Q P. Code :- 54970

(1)

(3 Hours) [Total Marks: 100]

Note: (*i*) All questions are compulsory.

(*ii*) Figures to the right indicate marks for respective parts.

Q.1	Choose correct alternative in each of the following (20)			
i.	Consider the following statements: I. Elementary matrix is invertible. II. A square matrix A is invertible iff it is row equivalent to the zero matrix. III. Any n-dimensional real vector space is isomorphic to $\mathbb{R}^n$. Then			
	(a) only (I) is true		(b)	(I) and (III) are true.
	(c) (II), and (III) are true		(d)	None of the above
	Ans	Solution: (b) (I) and (III) are true.		
ii.	Rank of a matrix A is			
	(a)	Row rank of A	(b)	Column rank of A
	(c)	Number of non-zero rows in row-echelon matrix	(d)	All of the above
	Ans	Solution: (d) All of the above.		
iii.	If I_{23}, $I_{13} \in M_3(\mathbb{R})$ then $I_{23}+I_{13}$ is			
	(a)	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	(b)	$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$
	(c)	$\begin{pmatrix}0&0&1\\0&0&1\\0&0&0\end{pmatrix}$	(d)	$\begin{pmatrix}0&0&0\\0&0&0\\1&1&0\end{pmatrix}$
	Ans	Solution: (c) $\begin{pmatrix}0&0&1\\0&0&1\\0&0&0\end{pmatrix}$		
iv.	Det $\begin{pmatrix}3&0&0&0\\0&0&1&1\\1&1&0&0\\0&5&2&0\end{pmatrix}$ is			
	(a)	0	(b)	3
	(c)	15	(d)	6
	Ans	(d)		
v.	If $A=\begin{pmatrix}6&9&3\\5&1&4\\1&2&1\end{pmatrix}$ and $B=\begin{pmatrix}2&3&1\\5&1&4\\1&2&1\end{pmatrix}$ then which of the following is true			
	(a)	$detA = detB$	(b)	$detA = 2detB$
	(c)	$detA = 3detB$	(d)	$3detA = detB$
	Ans	(a)		
vi.	Let $A = (a_{ij})$ be a nxn matrix. Which of the following is true			
	(a)	$detA = \sum_{\sigma \in S_n} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$		

	(b)	$$detA = \sum_{\sigma \epsilon S_n} sign(\sigma) a_{11} a_{22} \dots a_{nn}$$
	(c)	$$detA = \sum_{\sigma \epsilon S_n} (-1)^n sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$$
	(d)	none of these
	Ans	(c)
vii.	Given that u, v, w are linearly independent vectors of $\mathbb{R}^3$, which of the below is false	
	(a)	Volume of the parallelepiped obtained by the vectors $u, 2v, w$ is the same as that obtained by u, v, w
	(b)	Volume spanned by u, v, w is the same as spanned by $u, u+v, u+v+w$
	(c)	Volume spanned by u, v, w is the determinant of the matrix taking u, v, w as column vectors.
	(d)	Volume of parallelepiped using vectors $u+v, v, w$ is same as that spanned by $u, v-w, w$.
	Ans	(a)
viii.	Which of the following is an inner product on $\mathbb{R}^2$where $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$? (a) $\langle x, y\rangle = (x_1 + 2x_2)y_1 + (2x_1 + 5x_2)y_2$ (b) $\langle x, y\rangle = (2x_1 - 5x_2)y_2 - (x_1 + 2x_2)y_1$ (c) $\langle x, y\rangle = 3x_1y_1 - 2x_2y_2$ (d) $\langle x, y\rangle = x_1^2y_1^2 + x_2^2y_2^2$	
	Ans	(a)
ix.	Let V be a finite dimensional inner product space and W be a subspace of V and $W^{\perp}$ be the orthogonal complement of W in V. If $\dim V = n$, $\dim W = r$, then $\dim W^{\perp}$ is	
	(a) r	(b) $n - r$
	(c) n	(d) None of these
	Ans	(b)
x.	Consider the following sets of vectors in the inner product space $(P_2, \langle,\rangle)$where$\langle a_0 + a_1x + a_2x_1^2,\ b_0 + b_1x + b_2x^2\rangle = a_0b_0 + a_1b_1 + a_2b_2$ (i) $\left\{\frac{2}{3} - \frac{2}{3}x + \frac{1}{3}x^2,\ \frac{2}{3} + \frac{1}{3}x - \frac{2}{3}x^2,\ \frac{1}{3} + \frac{2}{3}x + \frac{2}{3}x^2\right\}$ (ii) $\left\{1\ ,\ \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}x^2,\ x^2\right\}$ (iii) $\left\{-1\ ,\ \frac{-1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}x^2,\ \frac{-1}{\sqrt{2}}x^2\right\}$	
	(a) All are orthogonal sets	(b) (ii) and (iii) are orthogonal sets
	(c) Only (i) is an orthogonal set	(d) (i) and (iii) are orthogonal sets
	Ans	(c)
Q2.	Attempt any **ONE** question from the following:	(08)
a)	i.	State and prove Rank-Nullity theorem
	Ans	Solution: dim ker(T)=m, dim(V)=n Marks:(01) Let $\{v_1, v_2, \dots v_m\}$ be a basis of kerT. — 1

