(x - 20)$$

$$(y - 100) = 3.2x - 64$$

$$\boldsymbol{y = 3.2x + 36}$$

ii) To find the regression equation of x on y

$$b_{xy} = r \times \frac{\sigma_x}{\sigma_y} = 0.8 \times \frac{3}{12} = 0.2$$

Regression equation of x on y is $(x - \bar{x}) = b_{xy}(y - \bar{y})$

$$(x - 20) = 0.2(y - 100)$$

$$(x - 20) = 0.2y - 20$$

$$\boldsymbol{x = 0.2y}$$

Example 9.3.3.2 :

Given the two regression lines as $x - 4y = 5$ and $x - 16y = -64$, find the average of x and y and correlation coefficient between x and y i.e. to find value of $\bar{x}, \bar{y}$ and r

Solution :

To find the mean of x and y

As per properties, $(\bar{x}, \bar{y})$ will lie on both the regression. thus,

$\bar{x} - 4\bar{y} - 5 = 0$ and $\bar{x} - 16\bar{y} = 64 = 0$

Solve the given two equations simultaneously.

$$\bar{x} - 4\bar{y} = 5 \qquad \text{---(1)}$$

$$\bar{x} - 16\bar{y} = -64 \qquad \text{---(2)}$$

$$- \quad + \quad +$$

$$12\bar{y} = 69$$

$\boldsymbol{\bar{y} = 5.75}$

substituting in equation (1)

$\bar{x} - 4(5.75) = 5$

$\boldsymbol{\bar{x} = 28}$

$\therefore \boldsymbol{\bar{x} = 28, \bar{y} = 5.75}$

To find correlation coefficient we have two lines of regressions. Which line is regression line y on x and which is x and which is x ob y in not known.

Let us assume that $x - 4y = 5$ be regression line of y on x and $x - 16y = -64$ be the regression line of x on y

Hence we get

$y = -\frac{5}{4} + \frac{1}{4}x$ and $x = 16y - 64$

$\therefore\ b_{yx} = \frac{1}{4}$ and $b_{xy} = 16$

We require to check whether the assumption is correct or not ?

Check 1: Signs: Both regression coefficients are positive.

Check 2: Product: Product of two regression coefficients.

$b_{yx} \times b_{xy} = \frac{1}{4} \times 16 = 4 > 1$ Which is greater than 1

$\therefore$ Assumption is wrong.

$\therefore$ Let $x - 4y = 5$ be regression line of x on y

and $x - 16y = -64$ be the regression line of y on x

$\therefore\ b_{xy} = 4$ and $b_{yx} = \frac{1}{16}$

Check 1: Signs: Both regression coefficients are positive.

Check 2: Product: Product of two regression coefficients.

$b_{yx} \times b_{xy} = \frac{1}{16} \times 4 = 0.25 < 1$ Which is less than 1

$\therefore$ This assumption is correct.

Now, $r = \pm\sqrt{b_{xy} \times b_{yx}} = \pm\sqrt{4 \times \frac{1}{16}} = \pm\frac{1}{2}$

The sing of correction coefficient is same as sign of both regression coefficients

$\therefore\ r = \frac{1}{2}$ or 0.5

9.4 Polynomial Regression

In statistics, polynomial regression is a form of regression analysis in which the relationship between the independent variable x and the dependent variable y is modeled as an n^{th} degree polynomial in x.

Polynomial regression fits a nonlinear relationship between the value of x and the corresponding conditional mean of y, and has been used to describe non linear phenomena such as growth rate of tissues, the distribution of carbon isotopes in lake sediments etc.

9.5 Non- liner Regression

Regression will be called non-linear if there exists a relationship other than a straight line between the variables under consideration.

Nonlinear regression is a form of regression analysis in which observational data are modeled by a function which is a nonlinear combination of the model parameters and depends on one or more independent variables.

9.5.1 Polynomial fit :

Let $y = a + bx + cx^2$ be second degree parabolic curves of regression of y on x to be fitted for the data (x_i, y_i), where $i = 1,2,3 \dots n$

To fit parabolic curves for x on y,

Consider $\hat{y} = a + bx + cx^2$

For each $x_i, \hat{y} = a + bx_i + cx_i^2$

$\therefore$ Error inestimation is $y_i - \hat{y_i}$ and Summation of squares of error is

$$\varphi = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{y_i})^2$$

$\therefore \varphi = \sum_{i=1}^{n}(y_i - a - bx_i - cx_i^2)^2 \qquad \dots \mathbf{(1)}$

To find values of a, b, c such that φ is minimum

$\therefore \frac{\partial\varphi}{\partial a} = 0,\ \frac{\partial\varphi}{\partial b} = 0$ and $\frac{\partial\varphi}{\partial c} = 0$

Differentiating φ w.r.t. a, we get

$\frac{\partial\varphi}{\partial a} = 2\ \sum(y_i - a - bx_i - cx_i^2)(-1)$

$\therefore \sum(y_i - a - bx_i - cx_i^2) = 0$

$\sum y_i - na - b\sum x_i - c\sum x_i^2 = 0$

$\therefore \boldsymbol{\sum y_i = na + b\sum x_i + c\sum x_i^2} \qquad$ **...(2)**

Again differentiating φ w.r.t. b, we get

$$\frac{\partial \varphi}{\partial b} = 2\sum(y_i - a - bx_i - cx_i^2)(-x_i)$$

$$\therefore \sum(y_i - a - bx_i - cx_i^2)(x_i) = 0$$

$$\sum x_i y_i - a\sum x_i - b\sum x_i^2 - c\sum x_i^3 = 0$$

$$\therefore \boldsymbol{\sum x_i y_i = a\sum x_i + b\sum x_i^2 + c\sum x_i^3} \qquad \textbf{...(3)}$$

Again differentiating φ w.r.t. c, we get

$$\frac{\partial \varphi}{\partial c} = 2\sum(y_i - a - bx_i - cx_i^2)(-x_i^2)$$

$$\therefore \sum(y_i - a - bx_i - cx_i^2)(x_i^2) = 0$$

$$\sum x_i^2 y_i - a\sum x_i^2 - b\sum x_i^3 - c\sum x_i^4 = 0$$

$$\therefore \boldsymbol{\sum x_i^2 y_i = a\sum x_i^2 + b\sum x_i^3 + c\sum x_i^4} \qquad \textbf{...(4)}$$

Equation (2) (3) and (4) are normal equations for fitting a second degree parabolic curve.

i.e. $\boldsymbol{\hat{y} = a + bx + cx^2}$

Example 9.5.1.1 :

Using method of least square fit a second degree parabola for the following data.

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

Solution :

The second-degree polynomial equation is $\boldsymbol{y = a + bx + cx^2}$ and Normal Equations are

$$\boldsymbol{\sum y = na + b\sum x + c\sum x^2}$$

$$\boldsymbol{\sum xy = a\sum x + b\sum x^2 + c\sum x^3}$$

$$\boldsymbol{\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4}$$

Consider the following table for calculation

x	y	x^2	xy	x^3	x^2y	x^4
1	2	1	2	1	2	1
2	6	4	12	8	24	16
3	7	9	21	27	63	81
4	8	16	32	64	128	256
5	10	25	50	125	250	625
6	11	36	66	216	396	1296
7	11	49	77	343	539	2401
8	10	64	80	512	640	4096
9	9	81	81	729	729	6561
45	**74**	**285**	**421**	**2025**	**2771**	**15333**

Here $\boldsymbol{n = 9}$,

$\boldsymbol{\sum x = 45, \sum y = 74, \sum x^2 = 285, \sum xy = 421, \sum x^3 = 2025, \sum x^2 y = 2771, \sum x^4 = 15333}$

Substituting these values into Normal Equations we get

$74 = 9a + 45b + 285c$ ...(1)

$421 = 45a + 285b + 2025c$ ...(2)

$2771 = 285a + 2025b + 15333c$ ...(3)

Solving equation (1), (2) and (3) (by simultaneously or use Cramer's rule)

$a = -0.9285,\ b = 3.5231,\ c = -0.2673$

Hence, The second degree equation is

$\boldsymbol{y = -0.9285 + 3.5231x - 0.2673x^2}$

9.6 Multiple Linear Regression

Multiple linear regression is a technique that uses two or more independent variables to predict the outcome of a dependent variable.

The technique enables analysts to determine the variation of the model and the relative contribution of each independent variable in the total variance.

Multiple regression can take two forms, i.e., linear regression and non-linear regression.

Consider such a linear function as $y = a + bx + cz$.

To fit multiple linear regression for x, y and z consider

$\hat{y} = a + bx + cz$

for each x_i, $\hat{y_i} = a + bx_i + cz_i$

$\therefore$ Error in estimation is $y_i - \hat{y_i}$ and summation of squares of error is

$$\varphi = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

$$\therefore \varphi = \sum_{i=1}^{n} (y_i - a - bx_i - cz_i)^2 \quad \ldots (\mathbf{1})$$

To find values of a, b, c such that φ is minimum

i.e. $\frac{\partial \varphi}{\partial a} = \frac{\partial \varphi}{\partial b} = \frac{\partial \varphi}{\partial c} = 0$

Differentiating equation (1) w.r.t. a we get

$\frac{\partial \varphi}{\partial a} = 2 \sum (y_i - a - bx_i - cz_i)(-1)$

$\therefore \sum y_i - na - b \sum x_i - c \sum z_i = 0$

$$\sum y_i = na + b \sum x_i + c \sum z_i \qquad \ldots (\mathbf{2})$$

Again differentiating equation (1) w.r.t b we get

$\frac{\partial \varphi}{\partial b} = 2 \sum (y_i - a - bx_i - cz_i)(-x_i)$

$\therefore \sum x_i y_i - a \sum x_i - b \sum x_i^2 - c \sum x_i z_i = 0$

$$\sum x_i y_i = a \sum x_i + b \sum x_i^2 + c \sum x_i z_i \qquad \ldots (\mathbf{3})$$

Again differentiating equation (1) with respect to c, we get

$\frac{\partial \varphi}{\partial c} = 2 \sum (y_i - a - bx_i - cz_i)(-z_i)$

$\therefore \sum z_i y_i - a\sum z_i - b\sum x_i z_i - c\sum z_i^2 = 0$

$$\sum z_i y_i = a\sum z_i + b\sum x_i z_i + c\sum z_i^2 \qquad \ldots(\mathbf{4})$$

Here Equation (2), (3), and (4) are the normal equations for fitting a multiple linear regression equation

$$\hat{y} = a + bx + cz$$

Example 9.6.1 :

Obtain a regression plane by using multiple regression to fit the following data

$\boldsymbol{x}$	0	1	2	3	4
$\boldsymbol{y}$	13	17	19	21	26
$\boldsymbol{z}$	1	2	3	4	5

Solution :

The multiple regression equation is $y = a + bx + cz$

and normal equations are

$\sum y = na + b\sum x + c\sum z$

$\sum xy = a\sum x + b\sum x^2 + c\sum xz$

$\sum yz = a\sum z + b\sum xz + c\sum z^2$

Consider the following table for calculation

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$	$\boldsymbol{xy}$	$\boldsymbol{x^2}$	$\boldsymbol{xz}$	$\boldsymbol{z^2}$	$\boldsymbol{yz}$
0	13	1	0	0	0	1	13
1	17	2	17	1	2	4	34
2	19	3	38	4	6	9	57
3	21	4	63	9	12	16	84
4	26	5	104	16	20	25	130
10	**96**	**15**	**222**	**30**	**40**	**55**	**318**

Here $n = 5$,

$\boldsymbol{\sum x = 10, \sum y = 96, \sum z = 15, \sum xy = 222, \sum x^2 = 30, \sum xz = 40, \sum z^2 = 55, \sum yz = 318}$

Substituting these values into Normal Equations we get,

$96 = 5a + 10b + 15c$ ...(1)

$222 = 10a + 30b + 40c$ ...(2)

$318 = 15a + 40b + 15c$ ...(3)

Solving equation (1), (2) and (3) (by simultaneously or use Cramer's rule)

$$a = 13.8, \qquad b = 3, \qquad c = 0$$

Hence, The required regression plane equation is

$\boldsymbol{y = 13.2 + 3x}$

9.7 General Linear Least Squares

The linear least square is the least squares approximation of linear function to data. It is a set of formulations for solving statistical problems involved in linear regression including variants for ordinary and generalized least squares.

9.7.1 Ordinary Least Square(OLS)

Ordinary least squares (OLS) is a type of linear least squares method for estimating the unknown parameters in a linear regression model. OLS chooses the parameters of a linear function of a set of explanatory variables by the principle of least squares: minimizing the sum of the squares of the differences between the observed dependent variable (values of the variable being observed) in the given dataset and those predicted by the linear function of the independent variable.

9.7.2 Generalized Least Square (GLS)

Generalized least squares (GLS) is a technique for estimating the unknown parameters in a linear regression model when there is a certain degree of correlation between the residuals in a regression model.

9.8 Summary

- A correlation or simple linear regression analysis can determine if two numeric variables are significantly linearly related. A correlation analysis provides information on the strength and direction of the linear relationship between two variables, while a simple linear regression analysis estimates parameters in a linear equation that can be used to predict values of one variable based on the other.

- Both correlation and regression can be used to examine the presence of a linear relationship between two variables providing certain assumptions about the data are satisfied. The results of the analysis, however, need to be interpreted with care, particularly when looking for a causal relationship or when using the regression equation for prediction. Multiple and logistic regression will be the subject of future reviews.

9.9 References

Following books are recommended for further reading :-

- Introductory Methods of Numerical Methods - *S. S. Shastri* ; PHI.
- Numerical Methods for Engineers - *Steven C. Chapra, Raymond P. Canale*; Tata Mc Graw Hill.

9.10 Exercises

Q.1) Calculate the product moment coefficient of correlation for the given data:

x	13	10	9	11	12	14	8
y	15	9	7	10	12	13	4

(Ans. $r = 0.948$)

Q.2) Calculate coefficient of correlation between the values of x and y given below.

x	66	70	70	68	74	70	72	75
y	68	74	66	69	72	63	69	65

(Ans. $r = -0.03$)

Q.3) Find the coefficient of correlation for the following data:

x	80	84	90	75	72	70	78	82	86
y	110	115	118	105	104	100	108	112	116

(Ans. $r = 0.987$)

Q.4) Find the coefficient of correlation for the following data by using assumed mean method

x	212	214	205	220	225	214	218
y	500	515	511	530	522	516	525

(Ans. $r \approx 0.668$)

Q.5) Fit the straight line by using least square method for the following data:

x	1	2	3	4	5	6	7
y	0.5	2.5	2.0	4.0	3.5	6.0	5.2

Q.6) Find the best fit values of a and b so that $y = a + bx$ fits the data given in the table:

x	0	1	2	4
y	1	1.8	3.3	4.5

Q.7) Consider the data below:

x	1	2	3	4
y	1	7	11	21

Use linear least-Square regression to determine function of the from $y = be^{mx}$ the given data by specifying b and m

(Hint: Take natural log on both side of the function)

Q.8) Fit the second order polynomial to the given below:

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

Q.9) Fit the second order polynomial to the given below:

x	0	1	2	3	4
y	-4	-1	4	11	20

Q.10) Use the multiple regression to fit the following data

x	0	2	2.5	1	4	7
y	0	1	2	3	6	2
z	5	10	9	0	3	27

❖❖❖

Unit 4

10

LINEAR PROGRAMMING

Unit Structure

10.0 Objectives

This chapter would make you understand the following concepts:

- Sketching the graph for linear equations.
- Formulate the LPP
- Conceptualize the feasible region.
- Solve the LPP with two variables using graphical method.

10.1 Introduction

Linear programming (LP, also called linear optimization) is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships. Linear programming is a special case of mathematical programming (also known as mathematical optimization).

10.2 Common terminology for LPP

Linear programming is a mathematical concept used to determine the solution to a linear problem. Typically, the goal of linear programming is to maximize or minimize specified objectives, such as profit or cost. This process is known as optimization. It relies upon three different concepts: variables, objectives, and constraints.

"Linear programming is the analysis of problems in which a linear function of a number of variables is to be maximized (or minimized) when those variables are subject to a number of restrains in the form of linear inequalities."

Objective Function : The linear function, which is to be optimized is called objective function The objective function in linear programming problems is the real-valued function whose value is to be either minimized or maximized subject to the constraints defined on the given LPP over the set of feasible solutions. The objective function of a LPP is a linear function of the form $Z = a_1x_1 + a_2x_2 + a_3x_3 \dots . a_nx_n$

Decision Variables : The variables involved in LPP are called decision variables denoted them as (x, y)or (x_1, x_2) etc. here its refer to some quantity like units, item production on sold quantity, Time etc.

Constraints : The constraints are limitations or restrictions on decision variables. they are expressed in linear equalities or inequalities i.e. $=, \leq, \geq$

Non-negative Constraints : This is a condition in LPP specifies that the value of variable being considered in the linear programming problem will never be negative. It will be either zero or greater than zero but can never be less than zero, Thus it is expressed in the form of $x \geq 0$ and $y \geq 0$.

Feasible Solution : A feasible solution is a set of values for the decision variables that satisfies all of the constraints in an optimization problem. The set of all feasible solutions defines the feasible region of the problem. in graph the overlapping region is called feasible region

Optimum solution : An optimal solution to a linear program is the solution which satisfies all constraints with maximum or minimum objective function value.

10.3 Mathematical Formulation of L.P.P

To write mathematical formulation of L.P.P following steps to be remembered.

Step 1 : Identify the variables involved in LPP (i.e. Decision variables) and denote them as (x, y)or (x_1, x_2).

Step 2 : Identify the objective function and write it as a mathematically in terms of decision variables.

Step 3 : Identify the different constraints or restrictions and express them mathematically

Example 10.3.1 :

A bakery produces two type of cakes I and II using raw materials R_1 and R_2. One cake of type I is produced by using 4 units of raw material R_1 and 6 units of raw material R_2 and one cake of type II is produced by using 5 units of raw material R_1 and 9 units of raw material R_2. There are 320 units of R_1 and 540 units R_2 in the stock. The profit per cake of type I and type II is Rs. 200 and Rs. 250 respectively. How many cakes of type I and type II be produced so as to maximize the profit? formulate the L.P.P

Solution:

Let x be the number of cakes of type I and y be the number of cakes of type II to be produce to get maximum profit.

since the production value is never negative

$\therefore \boldsymbol{x \geq 0, y \geq 0}$

This is non- negative constraints.

∴ The profit earned by selling 1 cake of type I is Rs. 200. Hence the profit earned by selling x cakes is Rs. $200x$.
Similarly, the profit earned by selling 1 cake of type II is Rs. 250 and hence profit earned by selling y cake is Rs. $250y$.

∴ The Profit earned is

$\boldsymbol{Z = 200x + 250y}$

This is objective function.

Now, after reading the given data carefully we can construct the following table

Cake Type	**Raw Material required**		**Profit**
	R_1 (units)/Cake	**R_2 (Units)/Cake**	
I	4	6	**200**
II	5	9	**250**
Availability (units)	**320**	**540**	-

∴ According to the table

∴ 1 Cake of type I consumes 4 units of R_1 hence x cakes of type I will consume $4x$ units of R_1 and one cake of type II consume 5 units of R_1 hence y cakes of type II will consume $5y$ units of R_1. But maximum number of units available of R_1 is 320. Hence, the constraint is

$4x + 5y \leq 320.$

Similarly, 1 Cake of type I consumes 6 units of R_2 hence x cakes of type I will consume $6x$ units of R_2 and one cake of type II consume 9 units of R_2 hence y cakes of type II will consume $9y$ units of R_2. But maximum number of units available of R_2 is 540. Hence, the constraint is

$6x + 9y \leq 540$

Hence the mathematical formulation of the given L.P.P is to

Maximize $Z = 200x + 250y$

Subject to,

$$\mathbf{4x + 5y \leq 320}$$

$$\mathbf{6x + 9y \leq 540}$$

$$\mathbf{x \geq 0, y \geq 0}$$

Example 10.3.2 :

A manufacture produce Ball pen and Ink pen each of which must be processed through two machines A and B. Machine A has maximum 220 hours available and machine B has maximum of 280 hours available . Manufacturing a Ink pen requires 6 hours on machine A and 3 hours on machine B. Manufacturing a Ball pen requires 4 hours on machine A and 10 hours on machine B. If the profit are Rs. 55 for Ink pen and Rs. 75 for Ball pen. Formulate the LPP to have maximum profit.

Solution :

Let Rs. Z be the profit, Which can be made by manufacturing and selling say 'x' number of Ink pens and 'y' number of Ball pens.

Here variable x and y are decision variables.

Since profit per Ink pen and ball pen is Rs. 55 and Rs. 75 respectively and we want to maximize the Z, Hence the objective function is

Max $Z = 55x + 75y$

We have to find x and y that maximize Z

We can construct the following tabulation form of given data:

Machine	**Time in hours required for**		**Maximum available time in hours**
	1 Ink pen	**1 Ball pen**	
A	6	4	220
B	3	10	280

A Ink pen requires 6 hr on machine A and Ball pen requires 4 hr on machine A and maximum available time of machine A is 220 hr.

1st Constraint is

$6x + 4y \leq 220$

Similarly, A Ink pen requires 3 hr on machine B and Ball pen requires 10 hr on machine B and maximum available time of machine B is 280 hr.

