Examination : SYBA_Semester IV **Subject : Mathematics (Paper III)**
Exam Date : 04-05-2019 **Q.P.Code : 66042**

(3 Hours) [Total Marks: 100]

Note: (i) All questions are compulsory.
(ii)Figures to the right indicate marks for respective parts.

Q.1	Choose correct alternative in each of the following (20)			
$i.$	Let $a \in G$ such that O(a)= n then a^m=e then			
	(a)	$n\|m$	(b)	$m\backslash n$
(a)	(c)	m=n	(d)	None of the above
$ii.$	The set U_n is forms a group under the binary operation			
	(a)	'+'	(b)	'-'
(c)	(c)	'•'	(d)	None of the above
$iii.$	Let S_n denote the permutation group. Then $\|S_n\|=$			
	(a)	n	(b)	$2n$
(c)	(c)	$n!$	(d)	None the above
$iv.$	Let Z(G) be the center of the group G. Then Z(G) is defined as			
	(a)	$\{x \| \forall a \in G, ax = xa\}$	(b)	$\{x \| \forall a \in G, ax \neq xa\}$
(a)	(c)	$\{x \| \forall a \in G, ax \notin G\}$	(d)	None of the above
$v.$	Suppose G is a cyclic group such that G has exactly three subgroups viz. $G, \{e\}$ and a subgroup of order 5. Then the order of G is			
	(a)	5	(b)	10
(c)	(c)	25	(d)	125
$vi.$	The number of subgroups of $(\mathbb{Z}_{20}, +)$ is			
	(a)	6	(b)	5
(a)	(c)	3	(d)	2
$vii.$	The generators of $20\mathbb{Z} \cap 30\mathbb{Z}$ are			
	(a)	$60, -60$	(b)	$10, -10$
(a)	(c)	$20, 30$	(d)	None of the above
$viii.$	Let H be a subgroup of G and $a, b \in G$ then $aH = bH$ if and only if			
	(a)	$a \in H$	(b)	$ab \in H$
(c)	(c)	$a^{-1}b \in H$	(d)	None of these
$ix.$	Let $\phi: \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}$ is a homomorphism given by $\phi(x) = 3x$ then $ker\phi =$			
	(a)	$\{\bar{0}, \bar{1}\}$	(b)	$\{\bar{0}, \bar{4}, \bar{8}\}$
(b)	(c)	$\{\bar{0}, \bar{4}\}$	(d)	None of these
$x.$	The number of group homomorphism of V_4 (Klien's four group) is			
	(a)	4	(b)	2
(d)	(c)	3	(d)	6
Q2.	Attempt any **ONE** question from the following: (08)			
$a)$	i.	Let H and K be the subgroup of a group G. Then prove that HK is a subgroup of G if and only if HK=KH.		
		1 1		

	Let H, K and HK be subgroups of G. We will have to show that HK = KH Let any $x \in HK$ Since HK is a subgroup, $x^{-1} \in HK$ $\Rightarrow x^{-1} = ab$ where $a \in H, b \in K$ $\therefore x = (ab)^{-1}$ $\Rightarrow x = b^{-1}a^{-1} \in KH$ (as H and K are subgroups [illegible] ∈ K) By (1), (2) HK = KH if HK is a subgroup of G **Conversely** Suppose HK = KH We will prove that HK is a subgroup of G Consider any $x, y \in HK$ $x \in HK \Rightarrow x = h_1k_1$ for some $h_1 \in H, k_1 \in K$ $y \in HK \Rightarrow y = h_2k_2$ for some $h_2 \in H, k_2 \in K$ $xy^{-1} = (h_1k_1)(h_2k_2)$ $= (h_1k_1)(k_2^{-1}h_2^{-1})$ $= h_1(k_1k_2^{-1}h_2^{-1})$ $k_1, k_2 \in K$ and K is a subgroup $\Rightarrow k_1k_2^{-1} \in K$ Let $k_1k_2^{-1} = k_3$ for some $k_3 \in K$ $\therefore xy^{-1} = h_1k_3h_2^{-1}$ $= h_1(k_3h_2^{-1})$ Now, $k_3h_2^{-1} \in KH = HK$ $\Rightarrow k_3h_2^{-1} = h_3k_4$ for same $h_3 \in H, k_4 \in K$ $\therefore xy^{-1} = h_1(h_3k_4)$ $= (h_1h_3)k_4 \in HK$ (Thus, for any $x, y \in HK$ $xy^{-1} \in HK$ $\Rightarrow$ HK is a subgroup of G	1 1 1 1 1 1	
	ii.	Show that $(\mathbb{Z}, *)$ is a group where '$*$' is defined as $a * b = a + b - 4,\ a, b \in \mathbb{Z}$	
		(1) Consider any $a, b \in \mathbb{Z}$, $a * b = a + b - 4 \in \mathbb{Z}$ Implies $\mathbb{Z}$ is closed with respect to * (2) Consider any $a, b, c \in \mathbb{Z}$ $a * (b * c) = a * (b + c - 4)$ 1 2 3	1