(2)

		Extend the basis of ker(T) to basis of V. Marks: (01) Let $\{v_1, v_2, \ldots v_m, \ldots v_n\}$ be a basis of V. Step 1: $\{T(v_{m+1}), T(v_{m+2}) \ldots T(v_n)\}$ be linearly independent. Marks: (03) Step:2 $L\{T(v_{m+1}), T(v_{m+2}) \ldots T(v_n)\}$=ImT Marks: (03) Therefore dim ImT=n-m = dim(V)-dim ker(T). Therefore dim(V)=dim ker(T)+dim Im(T)	1 3 3
	ii.	Prove that the row and column rank of an mxn matrix are equal	
	Ans	Proof: Let $A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ $If\ A = 0, i.e. zero\ matrix, then\ row\ rank\ of\ A = column\ rank\ of\ A = 0.$ $So\ assume\ that\ A \neq 0.$ $Let\ row\ rank\ of\ A = r\ and\ column\ rank\ of\ A = r'$ $Since\ row\ rank\ of\ A = r, there\ are\ r\ linearly\ independentvecntors\ in\ matrix$ $A, which\ are\ the\ basis\ of\ row\ space.$ $Let\ these\ row\ are\ R_1, R_2, \ldots, R_r. Since\ these\ rows\ are\ the\ basis\ of\ row\ space,$ $of\ A\ can\ be\ expressed\ as\ a\ linear\ combination\ of\ these\ vectors, as\ follows$ $R_1 = b_{11}R_1 + b_{12}R_2 + \ldots + b_{1r}R_r$ $R_2 = (b_{21}R_1 + b_{22}R_2 + \ldots + b_{2r}R_r$ $\vdots$ $R_r = b_{r1}R_1 + b_{r2}R_2 + \ldots + b_{rr}R_r$ $\vdots$ $R_m = b_{m1}R_1 + b_{m2}R_2 + \ldots + b_{mr}R_r \forall b_{ij} \in R$ $\therefore Column\ rank\ of\ A \leq r = row\ rank\ of\ A$ .... (1) $\therefore r' \leq r$ $In\ similar\ manner\ by\ taking\ a\ basis\ of\ column\ space\ we\ can\ prove\ that$ .... (2) $r \leq column\ rank\ of\ A = r'$ $\therefore r \leq r'$ $\therefore From\ (1) and\ (2) we\ have$ $r \leq r'$ Marks(04) $i.e. row\ rank\ of\ A = column\ rank\ of\ A$	2 4 2
		(12)	
Q.2		Attempt any **TWO** questions from the following:	
b)	i.	Let T: V →V' be a linear transformation. Prove that ker(T) is a subspace of V and ImT is a subspace of V'.	3 3
	Ans	X,Yϵker(T) $(\alpha X + \beta Y)\epsilon$ker(T) X,Y$\epsilon$Im(T) $(\alpha X + \beta Y)\epsilon Im$(T)	