2nd Constraint is

$3x + 10y \leq 280$

Here the production of Ball pen and Ink pen can not be negative:

we have Non-Negative constraints as $x \geq 0, y \geq 0$

Hence the required formulation of LPP is as follows:

Max $Z = 55x + 75y$

Subject to,

$\boldsymbol{6x + 4y \leq 220}$

$\boldsymbol{3x + 10y \leq 280}$

$\boldsymbol{x \geq 0, y \geq 0}$

Example 10.3.3 :

In a workshop 2 models of agriculture tools are manufactured A_1 and A_2. Each A_1 requires 6 hours for I processing and 3 hours for II processing. Model A_2 requires 2 hours for I processing and 4 hours for II processing. The workshop has 2 first processing machines and 4 second processing machine each machine of I processing units works for 50 hrs a week. Each machine in II processing units works for 40 hrs a week. The workshop gets Rs. 10/- profit on A_1 and Rs. 14/- on A_2 on sale of each tool. Determine the maximum profit that the work shop get by allocating production capacity on production of two types of tool A and B

Solution:

Decision Variables :

1. Let the number of units of type A_1 model tools be x.

2. Let the number of units of type A_2 model tools be y.

Objective function :

The objective of the workshop is to obtain maximum profit by allocating his production capacity between $A_1$and A_2 and 10 Rs. per unit profit on model A_1 and 14 Rs. on model A_2

$\therefore Z = 10x + 14y$

Constraints :

1. For processing of A_1 tool and A_2 tools require $6 + 2 = 8$ hrs in I Processing unit $= 6x + 2y$

2. For processing of A_1type of tools and A_2 type requires $3 + 4 = 7$ hrs in II processing unit $= 3x + 4y$

Total machine hours available in I processing unit = $2 \times 50 = 100$ hrs per week.

Total machine hours available in II processing unit = $4 \times 40 = 160$ hrs per week.

Considering the time constraint, the constrain function can be written in the following way:

$6x + 2y \leq 100$

$3x + 4y \leq 160$

Non-negative constraint :

There is no possibility of negative production in the workshop

$\therefore$ The non-negative function will be

$x \geq 0, y \geq 0$

Mathematical form of the production of 2 types of tools in the work shop to maximize profits under given constraints will be in the following way.

Maximize $Z = 10x + 14y$

Subject to,

$6x + 2y \leq 100$

$3x + 4y \leq 160, x \geq 0, y \geq 0$

Example 10.3.4 :

Diet for a sick person must contain at least 400 units of vitamins, 500 units of minerals and 300 calories. Two foods F_1 and F_2 cost Rs. 2 and Rs. 4 per unit respectively. Each unit of food F_1 contains 10 units of vitamins, 20 unit of minerals and 15 calories, whereas each unit of food F_2 contains 25 units of vitamins, 10 units of minerals and 20 calories. Formulate the L.P.P. to satisfy sick person's requirement at minimum cost.

Solution :

After reading this carefully, Tabulation form of give data:

Micronutrients	F_1	F_2	**Minimum units requirement**
vitamins	10	25	400
minerals	20	10	500
calories	15	20	300
Cost	2	4	

Decision Variables :

Let x for F_1 and y for F_2

Objective function :

We have to find minimum the cost for a diet hence the objective function in terms of decision variables is

Minimize $Z = 2x + 4y$

Constraints :

First constraints : $10x + 25y \geq 400$

Second Constraints : $20x + 10y \geq 500$

Third constraints : $15x + 20y \geq 300$

Non-negative constraint

There is no possibility of negative food quantity of the diet

hence, $x \geq 0, y \geq 0$

Mathematical formulation of LPP can be written as

Minimize $Z = 2x + 4y$

Subject to,

$$\mathbf{10x + 25y \geq 400}$$

$$\mathbf{20x + 10y \geq 500}$$

$$\mathbf{15x + 20y \geq 300, \quad x \geq 0, y \geq 0}$$

Example 10.3.5 :

A garden shop wishes to prepare a supply of special fertilizer at a minimal cost by mixing two fertilizer, A and B. The mixture is contains : at least 45 units of phosphate, at least 36 units of nitrate at least 40 units of ammonium. Fertilizer A cost the shop Rs 0.97 per Kg. fertilizer B cost the shop Rs.1.89 per Kg. Fertilizer A contains 5 units of phosphate and 2 units of nitrate and 2 units of ammonium, fertilizer B contains 3 units of phosphate and 3 units of nitrate and 5 units of ammonium. How many pounds of each fertilizer should the shop use in order to minimum their cost ?

Solution:

After reading this carefully, Tabulation form of give data:

Contains	Fertilizer type		Minimum units requirement
	A	B	
Phosphate	5	3	45
Nitrate	2	3	36
Ammonium	2	5	40
Cost	0.97	1.89	–

Decision Variables :

Let x for A and y for B

Objective function :

We have to find minimum the cost, Hence the objective function in terms of decision variables is

Minimize $Z = 0.97x + 1.89y$

Constraints :

First constraints : $5x + 3y \geq 45$

Second Constraints : $2x + 3y \geq 36$

Third constraints : $2x + 5y \geq 40$

Non-negative constraint

There is no possibility of negative supply of fertilizer

hence, $x \geq 0, y \geq 0$

Mathematical formulation of LPP can be written as

Minimize $\boldsymbol{Z = 0.97x + 1.89y}$

Subject to,

$\boldsymbol{5x + 3y \geq 45}$

$\boldsymbol{2x + 3y \geq 36}$

$\boldsymbol{2x + 5y \geq 40,}$

$\boldsymbol{x \geq 0, y \geq 0}$

Example 10.3.6 :

A printing company prints two types of magazines A and B the company earns Rs. 25 and Rs. 35 on each copy of magazines A and B respectively. The magazines are processed on three machines. Magazine A requires 2 hours on machine I, 4 hours on machine II and 2 hours on machine III. Magazine B requires 3 hours on machine I, 5 hours on machine II and 3 hours on machine III. Machines I, II and III are available for 35, 50 and 70 hours per week respectively formulate the L.P.P. so as to maximize the total profit of the company.

Solution :

Tabulation form of give data:

Machine	**Magazine**		**Maximum availability**
	A	**B**	
I	2	3	35
II	4	5	50
III	2	3	70

Decision Variables :

Let x number of copies of magazine A and y number of copies of magazine B to be printed to get maximum profit.

Objective function :

We have to Maximize the Profit, Hence the objective function in terms of decision variables is

$$\text{Maximize } Z = 25x + 35y$$

Constraints :

First constraints : $2x + 3y \leq 35$

Second Constraints : $4x + 5y \leq 50$

Third constraints : $2x + 3y \leq 70$

Non-negative constraint

There is no possibility of negative production of magazines

$\therefore$ The non-negative function will be

$x \geq 0, y \geq 0$

Mathematical formulation of LPP can be written as

Maximize $\boldsymbol{Z = 25x + 35y}$

Subject to,

$\boldsymbol{2x + 3y \leq 35}$

$\boldsymbol{4x + 5y \leq 50}$

$\boldsymbol{2x + 3y \leq 70}$

$\boldsymbol{x \geq 0, y \geq 0}$

Example 10.3.7 :

A food processing and distributing units has 3 production units A,B,C in three different parts of a city. They have five retails out lest in the city P, Q, R, S and T to which the food products are transported regularly. Total stock available at the production units is 500 units which is in the following ways : A=200 units; B=120 units and C= 180 units. Requirement at the retails outlets of the industry are: A=125, B=150, C=100, D=50, E=75. Cost of transportation of products from different production centers to different retail outlets is in the following way:

	P	Q	R	S	T
A	2	12	8	5	6
B	6	10	10	2	5
C	12	18	20	8	9

How the industry can minimize the cost on transportation of products. Formulate the linear programming problem.

Solution :

The objective of the industry is to minimize the possible cost on transportation

Let Z be the objective function.

Let x_1, x_2, x_3, x_4 and x_5 are decision variables

Tabulation form of the given data

Production centers	**Decision variables**	**Retails out lets**					**No. of units can be supplied**
		x_1	x_2	x_3	x_4	x_5	
A	x_1	2	12	8	5	6	200
B	x_2	6	10	10	2	5	120
C	x_3	12	18	20	8	9	180
Units of demand		**125**	**150**	**100**	**50**	**75**	**500**

Objective function :

The objective is to minimize the cost

$$\text{Minimize cost} = 2x_{11} + 12x_{12} + 8x_{13} + 5x_{14} + 6x_{15} + 6x_{21} + 10x_{22} + 10x_{23} + 2x_{24} + 5x_{25} + 12x_{31} + 18x_{32} + 20x_{33} + 8x_{34} + 9x_{35}$$

Constraint Function :

Supply constraint,

In the problem requirement at the retail out lets and the supply at the production units is the same. therefore supply constraint will be

$$A = x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 200$$

$$B = x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 120$$

$$C = x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 180$$

Demand Constraints :

P = $x_{11} + x_{12} + x_{13} = 125$

Q = $x_{21} + x_{22} + x_{23} = 150$

R = $x_{31} + x_{32} + x_{33} = 100$

S = $x_{41} + x_{42} + x_{43} = 50$

T = $x_{51} + x_{52} + x_{53} = 75$

Non- Negative function:

$x_{11}, x_{12}, x_{13}, x_{14}, x_{15} \geq 0$

10.4 Graphical Method

Graphical method, the inequality constraints are taken as equalities. Each equality constraints is drawn on the graph paper which forms a straight line. Lines are drawn equal to the constrains. Then the region which satisfies all inequality is located. this region is known as feasible region. Solution determine with regard to this region is called the feasible solution. Accordingly to Hadley "if an optimum (maximum or minimum) value of a linear programming problem exits then it must correspond to one of the corner points of the feasible region" Feasible region corresponding to a linear programming problem can be located by constructing graph as given below.

Steps for solving L.P.P. graphically

1. Formulate the mathematical linear programming problem. there will be 2 variables x and y.

2. Since both x and y are non negative, graphic solution will be restricted to the first quadrant.

3. Choose an appropriate scale for x and y axis.

4. Each inequality in the constraints equation can be written as equality. Example. $x + y \leq 70$ them make it $x + y = 70$

5. Given any arbitrary value to one variable and get the value of other variable by solving the equation. Similarly given another arbitrary value to the variable and find the corresponding value of the other variable. Example $x + y = 7$ at any point on x axis y will be 0 the $x + y = 7$ and $x = 7$ then at any point on y axis x will be zero $y = 5$, thus $(0,5)$ is a point on x axis and $(0,5)$ is a point on y axis that satisfies the equation.

6. Now plot these two sets of values connect these points by straight line. That divides the first quadrant into two parts. Since the constraints is an inequality, one of the two sides satisfies inequality

7. Repeat these steps for every constraints stated in the linear programming problem.

8. There forms a common area called feasible area.

9. For greater than or greater than equal to constraints, the feasible region will be the area which lies above the constrains.

10. For less than or less than equal, the area is below these lines.

Example 10.4.1 :

Solve the following linear programming by graphical method.

Maximize $Z = 5x + 6y$

Subject to, $2x + 4y \leq 16$

$3x + y \leq 12$

$3x + 3y \leq 24, \ x \geq 0, y \geq 0$

Solution :

By converting the inequality equation to equality equation we get:

$2x + 4y = 16$ ...(equation 1)

$3x + y = 12$ ...(equation 2)

$3x + 3y = 24$ ...(equation 3)

Equation (1)

$2x + 4y = 16$, When $x = 0$, then $y = \frac{16}{4} = 4$, When $y = 0$ then $x = \frac{16}{2} = 8$,

x	0	8
y	4	0

Equation (2)

$3x + y = 12$,When $x = 0$ then $y = 12$, When $y = 0$ then $x = \frac{12}{3} = 4$,

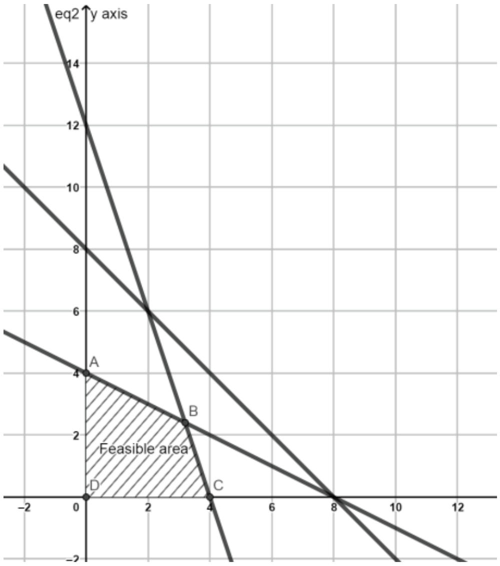
x	0	4
y	12	0

Equation (3)

$3x + 3y = 24$,When $x = 0$ then $y = \frac{24}{3} = 8$, When $y = 0$ then $x = \frac{24}{3} = 8$,

x	0	8
y	8	0

By above equation coordinates we can draw the straight lines to obtain feasible area



Thus the feasible region is OABC

Where $A(0,4)$, $B(3.2,2.4)$, and $C(4,0)$

Consider, $\quad Z = 5x + 6y$

at $A(0,4)$, $\quad Z = 5(0) + 6(4) = 24$

at $B(3.2,2.4)$, $Z = 5(3.2) + 6(2.4) = 30.4$

at $C(4,0)$, $\quad Z = 5(4) + 6(0) = 20$

Thus Z is maximum at point B, Thus the solution is $x = 3.2$ and $y = 2.4$

Example 10.4.2 :

Solve LPP graphically, Maximum $Z = 10x + 5y$

Subject to, $\quad x + y \leq 5$,

$2x + y \leq 6$,

$x \geq 0, y \geq 0$

Solution : Converting the given constraints in to equations (or ignore the inequality to find the co-ordinates for sketching the graph) we get,

$x + y = 5$ --- Equation (1)

$2x + y = 6$ --- Equation (2)

Consider the equation (1)

$x + y = 5$, When $x = 0$ then $y = 5$ and $y = 0$ then $x = 5$

x	0	5
y	5	0

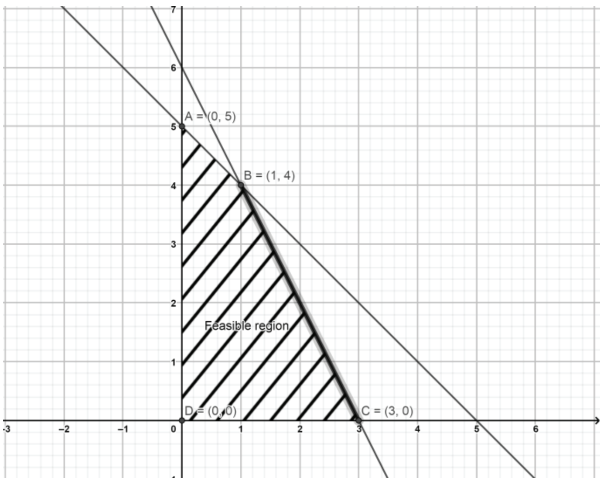
Plot (0,5) and (5,0) on the graph. Join the points by a straight line and shade the region represented by, $x + y \leq 5$

Then, Consider the equation (2)

$2x + y = 6$, When $x = 0$ then $y = 6$ and $y = 0$ then $x = 3$

x	0	3
y	6	0

Plot (0,6) and (3,0) on the graph. Join the points by a straight line and shade the region represented by, $2x + y \leq 6$



From the graph we get the feasible region ABCD is a feasible region where $D(0,0), A(0,5), B(1,4), C(3,0)$.

Consider the value of Z at different values of corner points of feasible region.

Corner point	$Z = 10x + 5y$
D	0
A	25
B	30
C	30

$\therefore$ Here Z attained maximum value at two points which is B and C

in this case any point which lies on set BC is also another optimal solution

$\therefore$ There are infinitely many optimal solution.

Example 10.4.3 :

Solve the following LPP graphically

Min $Z = 2x + 3y$

Subject to, $x + y \leq 3$

$2x + 2y \geq 10$

$x \geq 0, y \geq 0$

Solution :

Converting the given constraints in to equations (or ignore the inequality to find the co-ordinates for sketching the graph) we get,

$x + y = 3$ --- Equation (1)

$2x + 2y = 10$ --- Equation (2)

Consider the equation (1) $x + y = 3$, When $x = 0$ then $y = 3$ and $y = 0$ then $x = 3$

x	0	3
y	3	0

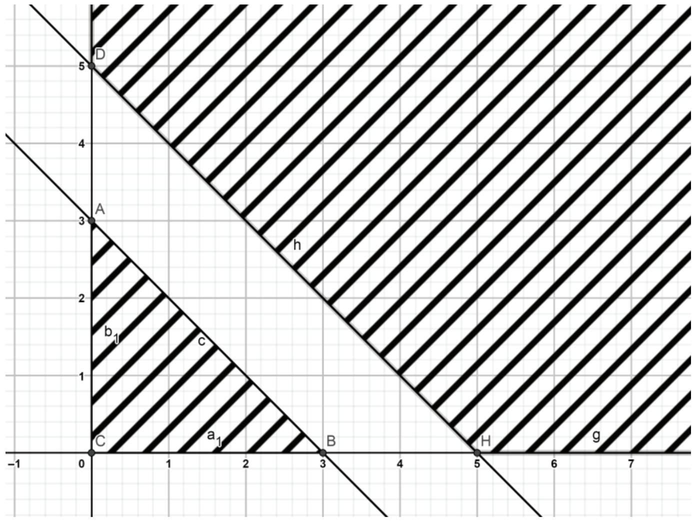
Plot (0,3) and (3,0) on the graph. Join the points by a straight line and shade the region represented by, $x + y \leq 3$

Then, Consider the equation (2)

$2x + 2y = 10$, When $x = 0$ then $y = 5$ and $y = 0$ then $x = 5$

x	0	5
y	5	0

Plot (0,5) and (5,0) on the graph. Join the points by a straight line and shade the region represented by, $2x + 2y \geq 10$



∴ This LPP does not have common feasible region and hence no optimum solution.

Example 10.4.4 :

Consider a calculator company which produce a scientific calculator and graphing calculator. long-term projection indicate an expected demand of at least 1000 scientific and 800 graphing calculators each month. Because of limitation on production capacity, no more than 2000 scientific and 1700 graphing calculators can be made monthly. To satisfy a supplying contract a total of at least 2000 calculator must be supplied each month. if each scientific calculator sold result in Rs. 120 profit and each graphing calculator sold produce Rs. 150 profit, how many of each type of calculator should be made monthly to maximize the net profit ?

Solution :

Decision variables

Let x is the number of scientific calculators produced and y is the number of graphing calculators produced.

Objective function

We have to find maximum profit. Hence objective function in a terms of decision variable is,

Maximize $Z = 120x + 150y$

Constraints

at least 1000 scientific calculators : $x \geq 1000$

at least 800 graphing calculators : $y \geq 800$

no more than 2000 scientific calculators : $x \leq 2000$

no more than 1700 graphing calculators : $y \leq 1700$

a total of at least 2000 calculators : $x + y \geq 2000$

and non-negative constraints : $x \geq 0, y \geq 0$

Hence the formulation of LPP can be written as

Maximize $Z = 120x + 150y$

Subject to, $1000 \leq x \leq 2000$

$800 \leq y \leq 1700$

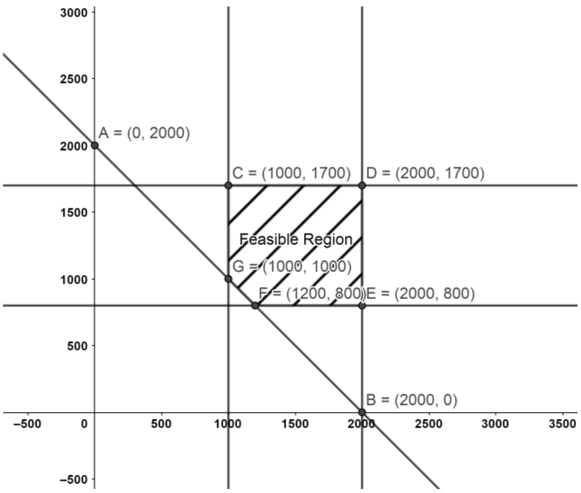
$x + y \geq 2000$

$x \geq 0, y \geq 0$

Consider, $x + y \geq 2000$

if $x = 0$ then $y = 2000$ and when $y = 0$ then $x = 2000$

∴ The point are $A(0{,}2000)$ and $B(2000{,}0)$



From the graph CDEFG is feasible region

To find the value of point G, Consider $x = 1000$ and $x + y = 2000$,

$\Rightarrow y = 1000$, Point G is (1000,1000)

To find the value of point F, Consider $y = 800$ in $x + y = 2000$,

$\Rightarrow y = 1200$, Point G is (1200,800)

Here CDEFG is feasible region.

Where $C(1000,1700), D(2000,1700), E(2000,800), F(1200,800)$ and $G(1000,1000)$

Now, $Z = 120x + 150y$

at $C(1000,1700)$, $\quad Z = 375000$

at $D(2000,1700)$, $\quad Z = 495000$

at $E(2000,800)$, $\quad Z = 360000$

at $F(1200,800)$, $\quad Z = 264000$

at $G(1000,1000)$, $\quad Z = 270000$

$\therefore$ Z is maximum at point $D(2000,1700)$

$\therefore$ The maximum value of $120x + 150y$ is 495000 at $(2000,1700)$.

$\therefore$ 2000 scientific and 1700 graphing calculators should be made monthly to maximize the net profit.

Example 10.4.5 :

Solve the following LPP graphically,

Minimize $\quad Z = 25x + 10y$

Subject to, $10x + 2y \geq 20$

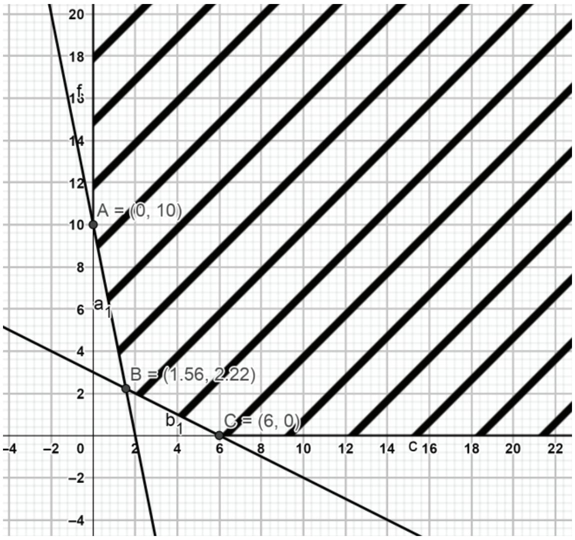
$$x + 2y \geq 6, \qquad x \geq 0, y \geq 0$$

Solution :

Consider the equation $10x + 2y = 20$, when $x = 0$ then $y = 10$ and when $y = 0$ then $x = 2$.