		$= a + (b + c - 4) - 4$ ··· (by d $= a + b + c - 8$ Also $(a * b) * c = (a + b - 4) * c$ $= a + b - 4 + c - 4$ $= a + b + c - 8$ $= a * (b * c) = (a * b) * c$ Implies * is associative in $\mathbb{Z}$ **(3)** Consider any $a \in \mathbb{Z}$. We have to find an element $e \in \mathbb{Z}$ such that $a * e = a$. i.e. $a + e - 4 = a$ $e = 4 \in \mathbb{Z}$ This $\exists\ e = 4 \in \mathbb{Z}$ such that $a * e = a + e - 4 = a + 4 - 4 = a$ Also, $e * a = e + a - 4 = 4 + a - 4 = a$ so, $4 \in \mathbb{Z}$ is the identity element w .r. t * **(4)** Consider any $a \in \mathbb{Z}$ We have to find $b \in \mathbb{Z}$ such that $a * b = e = 4$ i.e. $a + b - 4 = 4$ $a + b = 8$ $b = 8 - a \in \mathbb{Z}$ (as $a \in \mathbb{Z}$) Thus, $\forall\ a \in \mathbb{Z},\ \exists\ b = 8 - a \in \mathbb{Z}$ such that $a * b = a * (8 - a)$ $= a + 8 - a - 4$ $= 4$ Also, $b * a = (8 - a) * a$ $= 8 - a + a - 4$ $= 4$ Thus $\forall\ a \in \mathbb{Z}, \exists\ b = 8 - a \in \mathbb{Z}$ such that $a * b = b * a = e$ $\therefore$ $\mathbb{Z}$ satisfies all the properties of a group with respect to * $\therefore$ $(\mathbb{Z}, *)$ is a group Also, $a * b = a + b - 4 = b + a - 4$ $= b * a,\ \forall\ a, b \in \mathbb{Z}$ $(\mathbb{Z}, *)$ is an infinite Abelian group.	1 1
Q.2	Attempt any **TWO** questions from the following: (12)		
$b)$	i.	Let G be a group. Prove that p) Identity element of G is unique. q) The inverse of every element in G is unique.	
	Ans	p) Let there are two identities e and e' $xe = ex = e\ \ \forall x \in G$ $xe' = e'x = e'\ \ \forall x \in G$ $\therefore e'e = ee' = e$ and $ee' = e'e = e'$ $\therefore$e=e' q) Let $x \in G$ has two inverses y and y' $y = y + e = y + (x + y') = (y + x) + y' = e + y' = y'$	3 3
	ii.	Prove that $\mathbb{Z}_p^*$ is a group under multiplication modulo p, where p is prime.	
	Ans	T.P.T: - $\mathbb{Z}_p^*$ is a group. Closure: Let $\bar{a}, \bar{b} \in \mathbb{Z}_p^*$, $\bar{a}.\bar{b} = \overline{ab}$ If $\overline{ab} = \bar{0} \Rightarrow p\|ab \Longrightarrow p\|a$ or $p\|b \Longrightarrow \bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$, which is not true $\overline{ab} \neq \bar{0}$, $\therefore$ $\bar{a}.\bar{b} \in \mathbb{Z}_p^*$ Associative is trivial Inverse: Let $\bar{a} \in \mathbb{Z}_p^*, 1 \le a \le p - 1$ $\exists\ m, n \in \mathbb{Z}$, such that $am + pn = 1$ $\overline{am + pn} = \bar{1}$ $\bar{a}.\bar{m} = \bar{1}$ $\therefore$ $\bar{m}$ is the multiplicative inverseof$\bar{a}$. (Note that $\bar{m} \neq \bar{0}$)	3 3
	iii.	Let $GL_2(\mathbb{R})$ denote the group of all nonsingular 2×2 matrices with real entries.	