3

	ii.	If $T:V\rightarrow W$ is linear transformation, $B=\{v_1, v_2, \ldots\ldots v_n\}$ is linearly independent subset of V and $\ker T=\{0\}$ then show that $\{T(v_1), T(v_2), \ldots\ldots T(v_n)\}$ is linearly independent subset of W.	
	Ans	Solution: TPT $\{T(v_1), T(v_2), \ldots\ldots T(v_n)\}$ is linearly independent Let $b_1T(v_1)+b_2T(v_2)+\ldots+b_nT(v_n)=0$ Therefore $T(b_1v_1+b_2v_2+\ldots b_nv_n)=0$ since $\ker(T)=0$ $b_1v_1+b_2v_2+\ldots b_nv_n=0$ but $\{v_1, v_2, \ldots\ldots v_n\}$ therefore $b_1=b_2=\ldots b_n=0$	1 2 3
	iii.	Find the rank of a matrix A where $A=\begin{bmatrix}-1 & 2 & 3 & 1\\ 0 & -2 & -6 & -2\\ 1 & 2 & 1 & 1\end{bmatrix}$.	
	Ans	Solution: Row-echelon matrix of $A=\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$.	6
	iv.	$T:\mathbb{R}^3\rightarrow \mathbb{R}^3$ $T(x,y,z)=(x+y,y+z,x+z)$ be a linear transformation. Find the matrix of this linear transformation with respect to standard basis.	
	Ans	Solution: $\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 1 & 0 & 1\end{bmatrix}$	6
Q3.	Attempt any **ONE** question from the following:		(08)
$a)$	i.	$A=(a_{ij})_{n\times n}$ *then show that* $A\times(cofactorA)^t=(detA)I_n$. Hence obtain an expression for A^{-1}.	
	Ans	Using Laplace expansion of determinant $detA =$ $(-1)^{i+1}a_{i1}detA_{i1}+(-1)^{i+2}a_{i2}detA_{i2}+\cdots+(-1)^{n}a_{in}detA_{in}$ Whenever $i\neq j$, $(-1)^{i+1}a_{j1}detA_{i1}+(-1)^{i+2}a_{j2}detA_{i2}+\cdots+(-1)^{n}a_{jn}detA_{in}=0$ where A_{rs} *is the* $(n-1)\times(n-1)$obtained from A by deleting the r^{th}*row and* s^{th}*column of* A. $cofactorA=\begin{pmatrix}(-1)^{1+1}detA_{11} & \cdots & (-1)^{1+n}detA_{1n}\\ \vdots & \ddots & \vdots\\ (-1)^{n+1}detA_{n1} & \cdots & (-1)^{n+1}detA_{nn}\end{pmatrix}$ $\therefore (cofactorA)^t=\begin{pmatrix}(-1)^{1+1}detA_{11} & \cdots & (-1)^{n+1}detA_{n1}\\ \vdots & \ddots & \vdots\\ (-1)^{1+n}detA_{1n} & \cdots & (-1)^{n+1}detA_{nn}\end{pmatrix}$ $\therefore A\times(cofactorA)^t=$	2 1 1 2 1

(4)

$$\begin{pmatrix} \sum_{j=1}^{n}(-1)^{1+j}a_{1j}detA_{1j} & \cdots & \sum_{j=1}^{n}(-1)^{n+j}a_{1j}detA_{nj} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{n}(-1)^{1+j}a_{nj}detA_{1j} & \cdots & \sum_{j=1}^{n}(-1)^{n+j}a_{nj}detA_{nj} \end{pmatrix}$$ 1

$$= \begin{pmatrix} detA & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & detA \end{pmatrix} = (detA)I_n$$

$\therefore A^{-1} = \frac{1}{detA}(cofactorA)^t$ whenever $detA \neq 0$

ii. State and prove the Cramer's rule $for\ n \times n\ linear\ system\ AX = b$.