Consider the for 2^{nd} equation $x + 2y = 6$, when $x = 0$ then $y = 3$ and when $y = 0$ then $x = 6$.

Plotting the graph



From the graph we get the feasible region where $A(0,10), B(1.56,2.22), C(6,0)$,.

For Minimize $Z = 25x + 10y$

point	$Z = 25x + 10y$
A	100
B	61.2
C	150

∴ Minimum $Z = 61.2$ at point B i.e. $x = 1.56$ and $y = 2.22$

10.5 Summary

- The objective of this course is to present Linear Programming. Activities allow to train modeling practical problems in Linear programming. Various tests help understanding how to solve a linear program
- Linear programming is a mathematical technique for solving constrained maximization and minimization problems when there are many constraints and the objective function to be optimized as well as the constraints faced are linear (i.e., can be represented by straight lines). Linear programming has been applied to a wide variety of constrained optimization problems. Some of these are: the selection of the optimal production process to use to produce a product, the optimal product mix to produce, the least-cost input combination to satisfy some minimum product requirement, the marginal contribution to profits of the various inputs, and many others.

10.6 References

Following books are recommended for further reading :-

- Numerical Methods for Engineers - *Steven C. Chapra, Raymond P. Canale*; Tata Mc Graw Hill.
- Quantitative Techniques - *Dr. C. Satyadevi* ; S. Chand.

10.7 Exercise

Q. (1) A firm manufactures headache pills in two size A and B. size A contains 3 grains of aspirin, 6 grains of bicarbonate and 2 grain of codeine. Size B contains 2 grains of aspirin, 8 grains of bicarbonate and 10 grain of codeine. It is found by users that it requires at least 15 grains of aspirin, 86 grains of bicarbonate and 28 grain of codeine of providing immediate effect. it is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard LPP.

Q. (2) An animal feed company must produce 200lbs of mixture containing the ingredients A and B. A cost Rs. 5 per Lb. And B cost Rs. 10 per lb. Not more than 100 lbs. of A can used and minimum quantity to be used for B is 80lbs. Find how much of each ingredient should be used if the company wants to minimize the cost. Formulate the problem as a standard LPP.

Q. (3) A painter make two painting A and B. He spends 1 hour for drawing and 3 hours for coloring the painting A and he spends 3 hours for drawing and 1 hour for coloring the painting B. he can spend at most 8 hours and 9 hours for drawing and coloring respectively. The profit per painting of Type A is Rs. 4000 and that of type B is Rs. 5000. formulate as LPP to maximize the profit

Q. (4) A gardener wanted to prepare a pesticide using two solutions A and B. the cost of 1 liter of solution A is Rs. 2 and the cost of 1 liter of solutions B is Rs. 3. He wanted to prepare at least 20 liter of pesticide. The quality of solution A available in a shop is 12 liter and solution B is 15 liter. How many liter of pesticide the gardener should prepare so as to minimize the cost ? formulate the LPP

Q. (5) A bakery produce two kinds of biscuits A and B, using same ingredients L1 and L2. The ingredients L1 and L2 in biscuits of type A are in the ratio 4:1 and in biscuits of type B are in the ratio 9:1. The Profit per Kg for biscuits of type A and B is Rs. 8 per Kg and Rs. 10 per Kg respectively. The bakery had 90 Kg of L1 and 20 Kg of L2 in stock. How many Kg of biscuits A and B be produced to maximize total profit ?

Q. (6) Solve graphically the following LPP

Maximize $Z = 2x + 3y$

subject to, $x + 2y \geq 6$

$2x - 5y \leq 1, x \geq 0, y \geq 0$

Q. (7) Solve graphically the following LPP

Maximize $Z = 2x + 5y$

subject to, $4x + 2y \leq 80$

$2x + 5y \leq 180, x \geq 0, y \geq 0$

Q. (8) Solve graphically the following LPP

Maximize $Z = 2x + y$

subject to, $4x - y \leq 3$

$2x + 5y \leq 7, x \geq 0, y \geq 0$

Q. (9) Solve graphically the following LPP

Maximize $Z = 400x + 500y$

subject to, $x + 3y \leq 8$

$3x + y \leq 9, x \geq 0, y \geq 0$

Q. (10) Solve graphically the following LPP

Maximize $Z = 5000x + 4000y$

subject to, $6x + 4y \leq 24$

$x + 2y \leq 6$

$-x + y \leq 1$

$y \leq 2, \; x \geq 0, y \geq 0$

Q. (11) Solve graphically the following LPP

Minimize $Z = 30x + 50y$

subject to, $3x + 4y \geq 300$

$x + 3y \geq 210, x \geq 0, y \geq 0$

❖❖❖

UNIT 5

11

RANDOM VARIABLES

Unit Structure

11.0 Objectives

After going through this unit, you will be able to:

- Understand the concept of random variables as a function from sample space to real line
- Understand the concept of probability distribution function of a discrete random variable.
- Calculate the probabilities for a discrete random variable.
- Understand the Probability Distribution of Random Variable
- Understand the probability mass function and probability density function
- Understand the properties of random variables.
- Make familiar with the concept of random variable
- Understand the concept of cumulative distribution function.
- Understand the expected value and variance of a random variable with it s properties

Andrey Nikolaevich Kolmogorov (Russian: 25 April 1903 – 20 October 1987) who advances many scientific fields. In probability theory and related fields, a stochastic or random process is a mathematical object usually defined as a family of random variables. A stochastic process may involve several related random variables. A quotation attributed to Kolmogorov is " Every mathematician believes that he is ahead over all others . The reason why they don't say this in public, is because they are intelligent people."

11.1 Introduction

Many problems are concerned with values associated with outcomes of random experiments. For example we select five items randomly from the basket with known proportion of the defectives and put them in a packet. Now we want to know the probability that the packet contain more than one defective. The study of such problems requires a concept of a random variable. Random variable is a function that associates numerical values to the outcomes of experiments.

In this chapter, we considered discrete random variable that is random variable which could take either finite or countably infinite values. When a random variable X is discrete, we can assign a positive probability to each value that X can take and determine the probability distribution for X. The sum of all the probabilities associate with the different values of X is one. However not all experiments result in random variables that ate discrete. There also exist random variables such as height, weights, length of life an electric component, time that a bus arrives at a specified stop or experimental laboratory error. Such random variables can assume infinitely many values in some interval on the real line. Such variables are called continuous random variables. If we try to assign a positive probability to each of these uncountable values, the probabilities will no longer sum to one as was the case with discrete random variable.

11.2 Random Variable (R.V.) – Discrete and Continuous

11.2.1 Discrete Random Variable:

Sample space S or Ω contains non-numeric elements. For example, in an experiment of tossing a coin, $\Omega = \{ H, T \}$. However, in practice it is easier to deal with numerical outcomes. In turn, we associate real number with each outcome. For instance, we may call H as 1 and T as 0. Whenever we do this, we are dealing with a function whose domain is the sample space Ω and whose range is the set of real numbers . Such a function is called a random variable.

Definition 1 : Random variable : Let S / Ω be the sample space corresponding to the outcomes of a random experiment . A function $X: S \rightarrow R$ (where R is a set of real numbers) is called as a random variable.

Random variable is a real valued mapping. A function can either be one-to-one or many-to-one correspondence. A random variable assigns a real number to each possible outcome of an experiment. A random variable is a function from the sample space of a random experiment (domain) to the set of real numbers (co-domain).

Note: Random variable are denoted by capital letters X,Y, Z etc.., where as the values taken by them are denoted by corresponding small letters x, y, z etc.

Definition 2: Discrete random variable : A random variable X is said to be discrete if it takes finite or countably infinite number of possible values. Thus discrete random variable takes only isolated values.

Example1:

What are the values of a random variable X would take if it were defined as number of tails when two coins are tossed simultaneously?

Solutions:

Sample Space of the experiment (Tossing of two coins simultaneously) is,
$S / \Omega = \{ TT, TH, HT, HH \}$

Let X be the number of tails obtained tossing two coins.
$X : \Omega \rightarrow R$
$X(TT)=2$, $X(TH)=1$, $X(HT)=1$, $X(HH)=0$

Since the random variable X is a number of tails, it takes three distinct values $\{ 0, 1, 2 \}$

Remark: Several random variables can be defined on the same sample space Ω. For example in the Example1, one can define Y= Number of heads or Z= Difference between number of heads and number of tails.

Following are some of the examples of discrete variable.

a) Number of days of rainfalls in Mumbai.

b) Number of patients cured by using certain drug during pandemic.

c) Number of attempts required to pass the exam.

d) Number of accidents on a sea link road.

e) Number of customers arriving at shop.

f) Number of students attending class.

Definition 3: Let X be a discrete random variable defined on a sample space Ω. Since Ω contains either finite of countable infinite elements, and X is a function on Ω, X can take either finite or countably infinite values. Suppose X takes values x_1, x_2 x_3,.... then the set $\{ x_1, x_2\ x_3,...\}$ is called the range set of X.

In Example1, the range set of X = Number of tails is $\{ 0,1,2\}$

11.2.2 Continuous Random Variable:

A sample space which is finite or countably infinite is called as denumerable of countable. If the sample space is not countable then it is called **continuous.** For a continuous sample space Ω we can not have one to one correspondence between Ω and set of natural numbers { 1,2,…}

Random variable could also be such that their set of possible values is uncountable. Examples of such random variables are time taken between arrivals of two vehicles at petrol pump or life in hours of an electrical component.

In general if we define a random variable X(ω) as a real valued function on domain Ω. If the range set of X(ω) is continuous, the random variable is continuous. The range set will be a subset of real line.

Following are some of the examples of continuous random variable

a) Daily rainfall in mm at a particular place.

b) Time taken for an angiography operation at a hospital

c) Weight of a person in kg.

d) Height of a person in cm.

e) Instrumental error (measured in suitable units) in the measurement.

11.2.3 Distinction between continuous random variable and discrete random variable:

1) A continuous random variable takes all possible values in a range set. The ser is in the form of interval. On the other hand discrete random variable takes only specific or isolated values.

2) Since a continuous random variable takes uncountably infinite values, no probability mass can be attached to a particular value of random variable X. Therefore, $P(X = x) = 0$, for all x. However in case of a discrete random variable, probability mass is attached to individual values taken by random variable. In case of continuous random variable probability is attached to an interval which is a subset of R.

11.3 Probability Distributions of Discrete Random Variable

Each outcome i of an experiment have a probability P (i) associated with it. Similarly every value of random variable $X= x_i$ is related to the outcome i of an experiment. Hence, for every value of random variable x_i, we have a unique real value P(i) associated . Thus, every random variable X has probability P associated with it. This function P (X= x_i) from the ser of all events of the sample space Ω is called a probability distribution of the random variable.

The probability distribution (or simply distribution) of a random variable X on a sample space Ω is set of pairs (X= x_i, P (X= x_i)) for all x_i : ∈ x(Ω), where P (X= x_i) is the probability that X takes the value x_i .

Consider the experiment of tossing two unbiased coins simultaneously. X= number of tails observed in each tossing. Then range set of X = { 0, 1,2 }. Although we cannot in advance predict what value X will take, we can certainly state the probabilities with which X will take the three values 0,1,2. The following table helps to determine such probabilities.

Outcome	**Probability of Outcome**	**Value of X**
HH	1/4	0
TH	1/4	1
HT	1/4	1
TT	1/4	2

Following events are associated with the distinct values of X

(X = 0) ⇒ { HH}

(X = 1) ⇒ { TH,HT}

(X = 2) ⇒ { TT}

Therefore, probabilities of various values of X are nothing but the probabilities of the events with which the respective values are associated.

P (X = 0) ⇒ P { HH} = 1/4

P (X = 1) ⇒ P { TH, HT} = 1/4 + 1/4 = 1/2

P (X = 2) ⇒ P {TT} = 1/4

Example2:

A random variable is number of tails when a coin is flipped thrice. Find probability distribution of the random variable.

Solution:

Sample space Ω = { HHH, THH, HTH, HHT, TTH, THT, HTT, TTT }

The required probability distribution is

Value of Random Variable	$X = x_i$	0	1	2	3
Probability	$P(X = x_i)$	1/8	3/8	3/8	1/8

Example 3:

A random variable is sum of the numbers that appear when a pair of dice is rolled. Find probability distribution of the random variable.

Solution:

X(1,1) = 2, X(1, 2) = X(2, 1) = 3, X(1, 3) = X(3, 1) = X (2, 2) = 4 etc..;

The probability distribution is,

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$P(X = x_i)$	1/36	2/36	3/36	4/36	5/36	6/35	5/36	4/36	3/36	2/36	1/36

11.3.1 Probability Mass Function (p.m.f.)

Let X be a discrete random variable defined on a sample space Ω / S . Suppose $\{x_1, x_2, x_3, \ldots\ldots x_n\}$ is the range set of X. With each of x_i, we assign a number $P(x_i) = P(X = x_i)$ called the probability mass function (p.m.f.) such that,

$P(x_i) \geq 0$ for $i = 1, 2, 3 \ldots, n$ and (AND)

$\sum_{i=1}^{n} P(x_i) = 1$

The table containing, the value of X along with the probabilities given by probability mass function (p.m.f.) is called probability distribution of the random variable X.

For example,

$X = x_i$	X_1	X_2	…..	…..	X_i	…	….	X_n	Total
$P(X = x_i)$	P_1	P_2	….	….	P_i	…	…	P_n	1

Remark: Properties of a random variable can be studied only in terms of its p.m.f. We need not have refer to the underlying sample space Ω, once we have the probability distribution of random variable.

Example 4:

A fair die is rolled and number on uppermost face is noted. Find its probability distribution (p.m.f.)

Solution:

X= Number on uppermost face.

Therefore, Range set of X = {1, 2, 3, 4, 5, 6 }

Probability of each of the element is = 1 / 6

The Probability distribution of X is

$X = x_i$	1	2	3	4	5	6	Total
$P(X = x_i)$	1/ 6	1/ 6	1/ 6	1/ 6	1/ 6	1/ 6	1

Example 5:

A pair of fair dice is thrown and sum of numbers on the uppermost faces is noted. Find its probability distribution (p.m.f.).

Solution:

X = Sum of numbers on the uppermost faces.

Ω contains 36 elements (ordered pairs)

Range set of X = { 2, 3, 4, 5, 6, 7, 8,9, 10, 11, 12 }

Since, X (1, 1) = 2 and X (6, 6) = 12

Value of X	**Subset of Ω**	$P_i = P(X = i)$
2	{ (1,1) }	1/36
3	{ (1,2), (2,1) }	2/36
4	{ (1,3), (2,2), (3,1) }	3/36
5	{ (1,4), (2,3), (3,2), (4,1) }	4/36
6	{ (1,5), (2,4), (3,3), (4,2), (5,1) }	5/36
7	{ (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) }	6/36
8	{ (2,6), (3,5), (4,4), (5,3), (6,2) }	5/36
9	{ (3,6), (4,5), (5,4), (6,3) }	4/36
10	{ (4,6), (5,5), (6,4) }	3/36
11	{ (5,6) (6,5) }	2/36
12	{ (6,6) }	1/36

Example 6:

Let X represents the difference between the number of heads and the number of tails obtained when a fair coin is tossed three times. What are the possible values of X and its p.m.f.?

Solution:

Coin is fair. Therefore, probability of heads in each toss is $P_H = 1/2$. Similarly, probability of tails in each toss is $P_T = 1/2$.

X can take values n – 2r where n=3 and r = 0, 1, 2, 3.

e.g. X = 3 (HHH), X = 1 (HHT, HTH, THH), X = -1 (HTT, THT, HTT) and X = -3 (TTT)

Thus the probability distribution of X (possible values of X and p.m.f) is

X = x_i	-3	-1	1	3	Total
p.m.f. P(X = x_i)	1/ 8	3/ 8	3/ 8	1/ 8	1

Example 7:

Let X represents the difference between the number of heads and the number of tails obtained when a coin is tossed n times. What are the possible values of X ?

Solution:

When a coin is tossed n times, number of heads that can be obtained are n, n-1, n-2, ……,2, 1, 0. Corresponding number of tails are 0, 1, 2,……, n-2, n-1, n. Thus the sum of number of heads and number of tails must be equal to number of trials n.

Hence, values of X are from n to –n as n, n-2, n-4, ……, n-2r

where r = 0,1, 2, 3, ….., n

Note if n is even X has one of its value as zero also. However, if n is odd X has values -1, 1 but not zero.

11.4 Probability Distributions of Continuous Random Variable

In case of discrete random variable using p.m.f. we get probability distribution of random variable, however in case of continuous random variable probability mass is not attached to any particular value. It is attached to an interval. The probability attached to an interval depends upon its location.

For example, P (a < X < b) varies for different values of a and b. In other words, it will not be uniform. In order to obtain the probability associated with any interval, we need to take into account the concept of probability density.

11.4.1 Probability Density Function (p.d.f.)

Let X be a continuous random variable. Function *f(x)* defined for all real $x \in (-\infty, \infty)$ is called probability density function p.d.f. if for any set B of real numbers, we get probability,

$$P\{X \in B\} = \int_B f(x)dx$$

All probability statements about X can be answered in terms of *f (x)*. Thus,

$$P\{a \le X \le B\} = \int_a^b f(x)dx$$

Note that probability of a continuous random variable at any particular value is zero, since

$$P\{X = a\} = P\{a \le X \le a\} = \int_a^a f(x)dx = 0$$

11.5 Properties of Random variable and their probability distributions

Properties of a random variable can be studied only in terms of its p.m.f . or p.d.f. We need not have refer to the underlying sample space Ω, once we have the probability distribution of random variable. Since the random variable is a function relating all outcomes of a random experiment, the probability distribution of random variable must satisfy Axioms of probability. These in case of discrete and continuous random variable are stated as,

Axiom I : Any probability must be between zero and one.

For discrete random variable : $0 \le p(x_i) \le 1$

For continuous random variable : For any real number a and b

$0 \le P\{a \le x \le b\} \le 1$ **OR** $0 \le \int_a^b f(x)dx \le 1$

Axiom II : Total probability of sample space must be one

For discrete random variable : $\sum_{i=1}^{\infty} p(x_i) = 1$

For continuous random variable : $\int_{-\infty}^{\infty} f(x)dx = 1$

Axiom III : For any sequence of mutually exclusive events $E_1, E_2, E_3, \ldots\ldots$ i.e

$E_i \cap E_j = \Phi$ for $i \neq j$, probability of a union set of events is sum of their individual probabilities. This axiom can also be written as $P(E1 \cup E2) = P(E_1) + P(E_2)$ where E_1 and E_2 are mutually exclusive events.

For discrete random variable : $\boldsymbol{P}\left(\bigcup_{i=1}^{\infty} \boldsymbol{E}_i\right) = \sum_{i=1}^{\infty} P(E_i)$

For continuous random variable : $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ also,

$P(a \leq x \leq b \cup c \leq x \leq d) = \int_a^b f(x)dx + \int_c^d f(x)dx$

Axiom IV : $P(\Phi) = 0$

11.6 Cumulative Distribution Function (c.d.f.)

It is also termed as just a distribution function. It is the accumulated value of probability up to a given value of the random variable. Let X be a random variable, then the cumulative distribution function (c.d.f.) is defined as a function F(a) such that,

$F(a) = P\{X \leq a\}$

11.6.1 Cumulative Distribution Function (c.d.f.) for Discrete Random Variable

Let X be a discrete random variable defined on a sample space S taking values $\{x_1, x_2, \ldots, x_n\}$

With probabilities $p(x_i)$, $p(x_2)$, …. $p(x_n)$ respectively. Then cumulative distribution function (c.d.f.) denoted as F(a) and expressed in term of p.m.f. as

$$F(a) = \sum_{x_i \leq a} p(x_i)$$

Note:

1. The c.d.f. is defined for all values of x_i € R. However, since the random variable takes only isolated values, c.d.f. is constant between two successive values of X and has steps at the points x_iI, i = 1, 2, …, n. Thus, the c.d.f for a discrete random variable is a step function.

2. $F(\infty) = 1$ and $F(-\infty) = 0$

Properties of a random variable can be studied only in term of its c.d.f. We need not refer to the underlying sample space or p.m.f., once we have c.d.f. of a random variable.

11.5.2 Cumulative Distribution Function (c.d.f.) for Continuous Random Variable

Let X be a continuous random variable defined on a sample space S which p.d.f. *f(x)*. Then cumulative distribution (c.d.f.) denoted as F (a) and expressed in term of p.d.f. as,

$$F(a) = \int_{-\infty}^{\infty} f(x)dx$$

Also, differentiating both sides we get,

$\frac{d}{da}F(a) = f(a)$, thus the density is the derivative of the cumulative distribution function.

11.7 Properties of Cumulative Distribution Function (c.d.f.)

1. F (x) is defined for all x € R, real number.
2. $0 \leq F(x) \leq 1$
3. F (x) is a non-decreasing function of x. [if a < b,then F (a) ≤ F (b).
4. $F(\infty) = 1$ and $F(-\infty) = 0$

   Where $F(-\infty) = \lim_{x \to -\infty} F(x)$, $F(\infty) = \lim_{x \to \infty} F(x)$
5. Let a and b be two real numbers where a < b ; then using distribution function, we can compute probabilities of different events as follows.

   i) $P(a < X \leq b) = P[X \leq b] - P[X \leq a]$
   $= F(b) - F(a)$

   ii) $P(a \leq X \leq b) = P[X \leq b] - P[X \leq a] + P(X=a)$
   $= F(b) - F(a) + P(a)$

   iii) $P(a \leq X < b) = P[X \leq b] - P[X \leq a] - P(X=b) + P(X=a)$
   $= F(b) - F(a) - P(b) + P(a)$

   iv) $P(a < X < b) = P[X \leq b] - P[X \leq a] - P(X=b)$
   $= F(b) - F(a) - P(b)$

v) $P(X > a) = 1 - P[X \leq a] = 1 - F(a)$

vi) $P(X \geq a) = 1 - P[X \leq a] + P[X=a] = 1 - F(a) + P(a)$

vii) $P(X=a) = F(a) - \lim_{n\to\infty} F\left(a - \frac{1}{n}\right)$

viii) $P(X < a) = \lim_{n\to\infty} \left(a - \frac{1}{n}\right)$

Example 8:

The following is the cumulative distribution function of a discrete random variable.