		determine whether $H = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}) \mid ad - bc = 1\right\}$ is a subgroup of the group $GL_2(\mathbb{R})$.	
	Ans	Let $A, B \in H \therefore |A| = 1$ and $|B| = 1$, for some $n, m \in \mathbb{Z}$ $|AB^{-1}| = |A||B|^{-1} = 1$ $\therefore AB^{-1} \in H$ by 1-step test .	2 2 2
	iv.	Define order of an element in a group G. Further prove that for $a \in G$,if $O(a) = nm$ then $O(a^n) = m$	
	Ans	Order of an element 'a' of group G is defined as Least positive integer n such that $a^n = e$ Let $O(a^n) = t$ $(a^n)^t = e$ $a^{nt} = e$ $nm \mid nt$ $\therefore m \mid t$…..(1) $(a^n)^m = a^{nm} = e$ But $O(a^n) = t$ $\therefore t \mid m$…..(2) From (1) and (2) $t = m$	1 5
Q3.	Attempt any **ONE** question from the following: (08)		
$a)$	i.	Let G be a finite cyclic group of order n generated by $'a'$ then prove that, a^m is also a generator of G if and only if $(m, n) = 1$	
		$(\Longrightarrow)$ $G = < a^m >$ T.P.T: $(m, n) = 1$ Let $(m, n) = d$, suppose $d > 1$ $d \mid m$, $d \mid n$ $m = dd_1, n = dd_2$ for some $d_1, d_2 \in \mathbb{Z}$ $(a^m)^{d_2} = a^{md_2} = a^{dd_1d_2} = a^{d_1.(dd_2)} = (a^n)^{(d_1)} = e$ $(a^m)^{d_2} = e$ &$d_2 < n \therefore O(a^m) < n$ which is contradiction to the fact that $G = < a^m > \therefore O(G) = O(a^m) = n$ Assumption was wrong . Hence, $d = 1$ i.e $(m, n) = 1$ $(\Longleftarrow)(m, n) = 1, \therefore \exists x, y \in \mathbb{Z}\ s.t\ mx + ny = 1$ T.P.T $G = < a^m >$ $i.e\ T.P.T\ < a > = < a^m >$ $< a > \supseteq < a^m >$ is obvious $a = a^1 = a^{mx+ny} = a^{mx}a^{ny} = a^{mx} = (a^m)^x$ $\therefore a \in < a^m >$ $\therefore < a > \subseteq < a^m >$	
	ii.	Let G be a cyclic group of order 18 generated by 'a'. List all generators of G and all subgroups of G.	
		$G = < a >, O(G) = O(a) = 18$ Generators $a^1, a^5, a^7, a^{11}, a^{13}, a^{17}$ Subgroups $1 \mid 18$ $\exists!$ subgroup of order 1 namely $H_1 = < a^{\frac{18}{1}} > = \{e\}$ $2 \mid 18$ $\exists!$ subgroup of order 2 namely $H_2 = < a^{\frac{18}{2}} > = < a^9 > = \{a^9, e\}$ $3 \mid 18$ $\exists!$ subgroup of order 3 namely $H_3 = < a^{\frac{18}{3}} > = < a^6 > = \{a^6, a^{12}, e\}$ $6 \mid 18$ $\exists!$ subgroup of order 6 namely $H_4 = < a^{\frac{18}{6}} > = < a^3 > = \{a^3, a^6, a^9, a^{12}, a^{15}, e\}$ $9 \mid 18$ $\exists!$ subgroup of order 9 namely $H_5 = < a^{\frac{18}{9}} > = < a^2 > = \{a^2, a^4, a^6, a^8, a^{10}, a^{12}, a^{14}, a^{16}, e\}$ $18 \mid 18$ $\exists!$ subgroup of order 18 namely $H_6 = < a^{\frac{18}{18}} > = < a^1 > = G$	
Q3.	Attempt any **TWO** questions from the following: (12)		
$b)$	i.	Let G be an infinite cyclic group generated by $'a'$. Show that G has infinitely many distinct subgroups.	
		G is an infinite cyclic group generated by $'a'$ $\therefore G = < a >, O(a) = infinite$ Proof of Claim1: All distinct powers of a are distinct group elements Proof of Claim2: Each Distinct positive powers of a generates distinct subgroups of G	