Ans

Cramer's rule

If A is invertible, then the solution to the system of equations

$$AX = b\ is\ given\ by\ x_j = \frac{Det[A_j(b)]}{Det(A)}\ \ for\ 1 \le j \le n$$ 2

Where $A_j(b) is\ matrix\ A\ with\ its\ jth\ column\ replaced\ by\ b$

Denote the columns of A by $A^1, A^2, \ldots, A^n$ and columns of I(Identity matrix) by $E^1, E^2, \ldots E^n$

Let $I_j(X)\ denote\ the\ identity\ matrix\ with\ jth\ column\ replaced\ with$ the column vector X.

$\therefore AI_j(X) = A[E^1, E^2, \ldots X \ldots E^n] = [AE^1, AE^2, \ldots AX, \ldots AE^n]$ 2

$= [A^1, A^2, \ldots b, \ldots, A^n] = A_j(b)$

Taking determinants and using multiplicative property of determinant 2

$$DetADetI_j(X) = DetA_j(b)$$

But $DetI_j(X) = x_j where\ X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$

$$\therefore\ x_j = \frac{Det[A_j(b)]}{Det(A)}$$ 2

Q3. Attempt any **TWO** questions from the following: (12)

b) i. Prove that Area of a parallelogram spanned by (x_1, x_2)&(y_1, y_2) is $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$

Ans

From Euclidean geometry we know that area of a trapezium is half the product of sum of parallel sides and height and area of a triangle is half the product of base and height. 1

Consider the parallelogram formed by the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in the X-Y plane where $\overrightarrow{OA}$ is join of O(0,0) and A(x_1,x_2) and $\overrightarrow{OB}$ is join of O(0,0) and B(y_1,y_2). 1

$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ where C($x_1 + y_1$, $x_2 + y_2$)

Obtain a point P by dropping a perpendicular from A to X axis then P$(x_1, 0)$.

Obtain a point Q by dropping a perpendicular from C to X axis then Q(x_1+y_1 ,0).
Obtain a point C by dropping a perpendicular from join of C and Q to X axis call it D then D(x_1+y_1, y_2)

Area of a parallelogram spanned by $(x_1, x_2)\&(y_1, y_2)$
= Area of quadrilateral OQCB – Area Trapezium PQCA – Area of triangle OPA — 1

Also Area of quadrilateral OQCB = Area Triangle BDC + Area Trapezium OQDB
$=\frac{1}{2}x_1x_2+\frac{1}{2}(2x_1+y_1)y_2=\frac{1}{2}(x_1x_2+y_1y_2)+x_1y_2$ [from figure]

Area Trapezium PQCA= $x_2y_1+\frac{1}{2}y_1y_2$ — 1

Area Triangle OPA= $\frac{1}{2}x_1x_2$

$\therefore$ Area of a parallelogram spanned by $(x_1, x_2)\&(y_1, y_2)$

$$=\frac{1}{2}(x_1x_2+y_1y_2)+x_1y_2-\left(x_2y_1+\frac{1}{2}y_1y_2\right)-\frac{1}{2}x_1x_2$$ — 1

$$=x_1y_2-x_2y_1=\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$ — 1

ii. Prove that $\det(A^t)=det\,A$, $A\in M_n(\mathbb{R})$

Ans Let $A=(a_{ij})\in M_n(\mathbb{R})$ and $B=A^t=(b_{rs})=(a_{ji})$ — 2

$$detB=\sum_{\sigma\epsilon S_n}(-1)^n sign(\sigma)b_{1\sigma(1)}b_{2\sigma(2)}\dots b_{n\sigma(n)}$$ — 1

$$=\sum_{\sigma\epsilon S_n}(-1)^n sign(\sigma)a_{\sigma(1)(1)}a_{\sigma(2)(2)}\dots a_{\sigma(n)(n)}$$ — 1