$X = x_i$	-3	-1	0	1	2	3	5	8
F(x)	0.1	0.3	0.45	0.65	0.75	0.90	0.95	1.00

i) Find the p.m.f of X

ii) $P(0 < X < 2)$

iii) $P(1 \leq X \leq 3)$

iv) $P(-3 < X \leq 2)$

v) $P(1 \leq X < 1)$

vi) $P(X = \text{even})$

vii) $P(X > 2)$

viii) $P(X \geq 3)$

Solution:

i) Since $F(x_i) = \sum_{j=1}^{i} P_j$

$F(x_{i-1}) = \sum_{j=1}^{i-1} P_j$

$\therefore P_i = \sum_{j=1}^{i} P_j - \sum_{j=1}^{i-1} P_j = F(x_i) - F(x_{i-1})$

$\therefore$ The p.m.f of X is given by

$X = x_i$	-3	-1	0	1	2	3	5	8
F(x)	0.1	0.2	0.15	0.2	0.1	0.15	0.05	0.05

ii) $P(0 < X < 2) = F(2) - F(0) - P(2) = 0.75 - 0.45 - 0.1 = 0.2$

iii) $P(1 \leq X \leq 3) = F(3) - F(1) + P(1) = 0.9 - 0.65 + 0.2 = 0.45$

iv) $P(-3 < X \leq 2) = F(2) - F(-3) = 0.75 - 0.1 = 0.65$

v) $P(1 \leq X < 1) = F(1) - F(-1) - P(1) + P(-1) = 0.65 - 0.3 - 0.2 + 0.2 = 035$

vi) $P(X = \text{even}) = P(x=0) + P(x=2) + P(x=8) = 0.15 + 0.1 + 0.05 = 0.3$

vii) $P(X > 2) = 1 - F(2) = 1 - 0.75 = 0.25$ OR
$= P(x=3) + P(x=5) + P(x=8) = 0.15 + 0.05 + 0.05 = 0.25$

viii) $P(X \geq 3) = 1 - F(3) + P(3) = 1 - 0.9 + 0.15 = 0.25$

Example 9:

A random variable has the following probability distribution

Values of X	0	1	2	3	4	5	6	7	8
P(x)	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(1) Determine the value of a

(2) Find (i) $P(x < 3)$ (ii) $P(x \leq 3)$ (iii) $P(x > 7)$ (iv) $P(2 \leq x \leq 5)$ (v) $P(2 < x < 5)$

(3) Find the cumulative distribution function of x.

Solution:

1. Since p_i is the probability mass function of discrete random variable X,
   We have $\Sigma p_i = 1$
   $\therefore a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$
   $81a = 1$
   $a = 1/81$

2. (i) $P(x < 3) = P(x=0) + P(x=1) + P(x=2)$
   $= a + 3a + 5a$
   $= 9a = 9 * (1/81) = 1/9$

   (ii) $P(x \leq 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$
   $= a + 3a + 5a + 7a$
   $= 16a = 16 * (1/81) = 16/81$

   (iii) $P(x > 7) = P(x = 8) = 17a = 17 * (1/81) = 17/81$

   (iv) $P(2 \leq x \leq 5) = P(x=2) + P(x=3) + P(x=4) + P(x=5)$
   $= 5a + 7a + 9a + 11a = 32a = 32 * (1/81) = 32/81$

   (v) $P(2 < x < 5) = P(x = 3) + P(x = 4) = 7a + 9a = 16a = 16 *$
   $(1/81) = 16/81$

3. The distribution function is as follows:

X=x	0	1	2	3	4	5	6	7	8
F(x)= P (X≤ x)	a	$4a$	$9a$	$16a$	$25a$	$36a$	$49a$	$64a$	$81a$
(or) F(x)	$\frac{1}{81}$	$\frac{4}{81}$	$\frac{9}{81}$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	$\frac{81}{81}=1$

Example 10:

Find the probability between X = 1 and 2 i.e. P($1 \leq X \leq 2$) for a continuous random variable whose p.d.f. is given as

$f(x) = (\frac{1}{6}x + k)$ for $0 \leq x \leq 3$

Solution:

Now, p.d.f must satisfy the probability Axiom II. Thus

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_0^3 \left(\frac{1}{6}x + k\right) dx = \left[\frac{x^2}{12} + kx\right]_0^3 = \left[\frac{3^2}{12} + 3k\right] + 0 = 1$$

$\therefore 12k = 1$

$\therefore k = \frac{1}{12}$

Now, P ($1 \leq X \leq 2$) = $\int_1^2 f(x)dx = \int_1^2 \left(\frac{1}{6}x + \frac{1}{12}\right) dx = \frac{1}{3}$

Example 11:

A continuous random variable whose p.d.f. is given as

$$f(x) = \begin{cases} kx(2-x) & 0 < x < 2 \\ \mathbf{0} & otherwise \end{cases}$$

i) Find k

ii) Find P ($x < \frac{1}{2}$)

Solution:

i) Now, p.d.f must satisfy the probability Axiom II. Thus

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_0^2 kx(2-x)dx = \left[kx^2 - \frac{kx^3}{3}\right]_0^2 = 1$$

$\therefore k = \frac{3}{4}$

ii)

Now, P $(x < \frac{1}{2}) = \int_{-\infty}^{1/2} f(x)dx$

$$= \int_{-\infty}^{\frac{1}{2}} \frac{3}{4}x(2-x)dx \quad = \left[\frac{3}{4}x^2 - \frac{x^3}{3}\right]_0^{1/2} = \frac{5}{12}$$

Example 12:

A random variable is a number of tails when a coin is tossed three times. Find p.m.f. and c.d.f of the random variable.

Solution:

S / Ω = { TTT, HTT, THT, TTH, HHT, HTH,THH, HHH }

n(Ω) = 8

X = x*ᵢ*	0	1	2	3	Total
p.m.f. P(X = x*ᵢ*)	1/ 8	3/ 8	3/ 8	1/ 8	1
c.d.f F(*a*) = P{ X = xᵢ ≤ *a*}	1/ 8	4 / 8	7 / 8	1	

c.d.f. will be describe as follows:

$$F(a) = 0 \qquad -\infty < a < 0$$
$$= \frac{1}{8} \qquad 0 \le a < 1$$
$$= \frac{4}{8} \qquad 1 \le a < 2$$
$$= \frac{7}{8} \qquad 2 \le a < 3$$
$$= 1 \qquad 3 \le a < -\infty$$

Example 13:

A c.d.f. a random variable is as follows

$$F(a) = 0 \qquad -\infty < a < 0$$
$$= \frac{1}{2} \qquad 0 \le a < 1$$
$$= \frac{2}{3} \qquad 1 \le a < 2$$
$$= \frac{11}{12} \qquad 2 \le a < 3$$
$$= 1 \qquad 3 \le a < -\infty$$

Find i) $P(X < 3)$ ii) $P(X = 1)$

Solution:

i) $P(X < 3) = \lim_{n\to\infty} F\left(3 - \frac{1}{n}\right) = \frac{11}{12}$

ii) $P(X = 1) = P(X \leq 1) - P(X < 1)$

$= F(1) - \lim_{n\to\infty} F\left(3 - \frac{1}{n}\right) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

11.8 Expectation or Expected Value of Random Variable :

Expectation is a very basic concept and is employed widely in decision theory, management science, system analysis, theory of games and many other fields. It is one of the most important concepts in probability theory of a random variable.

Expectation of X is denoted by E(X). The expected value or mathematical expectation of a random variable X is the weighted average of the values that X can assume with probabilities of its various values as weights. Expected value of random variable provides a central point for the distribution of values of random variable. So expected value is a mean or average value of the probability distribution of the random variable and denoted as 'μ' (read as ' mew'). Mathematical expectation of a random variable is also known as its arithmetic mean.

$\mu = E(X)$

11.8.1 Expected Value of a Discrete Random Variable :

If X is a discrete random variable with p.m.f. $P(x_i)$, the expectation of X, denoted by E(X), is defined as,

$$E(X) = \sum_{i=1}^{n} x_i * P(x_i) \qquad \text{Where } x_i \text{ for i = 1, 2, …..n (values of } X)$$

11.8.2 Expected Value of a Continuous Random Variable :

If X is a discrete random variable with p.d.f. f(x), the expectation of X, denoted by E(X), is defined as,

$$E(X) = \int_{-\infty}^{\infty} f(x)dx$$

11.8.3 Properties of Expectation

1. For two random variable X and Y if E(X) and E(Y) exist, E(X + Y) = E(X) + E(Y) . This is known as addition theorem on expectation.

2. For two independent random variable X and Y, E(XY) = E(X).E(Y) provided all expectation exist. This is known as multiplication theorem on expectation.
3. The expectation of a constant is the constant it self. ie E(C) = C
4. E(cX) = cE(X)
5. E(aX+b) = aE(X) +b

11.9 Expectation of a Function

Let Y = g (X) is a function of random variable X, then Y is also a random variable with the same probability distribution of X.

For discrete random variable X and Y, probability distribution of Y is also P(x_i). Thus the expectation of Y is,

$$E(Y) = E[(g\ (x_i)] = \sum_{i=1}^{n} g(x_i\ ) * P(x_i)$$

For continuous random variable X and Y, probability distribution of Y is also f(x). Thus the expectation of Y is,

$$E(Y) = E[(g\ (x_i)] = \int_{-\infty}^{\infty} g(x) * f(x)dx$$

Example 14:

A random variable is number of tails when a coin is tossed three times. Find expectation (mean) of the random variable.

Solution:

S / Ω = { TTT, HTT, THT, TTH, HHT, HTH,THH, HHH }

n(Ω) = 8

X = x_i	0	1	2	3
p.m.f. P(X = x_i)	1/ 8	3/ 8	3/ 8	1/ 8
$\boldsymbol{x_i * P(x_i)}$	0	3 / 8	6 / 8	3 / 8

$E(X) = \sum_{i=1}^{4} x_i\ * P(x_i)$ Where x_i for i = 1, 2, …..n (values of X)

$$E(X) = 0+\frac{3}{8}+\frac{6}{8}+\frac{3}{8}=\frac{12}{8}=\frac{3}{2}$$

Example 15:

X is random variable with probability distribution

X = x_i	0	1	2
p.m.f. P(X = x_i)	0.3	0.3	0.4

Y = g(X) = 2X + 3

Find expected value or mean of Y . (i.e. E(*Y*))

Solution:

Y = g(X) = 2X + 3

When X=0, Y= ???

Y = 2X + 3 = 2(0) + 3 = 3,

Similarly,

when X = 1, Y = 5, when X = 2, Y = 7

X = x_i	0	1	2
Y = y_i	3	5	7
p.m.f. P(Y = y_i)	0.3	0.3	0.4

$E(Y) = E[(g(x_i)] = \sum_{i=1}^{n} g(x_i) * P(x_i) = \sum_{i=1}^{n} y_i * P(x_i)$

$E(Y) = 3\times0.3 + 5\times0.3 + 7\times0.4 = 5.2$

11.10 Variance of a Random Variable

The expected value of X (i.e. E(*X*)) provides a measure of central tendency of the probability distribution. However it does not provide any idea regarding the spread of the distribution. For this purpose, variance of a random variable is defined .

Let X be a discrete random variable on a sample space S. The variance of X denoted by *Var(X) or* σ^2 (read as 'sigma square) is defined as,

$$Var(X) = E[(X - \mu)^2] = E[(X - E(X))^2] = \sum_{i=1}^{n}(x_i - \mu)^2 P(x_i)$$

$$Var(X) = \sum_{i=1}^{n}(x_i)^2 P(x_i) - 2\mu \sum_{i=1}^{n} x_i P(x_i) + \mu^2 \sum_{i=1}^{n} P(x_i)$$

$$= E(X^2) - 2\mu E(X) + \mu^2 \quad \ldots\ldots\ldots\ldots\ldots..$$

$[E(X) = \sum_{i=1}^{n} x_i\, P(x_i)\ \ and \sum_{i=1}^{n} P(x_i) = 1]$

$= E(X^2) - 2\mu.\ \mu + \mu^2$

$= E(X^2) - \mu^2\ =\ E(X^2) - [E(X)]^2$

For continuous random variable,

$$Var(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx \ = \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2)\, f(x)dx$$

$$= \int_{-\infty}^{\infty} x^2\ f(x)dx \ - \int_{-\infty}^{\infty} 2\mu x\, f(x)dx\ + \int_{-\infty}^{\infty} \mu^2\ f(x)dx$$

$= E(X^2) - 2\mu\ E(X) - [E(X)]^2$

$= E(X^2) - 2\mu.\ \mu + \mu^2$

$= E(X^2) - \mu^2\ =\ E(X^2) - [E(X)]^2$

Since dimensions of variance are square of dimensions of X, foe comparison, it is better to take square root of variance. It is known as standard deviation and denoted be S.D.(X) or σ (sigma)

S.D.= σ = $\sqrt{\mathrm{Var(x)}}$

11.10.1 Properties of Variance:

1. Variance of constant is zero. ie Var(c) = 0
2. Var(X+c) = Var X

   Note: This theorem gives that variance is independent of change of origin.
3. Var (aX) = a^2 var(X)

   Note: This theorem gives that change of scale affects the variance.
4. Var (aX+b) = a^2Var(X)
5. Var (b-ax) = a^2 Var(x)

Example 16:

Calculate the variance of X, if X denote the number obtained on the face of fair die.

Solution:

X is random variable with probability distribution

$X = x_i$	1	2	3	4	5	6
p.m.f. P(X = x_i)	1/6	1/6	1/6	1/6	1/6	1/6
$x_i * P(x_i)$	1/6	2/6	3/6	4/6	5/6	6/6
${x^2}_i * P(x_i)$	1/6	4/6	9/6	16/6	25/6	36/6

$E(X) = \sum_{i=1}^{6} x_i * P(x_i)$ Where x_i for i = 1, 2, 3, 4, 5,6 (values of X)

$E(X) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{21}{6} = 3.5$

$E(X^2) = \sum_{i=1}^{6} {x^2}_i * P(x_i)$ Where x_i for i = 1, 2, 3, 4, 5,6 (values of X)

$E(X^2) = \frac{1}{6} + \frac{4}{6} + \frac{9}{6} + \frac{16}{6} + \frac{25}{6} + \frac{36}{6} = \frac{91}{6}$

$\sigma^2 = \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{105}{36} = 2.9167$

Example 17:

Obtain variance of r.v. X for the following p.m.f.

$X = x_i$	0	1	2	3	4	5
p.m.f. P(X = x_i)	0.05	0.15	0.2	0.5	0.09	0.01

Solution:

X is random variable with probability distribution

$X = x_i$	0	1	2	3	4	5
p.m.f. P(X = x_i)	0.05	0.15	0.2	0.5	0.09	0.01
$x_i * P(x_i)$	0	0.15	0.40	1.50	0.36	0.05
${x^2}_i * P(x_i)$	0	0.15	0.80	4.50	1.44	0.25

$E(X) = \sum_{i=0}^{5} x_i * P(x_i)$ Where x_i for i = 0,1, 2, 3, 4, 5 (values of X)

$E(X) = 2.46$

$E(X^2) = \sum_{i=0}^{5} {x^2}_i * P(x_i)$ Where x_i for i = 0,1, 2, 3, 4, 5 (values of X)

$E(X^2) = 7.14$

$\sigma^2 = \text{Var}(X) = E(X^2) - [E(X)]^2 = 7.14 - (2.46)^2 = 1.0884$

Example 18:

Obtain variance of r.v. X for the following probability distribution

P(x) = $\frac{x^2}{30}$, x = 0,1, 2,3, 4

Solution:

X is random variable with probability distribution P(x) = $\frac{x^2}{30}$, x = 0,1, 2,3, 4

E(X) = $\sum_{i=0}^{4} x_i * P(x_i)$ Where x_i for i = 0,1, 2, 3, 4 (values of X)

$= \sum_{i=0}^{4} x_i * \frac{x^2}{30} = \frac{1}{30} \sum_{i=0}^{4} x^3 = \frac{1}{30}(0 + 1 + 8 + 27 + 64) = \frac{100}{30} = \frac{10}{3}$ =3.33

E(X) = $\frac{10}{3}$

E(X^2) = $\frac{1}{30} \sum_{i=0}^{4} x^4$ Where x_i for i = 0,1, 2, 3, 4, 5 (values of X)

$E(X2) = \frac{1}{30} \sum_{i=0}^{4} x^4 = \frac{1}{30}(1 + 16 + 81 + 256) = \frac{354}{30}$ =11.8

E(X^2) = σ^2 = Var (X) = E(X^2) – [E(X)]^2 = $11.8 - (3.33)^2$ = 0.6889

11.11 Summary

In this chapter, random variables, its types with its Probability Distributions, expected value and variance is discussed.

Random variable : Let S / Ω be the sample space corresponding to the outcomes of a random experiment . A function X: S → R (where R is a set of real numbers) is called as a random variable.

A random variable X is said to be discrete if it takes finite or countably infinite number of possible values. A sample space which is finite or countably infinite is called as denumerable of countable. If the sample space is not countable then it is called **continuous.**

The probability distribution (or simply distribution) of a random variable X on a sample space Ω is set of pairs (X= x_i, P (X= x_i)) for all x_i : ∈ x(Ω), where P (X= x_i) is the probability that X takes the value x_i .

Let X be a discrete random variable defined on a sample space Ω / S . Suppose {x_1, x_2, x_3, ... x_n } is the range set of X. With each of x_i, we assign a number P(x_i) = P(X = x_i) called the probability mass function (p.m.f.) such that,

$P(x_i) \geq 0$ for $i = 1, 2, 3 \ldots, n$ and (AND)

$\sum_{i=1}^{n} P(x_i) = 1$

Probability Density Function (p.d.f.) Let X be a continuous random variable. Function *f(x)* defined for all real $x \in (-\infty, \infty)$ is called probability density function p.d.f. if for any set B of real numbers, we get probability,

$$P\{X \in B\} = \int_B f(x)dx$$

Axiom I : Any probability must be between zero and one.

Axiom II : Total probability of sample space must be one

Axiom III : For any sequence of mutually exclusive events $E_1, E_2, E_3, \ldots$ i.e

$E_i \cap E_j = \Phi$ for $i \neq j$

Cumulative Distribution Function (c.d.f.)

It is also termed as just a distribution function. It is the accumulated value of probability up to a given value of the random variable. Let X be a random variable, then the cumulative distribution function (c.d.f.) is defined as a function F(a) such that,

$$F(a) = P\{X \leq a\}$$

Expected Value of Random Variable expected value is a mean or average value of the probability distribution of the random variable $\mu = E(X)$

The variance of X denoted by *Var(X)* S.D.= $\sigma = \sqrt{\mathrm{Var}(x)}$

11.12 Reference

Fundamentals of Mathematical Statistics S. C. Gupta, V. K. Kapoor

11.13 Unit End Exercise

1 An urn contains 6 red and 4 white balls. Three balls are drawn at random. Obtain the probability distribution of the number of white balls drawn.

Hints and Answers:

X = x_i	0	1	2	3
p.m.f. P(X = x_i)	5/30	15/30	9/30	1/30

2 Obtain the probability distribution of the number of sixes in two tosses of a die

Hints and Answers:

X = x_i	0	1	2
p.m.f. P(X = x_i)	25/36	10/36	1/36

3 If the variable X denotes the maximum of the two numbers , when a pair of unbiased die is rolled , find the probability distribution of X.

Hints and Answers:

X = x_i	1	2	3	4	5	6
p.m.f. P(X = x_i)	1/36	3/36	5/36	4/36	9/36	11/36

4 A box of 20 mangoes contain 4 bad mangoes. Two mangoes are drawn at random without replacement from this box. Obtain the probability distribution of the number of bad mangoes in sample

Hints and Answers:

X = x_i	0	1	2
p.m.f. P(X = x_i)	95/138	40/138	3/138

5 Three cards are drawn at random successively, with replacement, from a well shuffled pack of 52 playing cards. Getting ‘a card of diamonds ‘ is termed as a success. Obtain the probability distribution of the number of the successes.