	ii.	Prove that $H = \left\{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{Z}\right\}$ is a cyclic subgroup of $GL_2(\mathbb{R})$.
		Let $n > 0$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-n} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}\right)^n = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$ $H = \langle\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\rangle$
	iii.	Prove that every subgroup of a cyclic group is cyclic.
		Let H be a subgroup of a cyclic group $G =< a >$ Claim: H is generated by a^m where m is the smallest positive integer such that $a^m \in H$ T.P.T $H =< a^m >$ $H \supseteq < a^m >$ ….(1) T.P.T $H \subseteq < a^m >$ Let $b = a^k \in H\ for\ some\ k$ $\exists!\, q, r\ s.t\ k = mq + r, where\ r = 0\ or\ r < m$ If $r < m$ then $a^r = a^{k-mq} = a^k(a^m)^{-q} \in H$ which is a contradiction because m is the smallest positive integer such that $a^m \in H$ $r = 0$ $k = mq$ $b = a^k = a^{mq} \in < a^m >$ $H \subseteq < a^m >$….(2) $H =< a^m > \cdots$from (1) and (2) ∴Every subgroup of a cyclic group is cyclic
	iv.	Let $U(n) = \{\bar{x} \mid x \in \mathbb{N}, (x, n) = 1, 1 \le x \le n\}$ under multiplication modulo n. Determine which of the following groups are cyclic. Justify your answer. (p) $U(6)$ (q) $U(7)$
		$U(6) = \{1,5\}$ $U(7) = \{1,2,3,4,5,6\}$ As $O(5) = 2\ \therefore U(6) =< 5 >$ As $O(3) = 7\ \therefore U(7) =< 3 >$
Q4.	Attempt any **ONE** question from the following: (08)	
a)	i.	Let H be a subgroup of a group G and $a, b \in G$ then show that (p) $a \in aH$ (q) $aH = bH\ or\ aH \cap bH = \emptyset$ (r) $\lvert aH\rvert = \lvert bH\rvert$
		(p) Since $e \in H \Longrightarrow ae \in aH \Longrightarrow a \in aH$ (q) case (i) If $aH \cap bH = \emptyset$ then done Case (ii) If $aH \cap bH \neq \emptyset$ Let $x \in aH \cap bH$ then for $h_1, h_2 \in H$ $\Longrightarrow x = ah_1$ and $x = bh_2 \Longrightarrow a = xh_1^{-1}$ Let $y \in aH \Longrightarrow y = ah = xh_1^{-1}h = bh_2h_1^{-1}h \in bH \Longrightarrow aH \subseteq bH$ Similarly one can show $bH \subseteq aH \Longrightarrow aH = bH$ (r) Define a map $f: aH \to bH$ by $f(ah) = bh$, $h \in H$ Show f is bijective map that gives $\lvert aH\rvert = \lvert bH\rvert$
	ii.	Define kernel of group homomorphism. If $f: G \longrightarrow G'$ is group homomorphism then show that $kerf$ is subgroup of G. Further f is injective if and only if $kerf = \{e\}$.
		$kerf = \{x \in G / f(x) = e'\} \subseteq G$ Since f is group homomorphism $\Longrightarrow f(e) = e' \Longrightarrow e \in kerf \Longrightarrow kerf \neq \emptyset$ Claim : $xy^{-1} \in kerf$, where $x, y \in kerf$ $x, y \in kerf \Longrightarrow f(x) = e', f(y) = e'$ Now $f(xy^{-1}) = f(x)f(y)^{-1} = e' \Longrightarrow kerf$ is subgroup of G. Claim : $kerf = \{e\}$ Let $x \in kerf \Longrightarrow f(x) = e' \Longrightarrow f(x) = f(e) \Longrightarrow x = e$ since f is injective $\therefore kerf = \{e\}$ Claim : f is injective Let $f(x) = f(y) \Longrightarrow f(x)f(y)^{-1} = e' \Longrightarrow f(xy^{-1}) = e'$ $\Longrightarrow xy^{-1} \in kerf = \{e\} \Longrightarrow xy^{-1} = e \Longrightarrow x = y \Longrightarrow f$ is injective
Q4.	Attempt any **TWO** questions from the following: (12)	
b)	i.	Let G be a group of prime order p. If H and K are subgroups of G then show that either $H \cap K = \{e\}$ or $H = K$.
		Since H and K be two subgroups of G.