Now if $\sigma=\begin{pmatrix} 1 & 2 & 3 & & n \\ \sigma(1) & \sigma(2) & \sigma(3) & & \sigma(n) \end{pmatrix}$

then $\sigma^{-1}=\begin{pmatrix} \sigma(1) & \sigma(2) & \sigma(3) & & \sigma(n) \\ 1 & 2 & 3 & & n \end{pmatrix}$ — 2

$$\therefore \det B=\sum_{\sigma\epsilon S_n}(-1)^n sign(\sigma^{-1})a_{1\sigma^{-1}(1)}a_{2\sigma^{-1}(2)}\dots a_{n\sigma^{-1}(n)}$$

$[using\ sign\,\sigma=sign\,\sigma^{-1}]$

$$\therefore \det B=\det A$$

iii. State the Laplace expansion formula for determinant.
Use Laplace expansion to find the determinant of the following matrix

$$\begin{pmatrix} 1 & -2 & -3 & 5 \\ 3 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

Ans For an $n\times n$ matrix $A=(a_{ij})$,
$detA=a_{i1}C_{i1}+a_{i2}C_{i2}+\dots+a_{in}C_{in}$ where C_{ij} is the Cofactor of a_{ij} — 2
Consider $i=3$,

(5)

		$C_{32} = (-1)^{3+2} det\begin{pmatrix} 1 & -3 & 5 \\ 3 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} = -(-1 + 3(3) + 5(5)) = -33$ $C_{33} = (-1)^{3+3} det\begin{pmatrix} 1 & -2 & 5 \\ 3 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} = 2 + 2(3) + 5(-4) = -12$ $\therefore detA = 0 + -1 \times (-33) + 1(-12) + 0 = 21$	1 1 2
	iv.	Find the inverse of the following matrix using adjoint $\begin{pmatrix} 5 & 7 & -2 \\ 3 & 2 & 1 \\ -2 & 4 & 6 \end{pmatrix}$	
	Ans	Let $A = \begin{pmatrix} 5 & 7 & -2 \\ 3 & 2 & 1 \\ -2 & 4 & 6 \end{pmatrix}$ Cofactor $A = \begin{pmatrix} 8 & -20 & 16 \\ -50 & 26 & -34 \\ 11 & -11 & -11 \end{pmatrix}$ $\therefore adjoint\ A = \begin{pmatrix} 8 & -20 & 16 \\ -50 & 26 & -34 \\ 11 & -11 & -11 \end{pmatrix}^t$ $\therefore A \times adjoint\ A = \begin{pmatrix} 5 & 7 & -2 \\ 3 & 2 & 1 \\ -2 & 4 & 6 \end{pmatrix}\begin{pmatrix} 8 & -50 & 11 \\ -20 & 26 & -11 \\ 16 & -34 & -11 \end{pmatrix}$ $= \begin{pmatrix} -132 & 0 & 0 \\ 0 & -132 & 0 \\ 0 & 0 & -132 \end{pmatrix}$ $\therefore A^{-1} = \frac{-1}{132}\begin{pmatrix} 8 & -50 & 11 \\ -20 & 26 & -11 \\ 16 & -34 & -11 \end{pmatrix}.$	2 2 2
Q4.	Attempt any **ONE** question from the following:		(08)
$a)$	i.	Define orthogonal & orthonormal sets. Find orthonormal set corresponding to $S = \{(1,-1,0),(1,0,1),(-1,2,0)\}$ using Gram-Schmidt Orthogonalisation Process in $\mathbb{R}^3$ with dot product.	
	Ans	Definition of orthogonal and orthonormal sets. $v_1 = (1,-1,0)$ $v_2 = (1,0,1) - \frac{\langle(1,0,1),(1,-1,0)\rangle}{\langle(1,-1,0),(1,-1,0)\rangle}(1,-1,0)$ $= \left(\frac{1}{2},\frac{1}{2},1\right)$ $v_3 = (-1,2,0) - \frac{\langle(-1,2,0),(\frac{1}{2},\frac{1}{2},1)\rangle}{\langle(\frac{1}{2},\frac{1}{2},1),(\frac{1}{2},\frac{1}{2},1)\rangle}\left(\frac{1}{2},\frac{1}{2},1\right) - \frac{\langle(-1,2,0),(1,-1,0)\rangle}{\langle(1,-1,0),(1,-1,0)\rangle}(1,-1,0)$ $= \left(\frac{1}{3},\frac{1}{3},\frac{-1}{3}\right)$ $\|v_1\| = \sqrt{2}, \quad \|v_2\| = \sqrt{\frac{3}{2}}, \quad \|v_3\| = \sqrt{\frac{1}{3}}$	1M 2M 3M