Hints and Answers:

X = x_i	0	1	2	3
p.m.f. P(X = x_i)	27/64	27/64	9/64	1/64

6 Determine ‘k’ such that the following functions are p.m.f.s

i) $f(x) = kx$, $x= 1 , 2 , 3 ,\ldots,10.$ [**Ans:** 1 /55]

ii) $f(x) = k.$ $x = 0 , 1 , 2 , 3$ [**Ans:** 3/19]

7 A random variable X has the following probability distribution.

$\mathbf{X = x_i}$	0	1	2	3	4	5	6
$\mathbf{P(X = x_i)}$	k	3k	5k	7k	9k	11k	13k

Find k, [**Ans:** 1/49]

i) Find $P(X \geq 2)$ [**Ans:** 45/49]

ii) Find $P(0 < X <5)$ [**Ans:** 24/49]

8 Given the following distribution function of a random variable X.

$\mathbf{X = x_i}$	-3	-2	-1	0	1	2	3
F(X)	0.05	0.15	0.38	0.57	0.72	0.88	1

Obtain:

i) p.m.f. of X [Ans :

$\mathbf{X = x_i}$	-3	-2	-1	0	1	2	3
P(X)	0.05	0.1	0.23	0.19	0.15	0.16	0.12

]

ii) $P (-2 \leq X \leq 1)$ [**Ans:** 0.67]

iii) $P (X > 0)$ [**Ans:** 0.43]

iv) $P(-1 < X < 2)$ [**Ans:** 0.34]

v) $P (-3 \leq X < -1)$ [**Ans:** 0.15]

vi) $P(-2 < X \leq 0)$ [**Ans:** 0.42]

$\mathbf{X = x_i}$	0	1	2	3	4	5	6
$\mathbf{P(X = x_i)}$	k	3k	5k	7k	9k	11k	13k

9 Continuous random variable X assumes values between 2 and 5 with p.d.f.

i) Find k [**Ans :** 2 /27]

ii) Find $P (x < 4)$ [**Ans:** 16 /27]

10 Following is the c.d.f. of a discrete random variable X.

X = x_i	1	2	3	4	5	6	7	8
F(X≤ *a*)	0.08	0.12	0.23	0.37	0.48	0.62	0.85	1

Find:

i) p.m.f. of X [**Ans :**

X = x_i	1	2	3	4	5	6	7	8
P(X)	0.08	0.04	0.11	0.08	0.17	0.14	0.23	0.15

]

ii) $P(X \leq 4)$ [**Ans :** 0.31 **]**

iii) $P(2 \leq X \leq 6)$ [A**ns :** 0.54 **]**

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Unit 5

12

DISTRIBUTIONS: DISCRETE DISTRIBUTIONS

Unit Structure

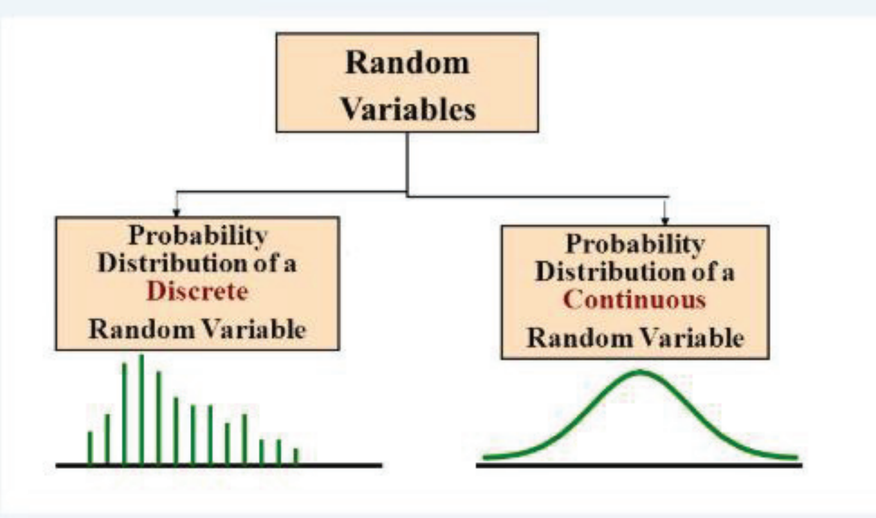
12.0 Objectives

After going through this unit, you will be able to:

- Understand the need of standard probability distribution as models
- Learn the Probability distributions and compute probabilities
- Understand the specific situations for the use of these models
- Learn interrelations among the different probability distributions.

12.1 Introduction

In previous unit we have seen the general theory of univariate probability distributions. For a discrete random variable, p.m.f. can be calculated using underlying probability structure on the sample space of the random experiment. The p.m.f. can be expressed in a mathematical form. This probability distribution can be applied to variety of real-life situations which possess some common features. Hence these are also called as 'Probability Models'.



Discrete distributions:	**Continuous distributions:**
Uniform, Bernoulli, Binomial, Poisson	Uniform distributions, Exponential

In this unit we will study some probability distributions. Viz. Uniform, Binomial, Poisson and Bernoulli distributions.

12.2 Uniform Distribution

Uniform distribution is the simplest statistical distribution. When a coin is tossed the likelihood of getting a tail or head is the same. A good example of a discrete uniform distribution would be the possible outcomes of rolling a 6-sided fair die. $\Omega = \{ 1, 2, 3, 4, 5, 6 \}$

In this case, each of the six numbers has an equal chance of appearing. Therefore, each time the fair die is thrown, each side has a chance of 1/6. The number of values is finite. It is impossible to get a value of 1.3, 4.2, or 5.7 when rolling a fair die. However, if another die is added and they are both thrown, the distribution that results is no longer uniform because the probability of the sums is not equal.

A deck of cards also has a uniform distribution. This is because an individual has an equal chance of drawing a spade, a heart, a club, or a diamond i.e. 1/52.

Consider a small scale company with 30 employees with employee id 101 to 130. A leader for the company to be selected at random. Therefore a employee id is selected randomly from 101 to 130. If X denotes the employee id selected then since all the id's are equally likely, the p.m.f. of X is given by,

$$P(x) = \frac{1}{30}; \qquad x = 101, 102, \ldots\ldots 150$$

$$= 0 \qquad \text{otherwise}$$

Such distribution is called as a discrete uniform distribution. The discrete uniform distribution is also known as the "equally likely outcomes" distribution.

The number of values is finite. It is impossible to get a value of 1.3, 4.2, or 5.7 when rolling a fair die. However, if another die is added and they are both thrown, the distribution that results is no longer uniform because the probability of the sums is not equal. Another simple example is the probability distribution of a coin being flipped. The possible outcomes in such a scenario can only be two. Therefore, the finite value is 2.

12.2.1 Definition: Let X be a discrete random taking values 1, 2, ……, n. Then X is said to follow uniform discrete uniform distribution if its p.m.f is given by

$$P(X = x) = \frac{1}{n} \qquad x = 1, 2,, \ldots n$$

$$= 0 \qquad \text{otherwise}$$

'n' is called as the parameter of the distribution. Whenever, the parameter value is known, the distribution is known completely. The name is 'uniform' as it treats all the values of the variable 'uniformly'. It is applicable where all the values of the random variable are equally likely.

Some examples or the situation where it applied

1. Let X denote the number on the face of unbiased die, after it is rolled.

$$P(X = x) = \frac{1}{6} \quad ; x = 1, 2, 3,4,5,6$$

$$= 0 \quad ; \text{otherwise}$$

2. A machine generates a digit randomly from 0 to 9

$$P(X = x) = \frac{1}{10} \quad ; x = 0, 1, 2, \ldots 9$$

$$= 0 \quad ; \text{otherwise}$$

12.2.2Mean and Variance of Uniform Distribution

Let X is said to follow Uniform discrete uniform distribution and its p.m.f is given by

$$P(X = x) = \frac{1}{n} \quad x = 1, 2,, \ldots n$$

$$= 0 \quad \text{otherwise}$$

Mean = E(X) = $\sum_{i=1}^{n} x_i * P(x_i) = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{n(n+1)}{2n} = \frac{n+1}{2}$

Variance *(X)* = E(X^2) – [E(X)]^2

E(X^2) =$\sum_{i=1}^{n} {x^2}_i * P(x_i) = \frac{1}{n}\sum_{i=1}^{n} {x^2}_i = \frac{(n+1)(2n+1)}{6}$

Variance *(X)* = E(X^2) – [E(X)]^2 = $\frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}$

Standard Deviation (S.D.) (X)= σ = $\sqrt{Var\ (X)} = \sqrt{\frac{n^2-1}{12}}$

12.2.3 Applications of Uniform Distribution

Example1: Find the variance and standard deviation of X, where X is the square of the score shown on a fair die.

Solutions: Let X is a random variable which shows the square of the score on a fair die.

X = { 1, 4, 9,16, 25, 36 } each on have the probability = $\frac{1}{6}$

Mean = E(X) = $\sum_{i=1}^{n} x_i * P(x_i) = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$

Variance *(X)* = $E(X^2) - [E(X)]^2$

$E(X^2) = \sum_{i=1}^{n} x^2{}_i * P(x_i) = \frac{1}{6}(1+16+81+256+625+1296) = \frac{2275}{6}$

Variance *(X)* = $E(X^2) - [E(X)]^2 = \frac{2275}{6} - \left[\frac{91}{6} * \frac{91}{6}\right] = \frac{5396}{36}$

Standard Deviation (S.D.) (X)= σ = $\sqrt{Var\ (X)} = \sqrt{\frac{5396}{36}}$

12.3 Bernoulli Distribution

A trial is performed of an experiment whose outcome can be classified as either success or failure. The probability of success is p ($0 \le p \le 1$) and probability of failure is (1-p). A random variable X which takes two values 0 and 1 with probabilities 'q' and 'p' i.e. P(x=1) = p and P(x=0) = q, p + q = 1 (i.e q = 1 –p), is called a Bernoulli variate and is said to be a Bernoulli Distribution, where p and q takes the probabilities for success and failure respectively. It is discovered by Swiss Mathematician James or Jacques Bernoulli (1654-1705). It is applied whenever the experiment results in only two outcomes. One is success and other is failure. Such experiment is called as Bernoulli Trial.

12.3.1 Definition: Let X be a discrete random taking values either success (1/ True / p) or failure (0 / False / q). Then X is said to follow Bernoulli discrete distribution if its p.m.f is given by

$$P(X = x) = p^x q^{1-x} \qquad x = 0, 1$$
$$= 0 \qquad \text{otherwise}$$

Note: $0 \le p \le 1$, $p + q = 1$

This distribution is known as Bernoulli distribution with parameter 'p'

12.3.2 Mean and Variance of Bernoulli Distribution

Let X follows Bernoulli Distribution with parameter ‘p’. Therefore its p.m.f. is given by

$$P(X = x) = p^x q^{1-x} \qquad x = 0, 1$$
$$= 0 \qquad \text{otherwise}$$

Note: $0 \le p \le 1$, $p + q = 1$

Mean = $E(X) = \sum_{i=0}^{1} x_i * P(x_i) = \sum_{i=0}^{1} x_i\, p^x q^{1-x}$

Substitute the value of x= 0, and x = 1, we get

$$E(X) = (0 \times p^0 \times q^{1-0}) + (1 \times p^1 \times q^{1-1}) = p$$

Similarly, $E(X^2) = \sum_{i=0}^{1} x_i^2 * P(x_i) = \sum_{i=0}^{1} x_i^2 p^x q^{1-x}$

Substitute the value of x= 0, and x = 1, we get

$E(X^2) = p$

Variance *(X)* $= E(X^2) - [E(X)]^2 = p - p^2 = p(1-p) = pq$ ……….. $(p + q = 1)$

Standard Deviation (S.D.) (X)= $\sigma = \sqrt{Var\ (X)} = \sqrt{pq}$

Note: if $p = q = \frac{1}{2}$ the Bernoulli distribution is reduced to a Discrete Uniform Distribution as

$$P(X = x) = \frac{1}{2} \qquad x = 0,1$$
$$= 0 \qquad \text{otherwise}$$

12.3.3 Applications of Bernoulli Distribution

Examples of Bernoulli’ s Trails are:

1) Toss of a coin (head or tail)

2) Throw of a die (even or odd number)

3) Performance of a student in an examination (pass or fail)

4) Sex of a new born child is recorded in hospital, Male = 1, Female = 0

5) Items are classified as ‘defective=0’ and ‘non-defective = 1’.

12.3.4 Distribution of Sum of independent and identically distributed Bernoulli Random variables

Let Y_i, i= 1,2,…n be 'n' independent Bernoulli random variables with parameter 'p' ('p' for success and 'q' for failure p + q = 1)

i.e. P [Y_i = 1] = p and P [Y_i = 0] = q, for i = 1,2,…n.

Now lets define, X which count the number of '1's (Successes) in 'n' independent Bernoulli trials,

$X = \sum_{i=1}^{n} Y_i$

In order to derive probability of 'x' successes in 'n' trials i.e. P [X = x]

Consider a particular sequence of 'x' successes and remaining (n – x) failures as

1 0 1 1 1 0 1 0 ……… 1 0

Here '1' (Success = p) occurs 'x' times and '0' (Failure = q) occurs (n – x) times.

Due to independence, probability of such a sequence is given as follows:

$$\underbrace{p\ p\ p \ldots\ldots p}_{x\ times} \qquad \underbrace{q\,q\,q\,q \ldots\ldots\ldots q}_{(n-x) times} = p^x q^{(n-x)}$$

However, the successes (1's) can occupy any 'x' places out of ' n' places in a sequence in $\binom{n}{x}$ ways, therefore

$$P(X = x) = \binom{n}{x} p^x q^{n-x} \quad x = 0, 1, 2\ldots..n$$

$$= 0 \qquad \text{otherwise}$$

Note: $0 \le p \le 1$, p + q = 1

This gives us a famous distribution called as **' Binomial Distribution'**

12.4 Binomial Distribution

This distribution is very useful in day to day life. A binomial random variable counts number of successes when 'n' Bernoulli trials are performed. A single success/failure experiment is also called a Bernoulli trial or Bernoulli experiment, and a sequence of outcomes is called a Bernoulli process; for a single trial, i.e., n = 1, the binomial distribution is a Bernoulli distribution.

Binomial distribution is denoted by $X \rightarrow B(n, p)$

Bernoulli distribution is just a binomial distribution with $n = 1$
i.e. parameters (1, p).

12.4.1 Definition:

A discrete random variable X taking values 0, 1, 2,…….n is said to follow a binomial distribution with parameters 'n' and 'p' if its p.m.f. is given by

$$P\,(X = x) = \binom{n}{x} p^x q^{n-x} \quad x = 0, 1, 2.....n$$

$$= 0 \quad \text{otherwise}$$

Note: $0 \leq p \leq 1, p + q = 1$

Remark:

1) The probabilities are term in the binomial expansion of $(p + q)^n$, hence the name ' Binomial Distribution ' is given

2) $\sum_{x=0}^{n} P(x) = \sum_{x=0}^{n} = \binom{n}{x} p^x q^{n-x} = (p + q)^n = 1$

3) The binomial distribution is frequently used to model the number of successes in a sample of size n drawn with replacement from a population of size N. If the sampling is carried out without replacement, the draws are not independent and so the resulting distribution is a hypergeometric distribution, not a binomial one

12.4.2 Mean and Variance of Binomial Distribution

Let X follows Binomial Distribution with parameters 'n' and 'p' if its p.m.f. is given by

$$P\,(X = x) = \binom{n}{x} p^x q^{n-x} \quad x = 0, 1, 2.....n$$

$$= 0 \quad \text{otherwise}$$

Note: $0 \leq p \leq 1, p + q = 1$

Mean = E(X) = $\sum_{i=0}^{1} x_i \ * P(x_i) = \sum_{i=0}^{1} x_i \ \binom{n}{x} p^x q^{n-x}$

Mean = E(X) = $\sum_{i=0}^{n} x_i \ * P(x_i)$

$$= \sum_{i=0}^{n} x_i \ \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{i=0}^{n} x_i \cdot \frac{n!}{x!\,(n-x)!} \ p^x q^{n-x}$$

Substitute the value of x= 0, we get first term as '0'

$$= \sum_{i=1}^{n} \cdot \frac{(n-1)!}{(x-1)!\,(n-x)!} \ p^x q^{n-x}$$

$= np \sum_{i=1}^{n} \cdot \frac{(n-1)!}{(x-1)!\,(n-1-(x-1)))!}\; p^{x-1} q^{n-x}$

$= np \sum_{i=1}^{n} \cdot \binom{n-1}{x-1} p^{x-1} q^{n-1-(x-1)}$

$= np\,(p+q)^{n-1}$ ----- Using Binomial Expansion

$= np \ldots\ldots\ldots\ldots\ldots\ldots..(p+q=1)$

Mean = E (X) = μ = np -------------------------------(1)

$E(X^2) = E[X(X-1)] + E[X]$ --------------------------(2)

$E[X(X-1)] = \sum_{i=0}^{n} x_i\,(x_i - 1)\binom{n}{x} p^x q^{n-x}$

$= \sum_{i=0}^{n} \frac{x(x-1)n!}{x!\,(n-x)!}\; p^x q^{n-x}$

$= n\,(n-1)\,p^2 \sum_{i=2}^{n} \cdot \binom{n-2}{x-2} p^{x-2} q^{n-2-(x-2)}$

$= n\,(n-1)\,p^2 (p+q)^{(n-2)}$

$= n(n-1)\,p^2$---------------------------------------(3)

Using (1), (2) and (3)

$E(X^2) = n(n-1)\,p^2 + np$ -------------------------(4)

Variance *(X)* $= E(X^2) - [E(X)]^2$

Variance *(X)* $= [n(n-1)\,p^2 + np] - (np)^2 = npq$

Standard Deviation (S.D.) (X)= σ = $\sqrt{Var(X)} = \sqrt{npq}$

NOTE: Binomial Theorem (Binomial Expansion) it states that, where n is a positive integer:

$(a+b)^n = a^n + ({}^nC_1)a^{n-1}b + ({}^nC_2)a^{n-2}b^2 + \ldots + ({}^nC_{n-1})ab^{n-1} + b^n$

$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = {}^nC_r$

12.4.3 Applications of Binomial Distribution

We get the Binomial distribution under the following experimental conditions.

1) The number of trials ‘ n’ is finite. (not very large)

2) The trials are independent of each other.

3) The probability of success in any trial‘ p’ is constant for each trial.

4) Each trial (random experiment) must result in a success or a failure (Bernoulli trial).

Following are some of the real life examples of Binomial Distribution

1. Number of defective items in a lot of n items produced by a machine
2. Number of mail births out of ‘n’ births in a hospital
3. Number of correct answers in a multiple choice test.
4. Number of seeds germinated in a row of ‘n’ planted seeds
5. Number of rainy days in a month
6. Number of re-captured fish in a sample of ‘n’ fishes.
7. Number of missiles hitting the targets out of ‘n’ fired.

In all above situations, ‘p’ is the probability of success is assumed to be constant.

Example 1:

Comment on the following: “ The mean of a binomial distribution is 5 and its variance is 9”

Solution:

The parameters of the binomial distribution are n and p

We have mean ⇒ np = 5

Variance ⇒ npq = 9

$\therefore q = \frac{\text{npq}}{\text{np}} = \frac{9}{5} > 1$

Which is not admissible since q cannot exceed unity. (p + q = 1) Hence the given statement is wrong.

Example 2:

Eight coins are tossed simultaneously. Find the probability of getting atleast six heads.

Solution:

Here number of trials, n = 8, p denotes the probability of getting a head.

P = 1 / 2 and q = 1 – p = 1 / 2

If the random variable X denotes the number of heads, then the probability of a success in n trials is given by

$$P(X = x) = \binom{n}{x} p^x q^{n-x} \qquad x = 0, 1, 2.....n$$

$$= \binom{8}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{8-x} = \binom{8}{x}\left(\frac{1}{2}\right)^8 = \frac{1}{2^8}\binom{8}{x}$$

Note: $0 \le p \le 1,\ p + q = 1$

Probability of getting at least 6 heads is given by

$$P(X \ge 6) = P(X = 6) + P(X = 7) + P(X = 8)$$

$$= \frac{1}{2^8}\binom{8}{6} + \frac{1}{2^8}\binom{8}{7} + \frac{1}{2^8}\binom{8}{8}$$

$$= \frac{1}{2^8}\left[\binom{8}{6} + \binom{8}{7} + \binom{8}{8}\right]$$

$$= \frac{1}{256}[\,28 + 8 + 1\,] = \frac{37}{256}$$

Example 3:

Ten coins are tossed simultaneously. Find the probability of getting (i) at least seven heads (ii) exactly seven heads (iii) at most seven heads

Solution:

Here number of trials, n = 10, p denotes the probability of getting a head.

P = 1 / 2 and q = 1 – p = 1 / 2

If the random variable X denotes the number of heads, then the probability of a success in n trials is given by

$$P(X = x) = \binom{n}{x} p^x q^{n-x} \qquad x = 0, 1, 2.....n$$

$$= \binom{10}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{10-x} = \binom{10}{x}\left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}}\binom{10}{x}$$

Note: $0 \le p \le 1,\ p + q = 1$

i) Probability of getting at least 7 heads is given by

$$P(X \geq 7) = P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

$$= \frac{1}{2^{10}}\binom{10}{7} + \frac{1}{2^{10}}\binom{10}{8} + \frac{1}{2^{10}}\binom{10}{9} + \frac{1}{2^{10}}\binom{10}{10}$$

$$= \frac{1}{2^{10}}\left[\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10} \right]$$

$$= \frac{1}{1024}[\, 120 + 45 + 10 + 1\,] = \frac{176}{1024}$$

ii) Probability of getting exactly 7 heads is given by .

$$P(X = 7) = \frac{1}{2^{10}}\binom{10}{7} = \frac{120}{1024}$$

iii) Probability of getting at most 7 heads is given by .

$$P(X \leq 7) = 1 - P(X > 7)$$

$$= 1 - \{\, P(X = 8) + P(X = 9) + P(X = 10)\, \}$$

$$= 1 - \frac{1}{2^{10}}\left[\binom{10}{8} + \binom{10}{9} + \binom{10}{10} \right]$$

$$= 1 - \frac{1}{1024}[\, 45 + 10 + 1\,] = 1 - \frac{56}{1024} = \frac{968}{1024}$$

Example 4:

20 wrist watches in a box of 100 are defective. If 10 watches are selected at random, find the probability that (i) 10 are defective (ii) 10 are good (iii) at least one watch is defective (iv)at most 3 are defective.