		By Lagrange's theorem , $o(H)\mid o(G)$ and $o(K)\mid o(G)$ $\Longrightarrow o(H)\mid p$ and $o(K)\mid p$ As p is prime. $o(H) = o(K) = 1\ or\ p$ If $o(H) = o(K) = 1 \Longrightarrow H = K = \{e\}$ ---------- (1) If $o(H) = o(K) = p = o(G)$, also $H\ and\ K \subseteq G$ gives $H = K = G$ ---- (2) (1) and (2) gives either $H \cap K = \{e\}$ or $H = K$.
	ii.	Let $G = \mathbb{R} \times \mathbb{R}$ be a group under binary operation $*$ defined by $(a,b) * (c,d) = (a+c, b+d)$ then show that $H = \{(a, 5a)/a \in \mathbb{R}\}$ is subgroup of G. Describe geometrically the left cosets $(2,3) + H$ in G.
		Clearly H is non−empty subset of G. Claim : $x * y^{-1} \in H$, where $x, y \in H$ Let $x = (a, 5a)$and $y = (b, 5b)$, $a, b \in \mathbb{R}$ Now $x * y^{-1} = (a, 5a) * (b, 5b)^{-1} = (a, 5a) * (-b, -5b) = (a-b, 5(a-b)) \in H$ as $a - b \in \mathbb{R}$ Therefore H is subgroup of G. Now the left cosets$(2,3) + H = \{(2+a, 3+5a)/a \in \mathbb{R}\}$ which is a straight line passing through the point $(2,3)$ and parallel to the line $y = 5x$.
	iii.	Let G be a group. Show that $f: G \longrightarrow G$ defined by $f(x) = x^{-1}$, $\forall x \in G$is an automorphism if and only if G is abelian.
		Consider $f(xy) = (xy)^{-1} = y^{-1}x^{-1} = f(y)f(x) = f(yx)$ as f is homomorphism Since f is injective, $xy = yx \Longrightarrow G$ is abelian. Conversely, Consider, as f is abelian $f(xy) = (\boldsymbol{xy})^{-1} = \boldsymbol{x^{-1}y^{-1}} = f(x)f(y) \Longrightarrow f$ is homomorphism Let $f(x) = f(y) \Longrightarrow x^{-1} = y^{-1} \Longrightarrow (x^{-1})^{-1} = (y^{-1})^{-1} \Longrightarrow x = y$ $\Longrightarrow f$ is injective Let $y \in G \Longrightarrow y^{-1} \in G$, Now $f(y^{-1}) = (y^{-1})^{-1} = y \Longrightarrow f$ is sujective Therefore $f$is automorphism.
	iv.	Let $f: G \longrightarrow G'$ is onto group homomorphism. then show that (p) $o(f(a)) \mid o(a), \forall a \in G$ (q) IfG is abelian then G' is also abelian.
		(p) Let $o(a) = n$ then $a^n = e$ Since f is homomorphism $\Longrightarrow [f(a)]^n = f(a^n) = f(e) = e'$ $\therefore o(f(a)) \mid n \Longrightarrow o(f(a)) \mid o(a), \forall a \in G$ (q) Claim : G' is abelian (i.e.) $xy = yx$, $\forall\ x, y \in G'$ Since f is onto $\exists\ a, b \in G$ such that $f(a) = x$, $f(b) = y$ Also G is abelian $\Longrightarrow ab = ba$ Now $xy = f(a)f(b) = f(ab) = f(ba) = f(b)f(a) = yx \Longrightarrow G'$ is abelian.
Q5.	Attempt any **FOUR** questions from the following: (20)	
$a)$	If α and β are disjoint permutations in S_n. Such that $o(\alpha) = m, o(\beta) = n$ then show that order of $(\alpha\ o \beta)$is $l.c.m[m, n]$.	
	Let $O(\alpha) = n_1, O(\beta) = n_2$ T.P.T $O(\alpha\beta) = n$ where $n = lcm(n_1, n_2)$ Let $O(\alpha\beta) = t$ $(\alpha\beta)^t = e$ $\alpha^t.\beta^t = e$ $\alpha^t = e\ or\ \beta^t = e$ ∵ α and β are disjoint $n_1 \mid t\ \ or\ \ n_2 \mid t$ $lcm(n_1, n_2) \mid t$ $n \mid t$ ….(1) 3 Consider $(\alpha\beta)^n = \alpha^n\beta^n$ (∵ α and β are disjoint cycles) $= e$ But $O(\alpha\beta) = t$ $\therefore t \mid n$….(2) From (1) and (2) $n = t \therefore O(\alpha\beta) = n$ where $n = lcm(n_1, n_2)$ 2	
$b)$	Construct composition table for $G = \{\overline{5}, \overline{15}, \overline{25}, \overline{35}\}$ under multiplication of residue classes modulo 40.	