(6)

		Orthonormal set is $\left\{\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}},0\right),\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{\sqrt{2}}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)\right\}$	2M
	ii.	(p) Let V be a real inner product space. Let u be an unit vector in V and $P_u(v)$ be the projection of v along u. Show that $\\|v-P_u(v)\\| \le \\|v-\alpha u\\| \forall \alpha \in \mathbb{R}$. (q) Find the closest approximation of (2,3,1) along (0,1,0) in $\mathbb{R}^3$ with Euclidean inner product.	
	Ans	(p) $P_u(v)=\frac{\langle u,v\rangle}{\langle u,u\rangle}u$ $\\|v-P_u(v)\\|^2=\langle v-P_u(v),v-P_u(v)\rangle$ $=\langle v-\frac{\langle u,v\rangle}{\langle u,u\rangle}u, v-\frac{\langle u,v\rangle}{\langle u,u\rangle}u\rangle$ $=\langle v-\langle u,v\rangle u, v-\langle u,v\rangle u\rangle$ ($\because u$ is unit vector) $=\langle v,v\rangle-2\langle u,v\rangle^2+\langle u,v\rangle^2$ $=\langle v,v\rangle-\langle u,v\rangle^2$----(1) We have $(\langle u,v\rangle-\alpha)^2\ge 0\ \forall\alpha\in\mathbb{R}$ $\Rightarrow \langle u,v\rangle^2-2\alpha\langle u,v\rangle+\alpha^2\ge 0\ \forall\alpha\in\mathbb{R}$ $\Rightarrow \langle u,v\rangle^2\ge 2\alpha\langle u,v\rangle-\alpha^2\ \forall\alpha\in\mathbb{R}$ $\Rightarrow -\langle u,v\rangle^2\le -2\alpha\langle u,v\rangle+\alpha^2\ \forall\alpha\in\mathbb{R}$ $\Rightarrow \langle v,v\rangle-\langle u,v\rangle^2\le\langle v,v\rangle-2\alpha\langle u,v\rangle+\alpha^2\ \forall\alpha\in\mathbb{R}$ $\Rightarrow \langle v,v\rangle-\langle u,v\rangle^2\le\langle v,v\rangle-2\alpha\langle u,v\rangle+\alpha^2\langle u,u\rangle\ \forall\alpha\in\mathbb{R}$ $\Rightarrow \langle v,v\rangle-\langle u,v\rangle^2\le\langle v-\alpha u, v-\alpha u\rangle\ \forall\alpha\in\mathbb{R}$ $\Rightarrow \\|v-P_u(v)\\|^2\le\\|v-\alpha u\\|^2\ \forall\alpha\in\mathbb{R}$ $\Rightarrow \\|v-P_u(v)\\|\le\\|v-\alpha u\\|\ \forall\alpha\in\mathbb{R}$ (q) from part (p), the closest approximation is projection. $P_u(v)=\frac{\langle u,v\rangle}{\langle u,u\rangle}u$ $=\langle(2,3,1),(0,1,0)\rangle(0,1,0)$ $=(0,3,0)$	1M 2M 3M 2M
Q4.	Attempt any **TWO** questions from the following:		(12)
b)	i.	State Cauchy-Schwarz inequality in a real inner product space. Further state and prove Triangle inequality.	
	Ans	Statement of Cauchy Schwarz inequality Statement of Triangle inequality $\\|x+y\\|^2=\langle x+y,x+y\rangle=\langle x,x\rangle+2\langle x,y\rangle+\langle y,y\rangle$ $\le\langle x,x\rangle+2\\|x\\|\\|y\\|+\langle y,y\rangle$ $=\\|x\\|^2+2\\|x\\|\\|y\\|+\\|y\\|^2=(\\|x\\|+\\|y\\|)^2$ Hence $\\|x+y\\|\le\\|x\\|+\\|y\\|$	2 1 3
	ii.	Prove that an orthogonal set in an inner product space V is linearly independent.	
	Ans	Let $S=\{x_1,x_2,\dots,x_n\}$ be an orthogonal set in real inner product space V. Hence $x_1,x_2,\dots,x_n$ are all non-zero vectors of V such that $\langle x_i,x_j\rangle=0$ for $i\ne j$, $i,j=1,2,\dots,n$ ....(*) Let $a_1x_1+a_2x_2+\dots+a_nx_n=0$ where $a_1,a_2,\dots,a_n\in\mathbb{R}$. For any $i=1,2,\dots,n$, consider $\langle x_i,a_1x_1+a_2x_2+\dots+a_nx_n\rangle=\langle x_i,0\rangle=0$	1