Solution:

20 out of 100 wrist watches are defective, so Probability of defective wrist watch p = 20 /100

$$p = \frac{20}{100} = \frac{1}{5} \quad \therefore q = 1 - p = 1 - \frac{20}{100} = \frac{80}{100} = \frac{4}{5}$$

Since 10 watches are selected at random, n = 10

$$P(X = x) = \binom{n}{x} p^x q^{n-x} \qquad x = 0, 1, 2\ldots..n$$

$$= \binom{10}{x}\left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{10-x}$$

i) Probability of selecting 10 defective watches

$$P(X = 10) = \binom{10}{10}\left(\frac{1}{5}\right)^{10}\left(\frac{4}{5}\right)^{10-10} = 1.\frac{1}{5^{10}}.1 = \frac{1}{5^{10}}$$

ii) Probability of selecting 10 good watches (i.e. no defective)

$$P(X = 0) = \binom{10}{0}\left(\frac{1}{5}\right)^{0}\left(\frac{4}{5}\right)^{10-0} = 1.1.\frac{4^{10}}{5^{10}} = \left(\frac{4}{5}\right)^{10}$$

iii) Probability of selecting at least one defective watch

$$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0)$$

$$= 1 - \binom{10}{0}\left(\frac{1}{5}\right)^{0}\left(\frac{4}{5}\right)^{10-0} = 1 - \left(\frac{4}{5}\right)^{10}$$

iv) Probability of selecting at most 3 defective watches

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$= \binom{10}{0}\left(\frac{1}{5}\right)^{0}\left(\frac{4}{5}\right)^{10} + \binom{10}{1}\left(\frac{1}{5}\right)^{1}\left(\frac{4}{5}\right)^{9} + \binom{10}{2}\left(\frac{1}{5}\right)^{2}\left(\frac{4}{5}\right)^{8} + \binom{10}{3}\left(\frac{1}{5}\right)^{3}\left(\frac{4}{5}\right)^{7}$$

$$= 1.1.\left(\frac{4}{5}\right)^{10} + 10.\frac{1}{5}\frac{4^9}{5^9} + 45.\frac{1}{5^2}\frac{4^8}{5^8} + 120.\frac{1}{5^3}\frac{4^7}{5^7}$$

$$= 0.859 \text{ (Approx.)}$$

Example 5:

With the usual notation find p for binomial random variable X if n = 6 and [9*P(X = 4) = P(X = 2)]

Solution:

The probability mass function of binomial random variable X is given by

$$P(X = x) = \binom{n}{x} p^x q^{n-x} \qquad x = 0, 1, 2.....n$$

Note: $0 \leq p \leq 1$, $p + q = 1$

Here n = 6,

$$P(X = x) = \binom{6}{x}(p)^x(q)^{6-x}$$

$$P(X = 4) = \binom{6}{4}(p)^4(q)^{6-4} = \binom{6}{4}(p)^4(q)^2$$

$$P(X = 2) = \binom{6}{2}(p)^2(q)^{6-2} = \binom{6}{2}(p)^2(q)^4$$

Given that, $9*P(X = 4) = P(X = 2)$

$$9 * \binom{6}{4}(p)^4(q)^2 = \binom{6}{2}(p)^2(q)^4$$

$$9 * 15* (p)^4(q)^2 = 15 * (p)^2(q)^4$$

$$9 (p)^4(q)^2 = (p)^2(q)^4$$

$$9 p^2 = q^2$$

Taking positive square root on both sides we get,

$$3p = q$$

$$3p = 1- p$$

$$4p = 1 \therefore p = \frac{1}{4}= 0.25$$

12.5 Poisson Distribution

Poisson distribution was discovered by a French Mathematician-cum-Physicist Simeon Denis Poisson in 1837. Poisson distribution is also a discrete distribution. He derived it as a limiting case of Binomial distribution. For n-trials the binomial distribution is $(p + q)^n$; the probability of x successes is given by P(X=x)= $\sum_{i=0}^{n} x_i \binom{n}{x} p^x q^{n-x}$ If the number of trials n is very large and the probability of success ' p' is very small so that the product np = m is non – negative and finite.

12.5.1 Definition:

A discrete random variable X, taking on one of the countable infinite values 0, 1, 2,……. is said to follow a Binomial distribution with parameters 'λ' or 'm', if for some m > 0, its p.m.f. is given by

$$P (X = x) = \frac{e^{-m}m^x}{x!} \qquad x = 0 , 1, 2….. \quad m > 0$$

$$= 0 \qquad \text{otherwise}$$

Note: $e^{-m} \geq 0,\ AND\ m^x \geq 0 , x! \geq 0,$

Hence, $\frac{e^{-m}m^x}{x!} \geq 0$

Note: $e^m = \frac{m^0}{0!} + \frac{m^1}{1!} + \frac{m^2}{2!} - - - - -- = \sum_{x=0}^{\infty} \frac{m^x}{x!}$

$e = \frac{1^0}{0!} + \frac{1^1}{1!} + \frac{2^2}{2!} - - - - -- = 2.71828$, 0!=1, 1!=1

Since, $e^{-m} \geq 0 \therefore P$ (X) ≥ 0 for all x and

$$\sum P(x) = \sum_{x=0}^{\infty} \frac{e^{-m}m^x}{x!} = e^{-m} , \sum_{x=0}^{\infty} \frac{m^x}{x!} = e^{-m}.e^m = 1$$

It is denoted by $X \rightarrow P(m)$ or $X \rightarrow P(\lambda)$

Since number of trials is very large and the probability of success p is very small, it is clear that the event is a rare event. Therefore Poisson distribution relates to rare events.

12.5.2 Mean and Variance of Poisson Distribution

Let X is a Poisson random variable with parameter 'm' (or 'λ'), if its p.m.f. is given as

$$P(X = x) = \frac{e^{-m}m^x}{x!} \qquad x = 0, 1, 2\ldots.. \quad m > 0$$

$$= 0 \qquad \text{otherwise}$$

Note: $e^{-m} \geq 0,\ AND\ m^x \geq 0, x! \geq 0,$

Hence, $\frac{e^{-m}m^x}{x!} \geq 0$

$\therefore$ Mean = $E(X) = \sum_{i=0}^{\infty} x_i\, P(x_i)$

$$= \sum_{i=0}^{\infty} x_i \frac{e^{-m}m^x}{x!}$$

The term corresponding to $x = 0$ is zero.

$$\therefore \quad = \sum_{i=1}^{\infty} m\frac{e^{-m}\, m^{x-1}}{(x-1)!} = me^{-m}\sum_{i=1}^{\infty}\frac{e^{-m}\, m^{x-1}}{(x-1)!} = me^{-m}.e^{m} = m$$

$\therefore$ Mean = $E(X) = \mu = m$

But, $E(X^2) = E[X(X-1)] + E[X]$

$$E[X(X-1)] = \sum_{i=0}^{\infty} x_i(x_i - 1)\ P(x_i) = \sum_{i=0}^{\infty} x_i(x_i - 1)\frac{e^{-m}m^x}{x!}$$

$$= m^2e^{-m}\sum_{i=2}^{\infty}\frac{m^{x-2}}{(x-2)!} = m^2e^{-m}.e^{m} = m^2$$

$E[X(X-1)] = m^2$

$E(X^2) = E[X(X-1)] + E[X] = m^2 + m$

Variance $(X) = E(X^2) - [E(X)]^2$

Variance $(X) = (m^2 + m) - (m)^2 = m$

Standard Deviation (S.D.) $(X) = \sigma = \sqrt{Var(X)} = \sqrt{m}$

Thus the mean and variance of Poisson distribution are equal and each is equal to the parameter of distribution ($'m'$ or $'\lambda'$)

12.5.3 Applications of Poisson Distribution

Some examples of Poisson variates are :

1) The number of blinds born in a town in a particular year.
2) Number of mistakes committed in a typed page.
3) The number of students scoring very high marks in all subjects.
4) The number of plane accidents in a particular week.
5) The number of defective screws in a box of 100, manufactured by a reputed company.
6) Number of accidents on the express way in one day.
7) Number of earthquakes occurring in one year in a particular seismic zone.
8) Number of suicides reported in a particular day.
9) Number of deaths of policy holders in one year.

Conditions:

Poisson distribution is the limiting case of binomial distribution under the following conditions:

1. The number of trials n is indefinitely large i.e., $n \to \infty$
2. The probability of success ‘ p’ for each trial is very small; i.e., $p \to 0$
3. $np = m$ (say) is finite, $m > 0$

Characteristics of Poisson Distribution:

Following are the characteristics of Poisson distribution

1. Discrete distribution: Poisson distribution is a discrete distribution like Binomial distribution, where the random variable assume as a countably infinite number of values 0,1,2 ….

2. The values of p and q: It is applied in situation where the probability of success p of an event is very small and that of failure q is very high almost equal to 1 and n is very large.

3. The parameter: The parameter of the Poisson distribution is m. If the value of m is known, all the probabilities of the Poisson distribution can be ascertained.

4. Values of Constant: Mean = m = variance; so that standard deviation = $\sqrt{m}$ Poisson distribution may have either one or two modes.

5. Additive Property: If X_1 and X_2 are two independent Poisson distribution variables with parameter m_1 and m_2 respectively. Then $(X_1+ X_2)$ also follows the Poisson distribution with parameter $(m_1 + m_2)$ i.e. $(X_1+ X_2) \rightarrow P(m_1 + m_2)$

6. As an approximation to binomial distribution: Poisson distribution can be taken as a limiting form of Binomial distribution when n is large and p is very small in such a way that product np = m remains constant.

7. Assumptions: The Poisson distribution is based on the following assumptions.

   i) The occurrence or non- occurrence of an event does not influence the occurrence or non-occurrence of any other event.

   ii) The probability of success for a short time interval or a small region of space is proportional to the length of the time interval or space as the case may be.

   iii) The probability of the happening of more than one event is a very small interval is negligible.

Example 1:

Suppose on an average 1 house in 1000 in a certain district has a fire during a year. If there are 2000 houses in that district, what is the probability that exactly 5 houses will have a fire during the year? [given that $e^{-2} = 0.13534$]

Solution:

Mean = np , n = 2000, $p = \frac{1}{1000}$

$m = np = 2000 * \frac{1}{1000} = 2$

∴ m = 2, now According to Poisson distribution

$$P(X = x) = \frac{e^{-m} m^x}{x!} \qquad x = 0, 1, 2….. \quad m > 0$$

$$\therefore \mathbf{P(X = 5)} = \frac{e^{-m} m^x}{x!} = \frac{e^{-2} 2^5}{5!}$$

$$\mathbf{P(X = 5)} = \frac{(0.13534) * 32}{120} = 0.36$$

Example 2:

In a Poisson distribution 3P(X=2) = P(X=4) Find the parameter ' m' .

Solution:

$$P(X = x) = \frac{e^{-m}m^{x}}{x!} \qquad x = 0, 1, 2\ldots.. \quad m > 0$$

Given, 3P(X=2) = P(X=4)

$$\therefore \; 3\,\frac{e^{-m}m^{2}}{2!} = \frac{e^{-m}m^{4}}{4!}$$

$$\therefore \; m^{2} = \frac{3\,.4!}{2!} = 36$$

$$\therefore \; m = \pm\, 6$$

Since mean is always positive $\therefore$ m = 6

Example 3:

If 2% of electric bulbs manufactured by a certain company are defective. Find the probability that in a sample of 200 bulbs i) less than 2 bulbs ii) more than 3 bulbs are defective.[e^{-4} = 0.0183]

Solution :

The probability of a defective bulb = p = 2 / 100 = 0.02

Given that n=200 since p is small and n is large, we use Poisson Distribution, mean m = np

m = np = 200 * 0.02 = 4

Now, Poisson Probability Function

$$P(X = x) = \frac{e^{-m}m^{x}}{x!} \qquad x = 0, 1, 2\ldots.. \quad m > 0$$

i) Probability of less than 2 bulbs are defective

P (X < 2) = P (X=0) + P (X = 1)

$$= \frac{e^{-4}4^{0}}{0!} + \frac{e^{-4}4^{1}}{1!} = e^{-4}\,(1 + 4) = e^{-4} * 5 = 0.0138 * 5 = 0.0915$$

ii) Probability of getting more than 3 defective bulbs

$P(X > 3) = 1 - P(X \leq 3)$

$= 1 - \{ P(X=0) + P(X = 1) + P(X=2) + P(X=3) \}$

$= 1 - e^{-4} (1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!})$

$= 1 - 0.0183*(1+4+8+10.67)$

$= 0.567$

Example 4:

In a company previous record show that on an average 3 workers are absent without leave per shift. Find the probability that in a shift

i) Exactly 2 workers are absent

ii) More than 4 workers will be absent

iii) At most 3 workers will be absent

Solution :

This is a case of Poisson distribution with parameter 'm=3'

i) $P(X = 2) = \frac{e^{-3}3^2}{3!} = 0.2241$

ii) $P(X > 4) = 1 - P(X \leq 4)$

$= 1 - \{ P(X=0) + P(X = 1) + P(X=2) + P(X=3) + P(X=4) \}$

$= 1 - \{ \frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!} + \frac{e^{-3}3^4}{4!} \} = 0.1845$

iii) $P(X \geq 3) = 1 - P(X \leq 3)$

$= 1 - \{ P(X=0) + P(X = 1) + P(X=2) + P(X=3) \}$

$= 1 - \{ \frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!} \} = 0.5767$

Example 5:

Number of accidents on Pune Mumbai express way each day is a Poisson random variable with average of three accidents per day. What is the probability that no accident will occur today?

Solution :

This is a case of Poisson distribution with m = 3,

$P(X = 0) = \frac{e^{-m}3^0}{0!} = e^{-3} = 0.0498$

Example 6:

Number of errors on a single page has Poisson distribution with average number of errors one per page . Calculate probability that there is at least one error on a page.

Solution :

This is a case of Poisson distribution with m = 1,

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{e^{-1}1^0}{0!} = 1 - e^{-1} = 0.0632$$

12.6 Summary

In this chapter, Discrete distribution, its types Uniform, Bernoulli, Binomial and Poisson with its mean , variance and its application is discussed.

Distribution	Definition	Mean E(X)	Variance (X)
Uniform	$P(X = x) = \frac{1}{n}$ $x = 1, 2,,\ldots.n$ $= 0$ otherwise	$\frac{n+1}{2}$	$\sqrt{\frac{n^2-1}{12}}$
Bernoulli	$P(X = x) = p^x q^{1-x}$ $x = 0, 1$ $= 0$ otherwise Note: $0 \leq p \leq 1$, $p + q = 1$	p	pq
Binomial	$P(X = x) = \binom{n}{x} p^x q^{n-x}$ $x = 0, 1, 2, \ldots n$ $= 0$ otherwise Note: $0 \leq p \leq 1$, $p + q = 1$	$\mu = np$	npq
Poisson	$P(X = x) = \frac{e^{-m}m^x}{x!}$ $x = 0, 1, 2\ldots.. m > 0$ $= 0$ Otherwise	$\mu = m$	m

12.7 Reference

Fundamentals of Mathematical Statistics S. C. Gupta, V. K. Kapoor

12.8 Unit End Exercise

1 A random variable X has the following discrete uniform distribution

If P(X) = , X= 0, 1 , …..n, Find E(X) and Var(X)

[**Hints and Answers**: n/2 , n(n+2/ 12]

2 Let X follow a discrete uniform distribution over 11, 12, …….,20.

Find i) $P(X > 15)$ ii) $P(12 \leq X \leq 18)$ iii) $P(X \leq 14)$ iv) Mean and S.D. of X

[**Hints and Answers**: i)0.5 ii) 0.6 iii)0.4 iv) Mean=15.5 and S.D.=2.8723]

3 Let X be the roll number of a student selected at random from 20 students bearing roll numbers 1 to 20. Write the p.m.f. of X.

[**Hints and Answers:**

P(X) = 1 /20 ; x=1,2,3,….,20

= 0 ; otherwise]

4 Let X and Y be two discrete uniform r.v.s assuming values 1,2,…..,10, X and Y are independent. Find p.m.f. of Z=X+Y. Also obtain i) P(Z=13) ii) $P(Z \leq 12)$.

Hints and Answers:

0.08 ii) 0.64

5 A radar system has a probability of 0.1 of detecting a certain target during a single scan. Find the probability that the target will be detected i) at least twice in four scans ii) at most once in four scans.

[**Hints and Answers:** i) 0.0523 ii) 0.9477]

6 If the probability that any person 65 years old will be dead within a year is 0.05. Find the probability that out of a group of 7 such persons (i) Exactly one, (ii) none, (iii) at least one, (iv) that more than one,

[**Hints and Answers:** i) 0.2573 ii) 0.6983 iii) 0.3017 iv) 0.9556]

7 For a B (5,*p*) distribution , P(X=1)=0.0768 , P(X=2)=0.2304. Find the value of *p*.

[**Hints and Answers:** 0.6]

8 Suppose $X \rightarrow B(n, p)$,

i) If E(X)=6 and Var(X)=4.2 , find n and p.

ii) If p= 0.6 E(X)=6 find n and Var(X).

iii) If n=25 , E(X)=10, find p and Var(X).

[**Hints and Answers:** i) 20, 0.3 ii) 10 , 2.4 iii) 0.4 , 6 **]**

9 In a summer season a truck driver experiences on an average one puncture is 1000 km . Applying Poisson distribution, find the probability that there will be

i) No Puncture ii) two punctures in a journey of 3000 km

[**Hints and Answers:** i) 0.049787 ii) 0.224042 **]**

10 A book contains 400 misprints distributed randomly throughout its 400 pages. What is probability that a page observed at random, contains at least two misprints?

[**Hints and Answers:** 0.264 **]**

❖❖❖

Unit 5

13

DISTRIBUTIONS: CONTINUOUS DISTRIBUTIONS

Unit Structure

13.0 Objectives

After going through this unit, you will be able to:

- Make familiar with the concept of continuous random variable
- Understand how to study a continuous type distribution
- Make Students to feel it very interesting and can find applications.
- Make the students familiar with applications and properties of exponential distribution.
- To understand why life time distributions of electronic equipments can be assumed to be exponential distributed.
- To understand the importance of normal distributions in the theory of distribution
- To study different properties of normal distribution useful in practice to compute probabilities and expected values.

13.1 Introduction

In continuous set up, many times it is observed that the variable follows a specific pattern. This behavior of the random variable can be described in a mathematical form called ‘ Probability Models’.

In this unit we will study important continuous probability distributions such as Uniform, Exponential, Normal

13.2 Uniform Distribution

In this distribution, the p.d.f. of the random variable remains constant over the range space of the variable.

13.2.1 Definition:

Let X be a continuous random variable on an interval *(c, d)*. X is said to be a uniform distribution with if the p.d.f. is constant over the range *(c, d)*. Further the probability distribution must satisfy the Axioms of probability.

Notation: X → U [c, d]

This distribution is also known as ‘Rectangular Distribution’, as the graph of for this describes a rectangle over the X axis and between the ordinates at X = c and X = d

Probability Density Function (p.d.f.)

p.d.f. of Uniform random variable is given by,

$$f(x) = \frac{1}{d - c} \quad if \quad c < x < d$$

$$= 0 \qquad \text{otherwise}$$

Note that, $0 \leq f(x) \leq 1$ Axiom I is satisfied

$\int_{-\infty}^{\infty} f(x)dx = \int_{c}^{d} \frac{1}{d-c}dx = \frac{1}{d-c}[x]_{c}^{d} = \frac{d-c}{d-c} = 1$ Axiom II is satisfied

(c, d) are parameters of the distribution.

Note: Whenever the value of parameter is known, the distribution is known completely. Knowing the parameter or parameters we can compute probabilities of all events, as well as quantities such as mean, variance etc. Different values of parameters specify different probability distributions of same kind.

A special case of uniform distribution very commonly used in digital electronics is a step function. This is a uniform distribution with parameters (0, 1) and its p.d.f. is

$$f(x) = \begin{cases} 1 & if \quad 0 < x < 1 \\ 0 & Otherwise \end{cases}$$

13.2.2 Mean and Variance of Uniform Distribution

Mean of Uniform random variable with parameter (c, d) is,

Mean = E(X) $= \int_{-\infty}^{\infty} xf(x)dx = \int_{c}^{d} \frac{1}{d-c} x dx = \frac{1}{d-c}\left[\frac{x^2}{2}\right]_{c}^{d} = \frac{d^2-c^2}{2(d-c)} = \frac{d+c}{2}$

Variance of Uniform random variable with parameter (c, d) is,

Variance (X) = $E(X^2) - [E(X)]^2$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \frac{1}{d-c}\int_{c}^{d} x^2 dx = \frac{1}{d-c}\left[\frac{x^3}{3}\right]_{c}^{d} = \frac{d^3-c^3}{3(d-c)} = \frac{d^2+dc+c^2}{3}$

Variance (X) = $E(X^2) - [E(X)]^2 = \frac{d^2+dc+c^2}{3} - \frac{(d+c)^2}{4} = \frac{(d-c)^2}{12}$

Standard Deviation (S.D.) (X)= $\sigma = \sqrt{Var\ (X)} = \sqrt{\frac{(d-c)^2}{12}} = \frac{d-c}{2\sqrt{3}}$

13.2.3 Applications of Uniform Distribution

1. It is used to represent the distribution of rounding-off errors.
2. It is used in life testing and traffic flow experiments
3. It is being the simplest continuous distribution is widely applied in research.
4. If a random variable Y follows any continuous distribution then is distribution function X= F (Y) can be shown to follow U[0, 1]. This result is useful in sampling from any continuous probability distributions.

Example1: If mean and variance of a U [c, d] random variable are 5 and 3 respectively, determine the values of c and d.

Solutions: Let X is a random variable which follows Uniform distribution with parameters (c,d)

$X \rightarrow U\,[c,d]$

Mean = E(X) = $\frac{d+c}{2}$ = 5 $\therefore$ c + d = 10

Variance *(X)* = $\frac{(d-c)^2}{12}$= *3* $\therefore (d-c)^2 = 36$ $\therefore$ d –c = 6

$\therefore$ c + d = 10 and d –c = 6 after solving these equations we get c =2 and d=8

Example2: X is continuous random variable with *f(x)* constant (*k*) over $a \le X \le b$ *and 0* elsewhere. Find i) p.d.f ii) Mean

Solutions:

i) P.d.f. is

$$f(x) = k \qquad For \quad a \le X \le b$$
$$= 0 \qquad \text{Elsewhere}$$

f(x) must satisfy Axioms of probability, Hence

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{a}^{b} kxdx = k\,(b-a) = 1$$

$\therefore$ k = $\frac{1}{(b-a)}$

Thus the p.d.f. is

$$f(x) = \frac{1}{b-a} \qquad For \quad a \le X \le b$$
$$= 0 \qquad \text{Elsewhere}$$

ii) Mean =

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{a}^{b} \frac{1}{b-a} xdx = \frac{1}{b-a}\int_{a}^{b} xdx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a}{2(b-a)} = \frac{a+b}{2}$$

Example3: Let X → U [-a, a]. Find the value of a such that P[|X|] > 1 $= \frac{6}{7}$

Solutions: P[|X|] > 1 = 1 - P[|X|] ≤ 1

= 1 - P [-1 ≤ X ≤ 1]

∴ P [-1 ≤ X ≤ 1] = $\int_{-1}^{1} xf(x)dx = \int_{-1}^{1} \frac{1}{2a} dx = \frac{1}{7}$

∴ a = 7

Example4: Amar gives a birthday party to his friends. A machine fills the ice cream cups. The quantity of ice cream per cup is uniformly distributed over 200 gms to 250 gms. i) what is the probability that a friend of Amar gets a cup with more than 230gms of ice cream? ii) If in all twenty five people attended the party and each had two cups of ice cream, what is the expected quantity of ice cream consumed in the party?