	5	15	25	35
5	25	35	5	15
15	35	25	15	5
25	5	15	25	35

35	15	5	35	25

$c)$	Show that every group of prime order p is cyclic.	
	Let G be a cyclic group of order p Let $a \neq e, a \in G$(Note : Such a choice is always possible) $O(a)\|O(G)$ $O(a)\|p$ $O(a) = 1 or\ p$ $If\ O(a) = 1, then\ a = e$, but e was non identity element If $O(a) = p, then\ G =< a >$	5
$d)$	Let G be a cyclic group of order 44. Find the number of elements of order 4 and the number of elements of order 11 in G. Clearly state the result used.	
	Result :If G is a cyclic group of order n generated by a then for every divisor d of n there are $\varphi(d)$ elements of order d 2 $4\|44$ ∴number of elements of order 4 $=\varphi(4) = 2$ $11\|44$ ∴number of elements of order 11 $=\varphi(11) = 10$	3
$e)$	Find all distinct left cosets of $H = \{\overline{1}, \overline{11}\}$ in $U(30)$.	
Ans	$U(30) = \{\overline{1}, \overline{7}, \overline{11}, \overline{13}, \overline{17}, \overline{19}, \overline{23}, \overline{29}\}$is a group under multiplication. Now all distinct left cosets of H in $U(30)$ are, $\overline{1}H = \overline{11}H = \{\overline{1}, \overline{11}\}$, $\overline{7}H = \overline{17}H = \{\overline{7}, \overline{17}\}$, $\overline{13}H = \overline{23}H = \{\overline{13}, \overline{23}\}$, $\overline{19}H = \overline{29}H = \{\overline{19}, \overline{29}\}$	
$f)$	Show that $f: GL_2(\mathbb{R}) \longrightarrow (\mathbb{R}^*, \cdot)$ defined by $f(A) = detA$ is a group homomorphism. Also find $kerf$. Is f an isomorphism? Justify.	
Ans	Now $f(AB) = \det(AB) = \det(A)\det(B) = f(A)\,f(B) \implies f$ is homomorphism $kerf = SL_2(\mathbb{R})$, since $kerf \neq \{e\} \implies f$an not isomorphism.	

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