(7)

		$\therefore \langle x_i, \sum_{j=1}^{n} a_j x_j \rangle = 0 \Rightarrow \sum_{j=1}^{n} \langle x_i, a_j x_j \rangle = 0$ $\Rightarrow \langle x_i, a_i x_i \rangle + \sum_{\substack{j=1 \\ j\neq i}}^{n} \langle x_i, a_j x_j \rangle = 0$ $\Rightarrow a_i \langle x_i, x_i \rangle + 0 = 0$ ……from (*) Since $x_i \neq 0, \langle x_i, x_i \rangle \neq 0 \quad \Rightarrow a_i = 0$ for all $i = 1,2, \ldots, n$ Hence, S is linearly independent set.	3 1 1
	iii.	Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Prove that the function given by $\langle x, y \rangle = 4x_1y_1 + 9x_2y_2$ is an inner product on $\mathbb{R}^2$.	
	Ans	Prove each of the following for $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$: $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$ $\langle x, y \rangle = \langle y, x \rangle$ $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ Hence given function is an inner product.	3 1 1 1
	iv.	Prove that $S = \{1, \sin x, \cos x\}$ is an orthogonal set of $C[-\pi, \pi]$ where $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$	
	Ans	Let $f(x) = 1, g(x) = \sin x, h(x) = \cos x$ for $x \in [-\pi, \pi]$ Prove $\langle f, g \rangle = 0$ $\langle g, h \rangle = 0$ $\langle f, h \rangle = 0$ Hence S is an orthogonal set.	2 2 2
Q5.	Attempt any **FOUR** questions from the following:	(20)	
a)	Any n-dimensional real vector space is isomorphic to $\mathbb{R}^n$.		
Ans	Let V be n dimensional vector space. Let $v_1, v_2, \ldots v_n$ be a basis of V Let $X \in V$ $X = c_1v_1 + c_2v_2 + \ldots c_nv_n$ Define a linear transformation $T: V \rightarrow \mathbb{R}^n$ $T(X) = (c_1, c_2, \ldots c_n)$ T is one-one and onto.		1 1 3
b)	Verify Rank-Nullity theorem for the following linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ $T(x, y, z) = (x+z, 2x+3y, 0)$		
Ans	Ker(T)=L{(1,-2/3,-1)} dim ker(T)=1 ImT=L{(1,2,0),(0,3,0),(1,0,0)} Above set is linearly dependent. Omit(1,2,0) Retain {(0,3,0),(1,0,0)} dim ImT=2 dim($\mathbb{R}^3$)=dim(kerT)+dim(ImT) M 3=1+2		2 2 1

c) Let (2,3,1), (0,1,1), (1,0,2) satisfy equation of a plane $ax + by + cz = 5$ then find a, b, c