Solutions: Let X = Quantity of ice cream per cup, X → U [c, d] → [200, 250]

i) P [X > 230] = 1 – P [X ≤ 230]

$= 1 - \frac{(230-200)}{(250-200)} = 1 - 0.6 = 0.4$

ii) On an average, quantity per cup is given by

Mean = (200 + 250) / 2 = 225 gms

∴ Total quantity consumed = 225 × 2 × 25

= 11250 gms = 11.25 kg

Example5: X is Uniform continuous random variable with p.d.f. given as,

$$f(x) = \frac{1}{8} \qquad For \quad 0 \le x \le 8$$

$$= 0 \qquad \text{Elsewhere}$$

Determine i) P(2 ≤ X ≤ 5) ii) P(3 ≤ X ≤ 7) iii) P(X ≤ 6)

iv) P(X > 6) v) P(2 ≤ X ≤ 12)

Solutions:

i) P(2 ≤ X ≤ 5) = $\int_2^5 \frac{1}{8} dx = \frac{1}{8}[x]_2^5 = \frac{3}{8}$

ii) P(3 ≤ X ≤ 7) = $\int_3^7 \frac{1}{8} dx = \frac{1}{8}[x]_3^7 = \frac{4}{8} = \frac{1}{2}$

iii) P(X ≤ 6) = $\int_{-\infty}^6 \frac{1}{8} dx = \int_0^6 \frac{1}{8} dx = \frac{1}{8}[x]_0^6 = \frac{6}{8} = \frac{3}{4}$

iv) P(X > 6) = 1 – P (X ≤ 6) = 1 - $\frac{3}{4} = \frac{1}{4}$

v) P(2 ≤ X ≤ 12) $= \int_2^{12} \frac{1}{8} dx = \int_2^8 \frac{1}{8} dx + \int_8^{12} (0) dx = \frac{6}{8} - 0 = \frac{3}{4}$

13.3 Exponential Distribution

Exponential distribution is one of the important distributions and has wide applications in operation research, life testing experiments, in reliability theory and survival analysis. Life time of an electronic component, time until decay of a radioactive element is modeled by the exponential distribution. Exponential distribution is closely related to Poisson distribution. If the number of occurrences of the event follows Poisson distribution then the distribution of the time that elapses between these successive occurrences follows an exponential distribution. For example, if the number of patients arriving at hospital follows Poisson distribution, then the time gap between the current arrival and the next arrival is exponential. This has specific application in Queuing Theory.

13.3.1 Definition: Let X be a continuous random variable taking non-negative values is said to follow exponential distribution with mean θ if its probability density function (p.d.f.) is given by

$$f(x) = \frac{1}{\theta} e^{-x/\theta} \quad ; x \geq 0,\ \theta > 0$$

$$= 0 \quad ; \text{otherwise}$$

Notation: X → Exp(θ)

Note : (1) $f(x) \geq 0 \quad \because x \geq 0,\ \theta > 0$ and $e^{-\frac{x}{\theta}} > 0$

(2) $\int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} \frac{1}{\theta} e^{-x/\theta} dx = \frac{1}{\theta}\left[-\frac{e^{-x/\theta}}{1/\theta}\right]_0^{\infty} = \left[-e^{-x/\theta}\right]_0^{\infty} = 1$

From **(1) and (2) it is clear that *f(x)* is p.d.f.**

(3) if θ = 1, then the distribution is called standard exponential distribution.

Hence probability density function is given by

$$f(x) = e^{-x} \quad ; x \geq 0,\ \theta > 0$$
$$= 0 \quad ; \text{otherwise}$$

13.3.2 Mean and Variance of Exponential Distribution

If X → Exp(θ), then its p.d.f is given by

$$f(x) = \frac{1}{\theta} e^{-x/\theta} \quad ; x \geq 0,\ \theta > 0$$
$$= 0 \quad ; \text{otherwise}$$

Mean of Exponential random variable with parameter (0,∞) is,

Mean = E(X) = $\int_0^\infty x f(x)dx = \int_0^\infty \frac{1}{\theta} e^{-x/\theta} x dx = \int_0^\infty \frac{x}{\theta} e^{-x/\theta} dx = \theta$

Variance of Exponential random variable with parameter (0,∞) is,

Variance *(X)* = E(X^2) – [E(X)]^2

E(X^2) = $\int_0^\infty x^2 f(x)dx = \int_0^\infty \frac{1}{\theta} e^{-x/\theta} x x^2 dx = \int_0^\infty \frac{x^2}{\theta} e^{-x/\theta} dx = 2\theta^2$

Variance *(X)* = E(X^2) – [E(X)]^2 = $2\theta^2 - \theta^2 = \theta^2$

Standard Deviation (S.D.) (X)= σ = $\sqrt{Var\ (X)} = \sqrt{\theta^2} = \theta$

Note: For Exponential distribution, mean and standard deviation are same.

13.3.3 Distribution Function of Exponential Distribution i.e. Exp (θ)

Let X → Exp(θ). Then the distribution function is given by

F_x (X) = P [X≤ x] = $\int_0^x f(t)dt = \int_0^x \frac{1}{\theta} e^{-t/\theta} dt = \frac{1}{\theta}\int_0^x e^{-t/\theta} dt = \frac{1}{\theta}\left[-\frac{e^{-t/\theta}}{1/\theta}\right]_0^x = \left[-e^{-t/\theta}\right]_0^x$

F_x (X) = 1- $e^{-x/\theta}$; x > 0, θ > 0

Note: When X is life time of a component, P [X > x] is taken as reliability function or survival function.

Here P [X > x] = 1 – P [X ≤ x] = 1 - F_x (X) = 1- $e^{-x/\theta}$; x > 0, θ > 0

13.3.4 Applications of Exponential Distribution

Exponential distribution is used as a model in the following situations:

1) Life time of an electronic component
2) The time between successive radioactive emissions or decay
3) Service time distribution at a service facility.
4) Amount of time until interrupt occurs on server
5) Time taken to serve customer at ATM counter.

Example1: Suppose that the life time of a XYZ company T.V. tube is exponentially distributed with a mean life 1600 hours. What is the probability that

i) The tube will work upto 2400 hours?

ii) The tube will survive after 1000 hours?

Solutions: Let X : Number of hours that T.V. tube work

$X \rightarrow Exp(\theta)$ where $\theta = 1600$

We know that if $X \rightarrow Exp(\theta)$ then

$P[X \leq x] = 1- e^{-x/\theta}$; $x > 0, \theta > 0$

i) $P[X \leq 2400] = 1- e^{-2400/1600} = 1 - e^{-1.5} = 1 - 0.223130 = 0.77687$

ii) $P[X > 1000] = - e^{-1000/1600} = e^{-0.625} = 0.5353319$

$(\because$ If $X \rightarrow Exp(\theta)$, $P[X > x] = 1- e^{-x/\theta} = e^{-x/\theta})$

Example2: The life time in hours of a certain electric component follows exponential distribution with distribution function.

$F(x) = 1- e^{-0.004x}$; $x \geq 0$

i) What is the probability that the component will survive 200 hours?
ii) What is the probability that it will fail during 250 to 350 hours?
iii) What is the expected life time of the component?

Solutions: Let X : Life time (in hours) of the electric component, $X \rightarrow Exp(\theta)$ and

$F(x) = 1- e^{-0.004x}$; $x \geq 0$

$\therefore e^{-0.004x} = e^{-x/\theta}$

$\therefore\ -0.004\, x = -x/\theta$

$\therefore\ 0.004 = 1/\theta$

$\therefore\ \theta = \frac{1}{0.004} = 250\ hours$

i) P [X > 200]= $e^{-x/\theta} = e^{-\frac{200}{250}} = e^{-0.8} = 0.449329$

ii) P [250 < X < 350] = P[X > 250] – P [X > 350]

$= e^{-\frac{250}{250}} - e^{-\frac{350}{250}} = e^{-1} - e^{-1.4}$

$= 0.367879 - 0.246597$

iii) Expected life time of the component = E(X) = θ = 250 hours

Example3: The life time of a microprocessor is exponentially distributed with mean 3000 hours. What is the probability that

i) The microprocessor will fail within 300 hours?

ii) The microprocessor will function for more than 6000 hours?

Solutions: Let X : Life time (in hours) of the microprocessor,
X → Exp(θ) and θ = 3000

i) P (Microprocessor will fail within 300 hours)

$= P(X \leq 300)$

$= \int_0^{300} \frac{1}{3000} e^{-\frac{x}{3000}} dx = 0.0951$

ii) P (Microprocessor will function for more than 6000 hours)

$= P(X \geq 6000)$

$= \int_{6000}^{\infty} \frac{1}{3000} e^{-\frac{x}{3000}} dx = e^{-2} = 0.1353$

13.4 Normal Distribution

In this section we deal with the most important continuous distribution, known as normal probability distribution or simply normal distribution. It is important for the reason that it plays a vital role in the theoretical and applied statistics. It is one of the commonly used distribution. The variables such as intelligence quotient, height of a person, weight of a person, errors in measurement of physical quantities follow normal distribution. It is useful in statistical quality control, statistical inference, reliability theory, operation research, educational and psychological statistics. Normal distribution works as a limiting distribution to several distributions such as Binomial, Poisson.

The normal distribution was first discovered by DeMoivre (English Mathematician) in 1733 as limiting case of binomial distribution. Later it was applied in natural and social science by Laplace (French Mathematician) in 1777. The normal distribution is also known as Gaussian distribution in honour of Karl Friedrich Gauss(1809).

13.4.1 Definition:

A continuous random variable X is said to follow normal distribution with parameters mean μ and standard deviation $\sigma^{2,}$ if its probability density function (p.d.f.) is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(x-\mu)^2} \;\; ; -\infty < x < \infty;\ -\infty < \mu < \infty;\ \sigma > 0$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \;\; ; -\infty < x < \infty;\ -\infty < \mu < \infty;\ \sigma > 0$$

Note:

1. The mean μ and standard deviation σ^2 are called the parameters of Normal distribution.

   Notation: The normal distribution is expressed by $\mathbf{X \rightarrow N(\mu, \sigma^2)}$

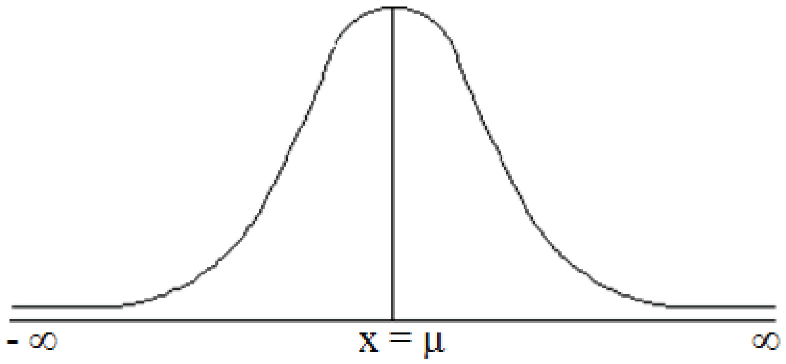
2. If $\mu = 0$ and $\sigma^2 = 1$,then the normal variable is called as standard normal variable.

   $\mathbf{X \rightarrow N(\mu, \sigma^2)}$ => $\mathbf{X \rightarrow N(0, 1)}$ **.** Generally it is denoted by Z. The p.d.f. of Z is

   $$f(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} \;\; ; -\infty < z < \infty;\ \ with\ \pi = 3.141159\ and\ e = 2.71828$$

   The advantage of the above function is that it doesn' t contain any parameter. This enable us to compute the area under the normal probability curve.

3. The probability density curve of $N(\mu, \sigma^2)$ is bell shaped, symmetric about μ and mesokurtic . Naturally, the curve of standard normal distribution is symmetric around zero.



4. The maximum height of probability density curve is = $1 / (\sigma \sqrt{2\pi})$

5. As the curve is symmetric about μ, the mean, median and mode coincide and all are equal to μ.

6. The parameter σ^2 is also the variance of X, hence the standard deviation (s.d.) (X) = σ

Relation between $N(\mu, \sigma^2)$ and $N(0, 1)$

If $X \rightarrow N(\mu, \sigma^2)$ then $Z = \frac{(x-\mu)}{\sigma} \rightarrow N(0, 1)$. This result is useful while computing probabilities of $N(\mu, \sigma^2)$ variable. The statistical table give probabilities of a standard normal i.e. N(0,1) variable.

13.4.2 Properties of Normal Distribution:

- The normal curve is bell shaped and is symmetric at x = μ
- Mean, median, and mode of the distribution are coincide
  i.e.,Mean = Median = Mode = μ
- It has only one mode at x = μ (i.e., unimodal)
- The points of inflection are at $x = \mu \pm \sigma$
- The x axis is an asymptote to the curve (i.e. the curve continues to approach but never touches the x axis)
- The first and third quartiles are equidistant from median.
- The mean deviation about mean is 0.8 σ
- Quartile deviation = 0.6745 σ
- If X and Y are independent normal variates with mean μ_1 and μ_2, and variance σ_1^2 and σ_2^2 respectively then their sum (X + Y) is also a normal variate with mean $(\mu_1 + \mu_2)$ and variance $(\sigma_1^2 + \sigma_2^2)$

- Area Property $P(\mu - \sigma < x < \mu + \sigma) = 0.6826$

$$P(\mu - 2\sigma < x < \mu + 2\sigma) = 0.9544$$

$$P(\mu - 3\sigma < x < \mu + 3\sigma) = 0.9973$$

13.4.3 Properties and Applications of Normal Distribution

Let X be random variable which follows normal distribution with mean μ and variance σ^2 .The standard normal variate is defined as $Z = \frac{(X - \mu)}{\sigma}$ which follows standard normal distribution with mean 0 and standard deviation 1 i.e. $Z \rightarrow$ **N (0,1).** The p.d.f. of Z is

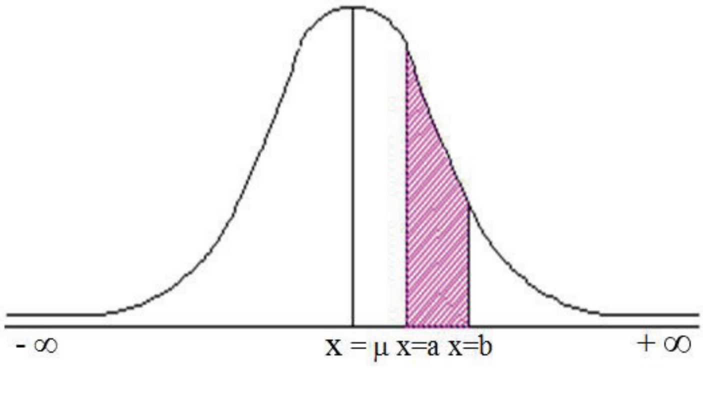
$$f(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} \; ; -\infty < z < \infty; \; with\, \pi = 3.141159 \; and \; e = 2.71828$$

The advantage of the above function is that it doesn't contain any parameter. This enable us to compute the area under the normal probability curve.

Area Properties of Normal Curve:

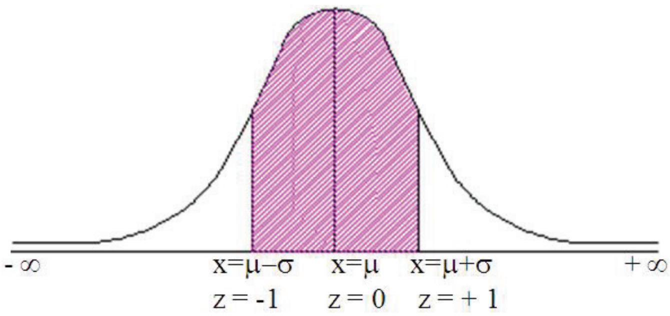
The total area under the normal probability curve is 1. The curve is also called standard probability curve. The area under the curve between the ordinates at x = a and x = b where a < b, represents the probabilities that x lies between x = a and x = b

i.e., $P(a \le x \le b)$



To find any probability value of X, we first standardize it by using $Z = \frac{(X - \mu)}{\sigma}$, and use the area probability normal table.

For e.g. the probability that the normal random variable X to lie in the interval (μ- σ, μ + σ) is given by



$P(\mu - \sigma < \xi < \mu+\sigma) = P(-1 \leq \zeta \leq 1)$

$= 2P(0 < z < 1)$ (Due to symmetry)

$= 2\ (0.3413)$ (from the area table)

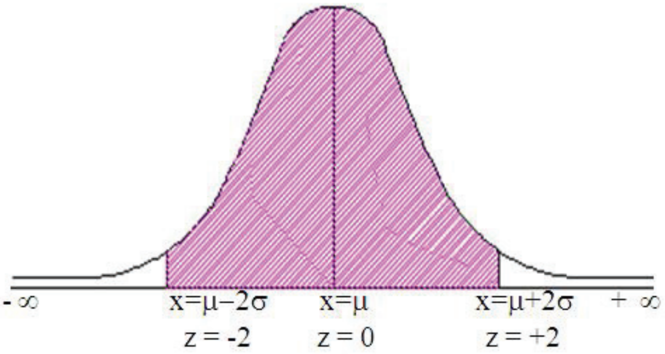
$= 0.6826$

$P(\mu - 2\sigma < \xi < \mu+2\sigma) = P(-2 < z < 2)$

$= 2P(0 < z < 2)$ (Due to symmetry)

$= 2\ (0.4772)$

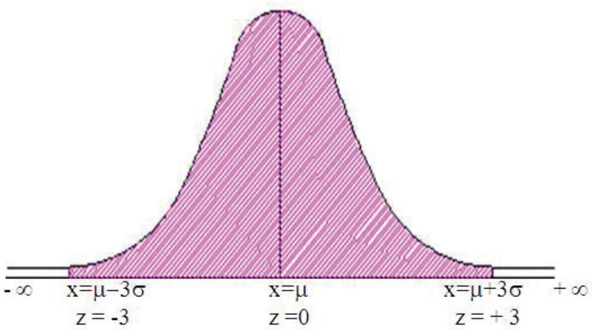
$= 0.9544$



$P(\mu - 3\sigma < \xi < \mu + 3\sigma) = P(-3 < \zeta < 3)$

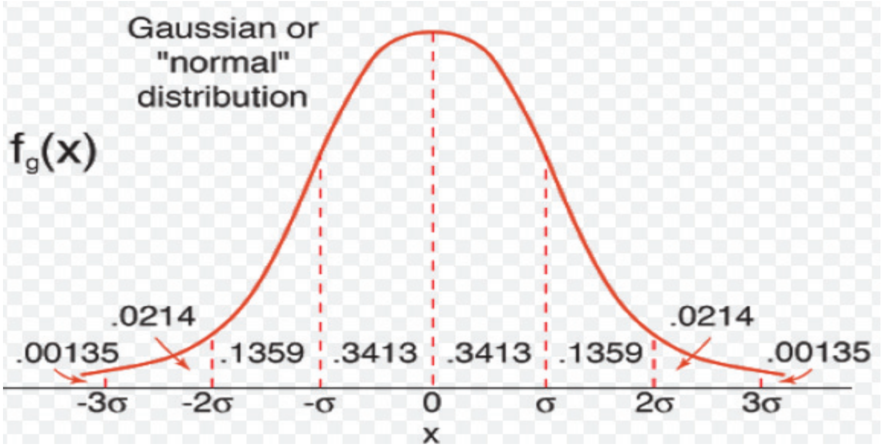
$= 2P(0 < z < 3)$ (Due to symmetry)

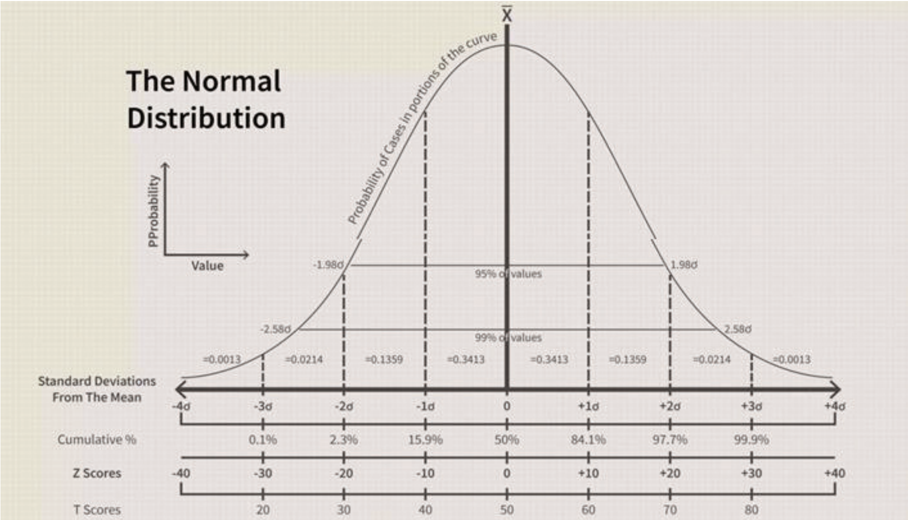
$= 2\ (0.49865) = 0.9973$



Thus we expect that the values in a normal probability curve will lie between the range

$\mu \pm 3\sigma$, though theoretically it range from $-\infty$ to ∞.

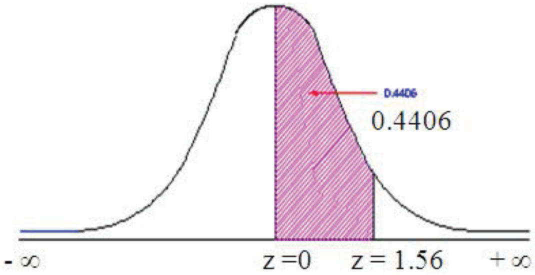




Example 1: Find the probability that the standard normal variate lies between 0 and 1.56

Solution: $P(0 < z < 1.56)$ = Area between $z = 0$ and $z = 1.56$

$= 0.4406$ (from table)

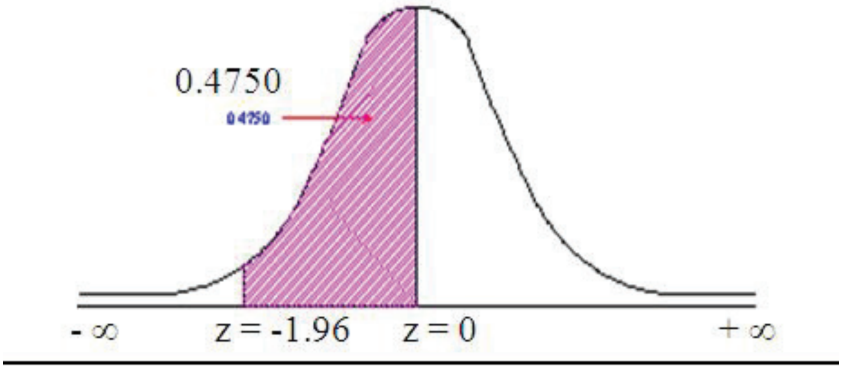


Example2: Find the area of the standard normal variate from –1.96 to 0.

Solution: Area between z = 0 & z =1.96 is same as the area z = –1.96 to z = 0

P(-1.96 < z < 0) = P(0 < z < 1.96) (by symmetry)

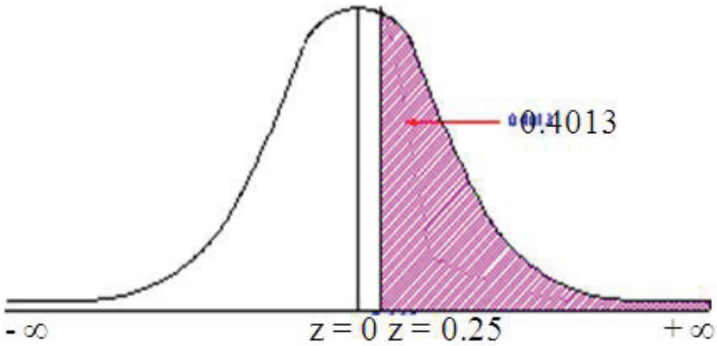
= 0.4750 (from the table)



Example 3: Find the area to the right of z = 0.25

Solution: P(z > 0.25) = P(0 < z < ∞) – P(0 < z < 0.25)

= 0.5000 - 0.0987 (from the table) = 0.4013

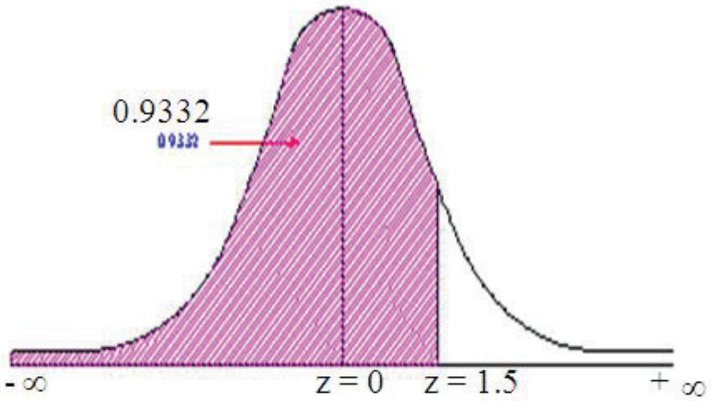


Example 4: Find the area to the left of z = 1.5

Solution: $P(z < 1.5) = P(-\infty < z < 0) + P(0 < z < 1.5)$

$= 0.5 + 0.4332$ (from the table)

$= 0.9332$



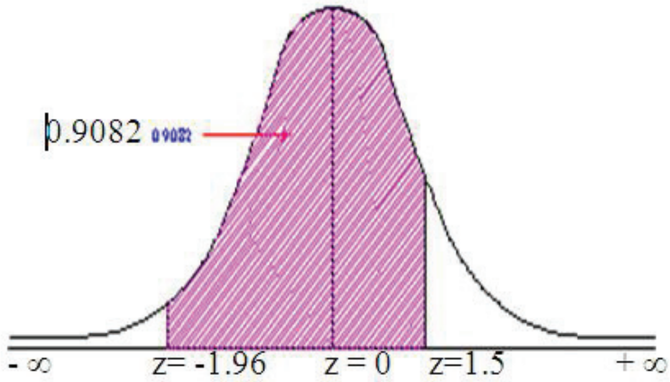
Example 5: Find the area of the standard normal variate between –1.96 and 1.5

Solution: $P(-1.96 < z < 1.5) = P(-1.96 < z < 0) + P(0 < z < 1.5)$

$= P(0 < z < 1.96) + P(0 < z < 1.5)$

$= 0.4750 + 0.4332$ (from the table)

$= 0.9082$



Example 6: Given a normal distribution with $\mu = 50$ and $\sigma = 8$, find the probability that x assumes a value between 42 and 64

Solution: Given that $\mu = 50$ and $\sigma = 8$ The standard normal variate $Z = \frac{(X - \mu)}{\sigma}$

If X = 42, $Z_1 = \frac{(X-\mu)}{\sigma} = \frac{(42-50)}{8} = \frac{-8}{8}$ = -1

If X = 64, $Z_2 = \frac{(X-\mu)}{\sigma} = \frac{(64-50)}{8} = \frac{14}{8}$ = 1.75

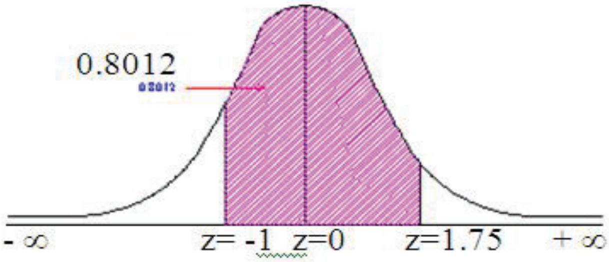
$\therefore$ P(42 < x < 64) = P(−1 < z <1.75)

= P(−1< z < 0) + P(0 < z <1.95)

= P(0<z<1) + P (0 < z <1.75) (by symmetry)

= 0.3413 +0 .4599 (from the table)

= 0 .8012



Example 7: Students of a class were given an aptitude test. Their marks were found to be normally distributed with mean 60 and standard deviation 5. What percentage of students scored.

i) More than 60 marks (ii) Less than 56 marks (iii) Between 45 and 65 marks

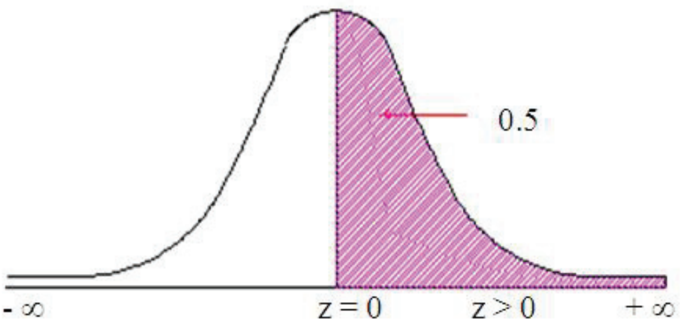
Solution: Given that μ = 60 and σ = 5 The standard normal variate $Z = \frac{(X-\mu)}{\sigma}$

i) If X = 60, $Z = \frac{(X-\mu)}{\sigma} = \frac{(60-60)}{5} = \frac{0}{5}$ = 0

$\therefore$ P(x > 60) = P(z > 0)

= P(0 < z < ∞) = 0.5000

Hence percentage (%) of students scored more than 60 marks is
0.5000 * 100 = 50%



ii) If X = 56, $Z = \frac{(X-\mu)}{\sigma} = \frac{(56-60)}{5} = \frac{-4}{5} = -0.8$

$\therefore$ P(x < 56) = P(z < -0.8)

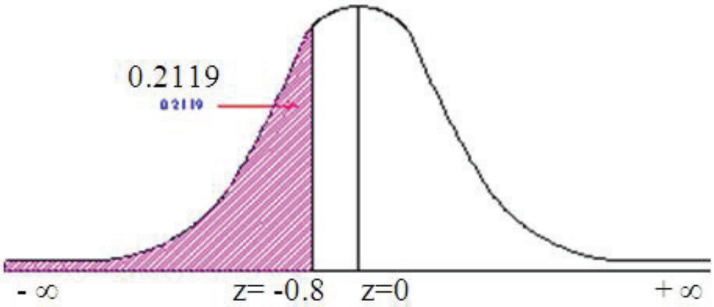
= P(- ∞ < z < 0) – P(–0.8 < z < 0) (by symmetry)

= P(0 < z < ∞) – P(0 < z < 0.8)

= 0.5 – 0.2881 (from the table)

= 0.2119

Hence the percentage of students score less than 56 marks is
0.2119 * (100) = 21.19 %



iii) If X = 45, $Z = \frac{(X-\mu)}{\sigma} = \frac{(45-60)}{5} = \frac{-15}{5} = -3$

X = 65, $Z = \frac{(X-\mu)}{\sigma} = \frac{(65-60)}{5} = \frac{5}{5} = 1$

P(45 < x < 65) = P(–3 < z < 1)

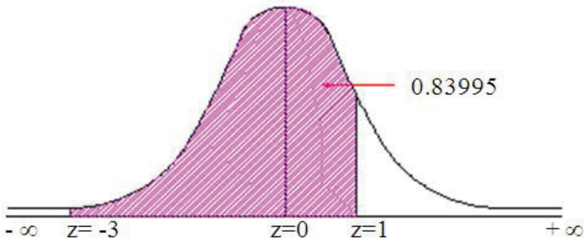
= P(–3 < z < 0) + P (0 < z < 1)

= P(0 < z < 3) + P(0 < z < 1) (by symmetry)

= 0.4986 + 0.3413 (from the table)

= 0.8399

Hence the percentage of students scored between 45 and 65 marks is 0.8399*(100) = 83.99 %



Example 8: X is normal distribution with mean 2 and standard deviation 3. Find the value of the variable x such that the probability of the interval from mean to that value is 0.4115

Solution: Given μ = 2, σ = 3

Suppose z_1 is required standard value,

Thus P $(0 < z < z_1) = 0.4115$

From the table the value corresponding to the area 0.4115 is 1.35 (i.e. $z_1 = 1.35$)

$$\boldsymbol{Z_1 = \frac{(X - \mu)}{\sigma} = \frac{(X - 2)}{3} = 1.35}$$

$(X – 2)= 3 * 1.35$

$X = 4.05 + 2 = 6.05$

Example 9: In a normal distribution 31 % of the items are under 45 and 8 % are over 64. Find the mean and variance of the distribution.

Solution: Let x denotes the items are given and it follows the normal distribution with mean μ and standard deviation σ

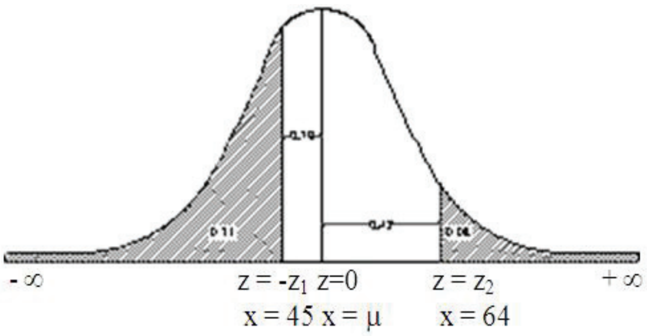
The points x = 45 and x = 64 are located as shown in the figure.

Since 31 % of items are under x = 45,

position of x into the left of the ordinate x = μ

Since 8 % of items are above x = 64,

position of this x is to the right of ordinate x = μ



When $X = 45$, $Z = \frac{(X-\mu)}{\sigma} = \frac{(45-\mu)}{\sigma} = -z_1$ (Say)

Since x is left of $x = \mu$, Z_1 is taken as negative

When $X = 64$, $Z = \frac{(X-\mu)}{\sigma} = \frac{(64-\mu)}{\sigma} = z_2$ (Say)

From the diagram $P(x < 45) = 0.31$

$P(z < -z_1) = 0.31$

$P(-z_1 < z < 0) = P(-\infty < z < 0) - P(-\infty < z < z_1)$

$P(-z_1 < z < 0) = 0.5 - 0.31 = 0.19$

$P(0 < z < z_1) = 0.19$ (by symmetry)

$z_1 = 0.50$ (from the table)

Also from the diagram $P(X > 64) = 0.08$

$P(0 < z < z_2) = P(0 < z < \infty) - P(z_2 < z < \infty)$

$= 0.5 - 0.08 = 0.42$

$z_2 = 1.40$ (from the table)

Substituting the values of z_1 and z_2 we get $Z = \frac{(X-\mu)}{\sigma} = \frac{(45-\mu)}{\sigma} = -0.50$

$$Z = \frac{(X-\mu)}{\sigma} = \frac{(64-\mu)}{\sigma} = 1.40$$

Solving $\mu - 0.50\,\sigma = 45$ (1)

$\mu + 1.40\,\sigma = 64$ ----- (2)

$(2) - (1) \Rightarrow 1.90\,\sigma = 19 \Rightarrow \sigma = 10$

Substituting $\sigma = 10$ in (1)

$\mu = 45 + 0.50\,(10) = 45 + 5.0 = 50.0$

Hence mean = 50 and variance = $\sigma^2 = 100$

13.4.4 Z table and its User Manual

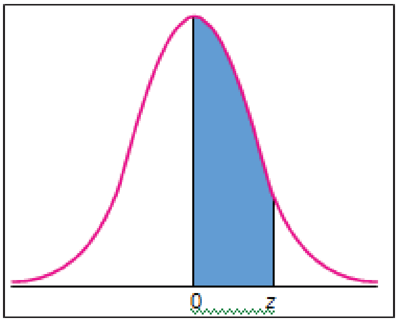


FIGURE 1

Area from $z = —2.34$ to 0 Equals Area from $z = 0$ to 2.34

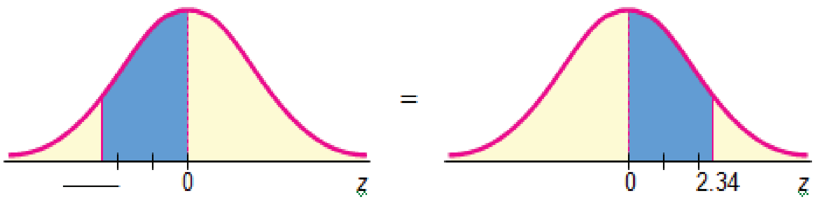


FIGURE 2

Patterns for Finding Areas Under the Standard Normal

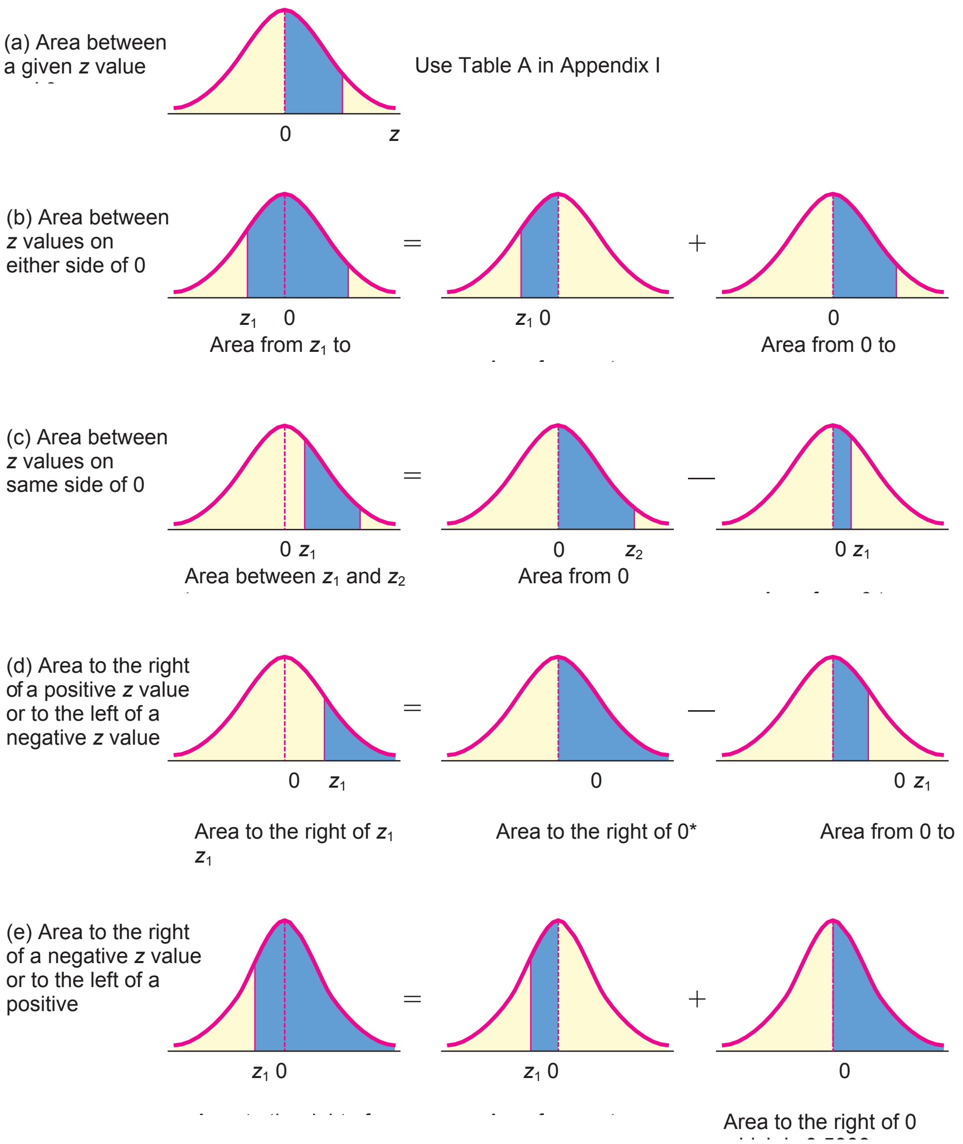


TABLE A Areas of a Standard Normal Distribution
(Alternate Version of Table)

The table entries represent the area under the standard normal curve from 0 to the specified value of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999

For values of z greater than or equal to 3.70, use 0.4999 to approximate the shaded area under the standard normal curve.

13.5 Summary

In this chapter, **Continuous Distributions**, its types Uniform, Exponential with its mean , variance and its application is discussed.

A special and very useful distribution called as Normal distribution and its application in day to day life also discussed.

Distribution	**Definition**	**Mean E(X)**	**Variance (X)**
Uniform	**X → U [c, d]** $f(x) = \dfrac{1}{d-c}$ if $c < x < d$ $= 0$ otherwise	$\dfrac{(d+c)}{2}$	$\dfrac{(d-c)^2}{2}$
Exponential	$f(x) = \dfrac{1}{\theta} e^{-x/\theta}$; $x \geq 0$, $\theta > 0$ $= 0$; otherwise	θ	θ^2

Normal distribution with parameters mean μ and standard deviation σ^2, **X → N(μ, σ²)**

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(x-\mu)^2} \quad ; -\infty < x < \infty; \; -\infty < \mu < \infty; \; \sigma > 0$$

13.6 Unit End Exercise

1 If X → U (2, 8) State Mean and Variance of X.

[**Hints and Answers**:Mean = 5, Variance = 3**]**

2 If X is a continuous random variable having uniform distribution with parameters 'a' and 'b' such that mean =1 and variance = 3.Find P(X <0)

[**Hints and Answers**: 1/3 **]**

3 On a route buses travels at an half hourly intervals, starting at 8.00 am. On a given day, a passenger arrives at the stop at a time X, which is uniformly distributed over the interval [8.15am, 8.45am**]**. What is the probability that the passenger will have to wait for more than 15 minutes for the bus?

[**Hints and Answers:** 1/2 **]**

4 The mileage which car owners get with certain kind of radial tyres (measured in '000' km) is a random variable with mean 40. Find the probabilities that one of these tyres will last.

i) At least 20000 km

ii) At most 30000 km

[**Hints and Answers:** i) 0.6065 ii) 0.5276 **]**

5 The lifetime of a microprocessor is exponentially distributed with mean 3000 hours. Find the probability that

i) The microprocessor will fail within 300 hours

ii) The microprocessor will function for more than 6000 hours

[**Hints and Answers:** i) 0.0951 ii) 0.1353 **]**

6 The time until next earthquake occurs in a particular region is assumed to be exponentially distributed with mean 1/2 per year. Find the probability that the next earthquake happens

i) Within 2 years

ii) After one and half year

[**Hints and Answers:** i) 0.9817 ii) 0.0497 **]**

7 Let X be normally distributed random variable with parameters (100,25) Calculate:

i) $P(X \geq 108)$ ii) $P(90 \leq X \leq 110)$

[**Hints and Answers:** i) 0.27425 ii) 0.9545]

8 If X is a standard normal variable [$X \rightarrow N(0,1)$], determine the following probabilities using normal probability tables;

(i) $P(X \geq 1.3)$ (ii) $P(0 \leq X \leq 1.3)$ (iii) $P(X \leq 1.3)$ (iv) $P(-2 \leq X \leq 1.3)$ (v) $P(0.5 \leq X \leq 1.3)$

[**Hints and Answers:** (i)0.09680 (ii) 0.4032 (iii)0.9032 (iv)0.88045 (v)0.21174]

9 The annual rainfall (in inches) in a certain region is normally distributed with mean = 40 and standard deviation = 4. What is the probability that in the current year it will have rainfall of 50 inches or more

[**Hints and Answers:** i) $P(X \geq 50)=0.0062097$]

10 The intelligent quotient (IQ) of adults is known to be normally distributed with mean 100 and variance 16. Calculate the probability that a randomly selected adult has IQ lying between 90 and 110.

[**Hints and Answers:** 0.98758]

❖❖❖