Ans

$$\therefore 2a + 3b + c = 5$$
$$0(a) + b + c = 5$$
$$a + 0(b) + 2c = 5$$
1

Let D = $\begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} = 4 - 3(-1) + 1(-1) = 6$ 1

$a = \frac{5}{6}\det\begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \frac{5}{6}(2 - 3(2-1) + 1(-1)) = \frac{-5}{3}$ 1

$b = \frac{5}{6}det\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \frac{5}{6}(2(1) - (-1) + 1(-1)) = \frac{5}{3}$ 1

$c = \frac{5}{6}det\begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \frac{5}{6}(2 - 3(-1) + 1(-1)) = \frac{10}{3}$ 1

d) Determine determinant of the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix}$

Ans

Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix}$ then $A^t = \begin{pmatrix} 1 & a & a^2 a^3 \\ 1 & b & b^2 b^3 \\ 1 & c & c^2 c^3 \\ 1 & d & d^2 d^3 \end{pmatrix}$ 1

Perform following column operations on 2nd, 3rd and 4th columns respectively on A^t

1. Multiply 1st column with a and subtract from second column
2. Multiply 2nd column with a and subtract from third column
3. Multiply 3rd column with a and subtract from fourth column

These column operations has no effect on the determinant 1

$$\therefore det\begin{pmatrix} 1 & a & a^2 a^3 \\ 1 & b & b^2 b^3 \\ 1 & c & c^2 c^3 \\ 1 & d & d^2 d^3 \end{pmatrix} = (b-a)(c-a)(d-a)det\begin{pmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{pmatrix}$$
1

Similarly perform column operations on 2nd and 4th column

1. Multiply 1st column with a and subtract from second column
2. Multiply 2nd column with a and subtract from third column

$\therefore$ Required determin = $(b-a)(c-a)(d-a)(c-b)(d-b)det\begin{pmatrix} 1 & c \\ 1 & d \end{pmatrix}$

Perform 2nd column operation –multiply 1st column with c and subtract from 2nd column finally gives

Required determinant = $(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$ [Using that $detA = detA^t$] 1

e) Let $V = M_{2x2}(\mathbb{R}), \langle A, B \rangle = tr(AB^t)$. Find the angle between $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

		1M
Ans	$\langle A,B\rangle = tr\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) = 1$ $\langle A,A\rangle = tr\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = 2$ $\langle B,B\rangle = tr\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) = 1$ $\|A\| = \sqrt{2}, \|B\| = 1$ $\cos\theta = \frac{\langle A,B\rangle}{\|A\|\|B\|} = \frac{1}{\sqrt{2}}$ $\therefore \theta = \frac{\pi}{4}$	1M 1M 1M 1M
f)	Find the orthogonal compliment of the following subspaces with respect to dot product. (i) $W = \left\{(x,y,z) \in \mathbb{R}^3 / x = y = \frac{z}{2}\right\}$ (ii) $W = \{(x,y) \in \mathbb{R}^2 / y = x\}$	
Ans	(i) $W^{\perp} = \{(a,b,c)/\langle(a,b,c),(x,x,2x)\rangle = 0 \forall x \in \mathbb{R}\}$ $ax + bx + 2cx = 0\ \forall x \in \mathbb{R}$ $x(a+b+2c) = 0 \forall x \in \mathbb{R}$ $a + b + 2c = 0$ $\therefore W^{\perp} = \{(x,y,z) \in \mathbb{R}^3 / x + y + 2z = 0\}$ (ii) $W^{\perp} = \{(a,b)/\langle(a,b),(x,x)\rangle = 0 \forall x \in \mathbb{R}\}$ $ax + bx = 0 \forall x \in \mathbb{R}$ $x(a+b) = 0 \forall x \in \mathbb{R}$ $a + b = 0$ $\therefore W^{\perp} = \{(x,y) \in \mathbb{R}^2 / y = -x\}$	3M 2M

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