① M-3 SOLUTION SET-3 — 00054563.

(3 Hours) [Total Marks: 100]

Note: (*i*) All questions are compulsory.
(*ii*)Figures to the right indicate marks for respective parts.

Q.1	Choose correct alternative in each of the following (20)			
i.	How many permutations of S_4 are expressed as composite of disjoint 2-cycles?			
	(a)	12	(b)	6
	(c)	3	(d)	10
	Ans	(c) 3		
ii.	Signature of a cycle of length r is			
	(a)	$(-1)^{(r)}$	(b)	r
	(c)	$(-1)^{(r+1)}$	(d)	None of these
	Ans	(c) $(-1)^{(r+1)}$		
iii.	The number of elements in A_6 is			
	(a)	6	(b)	720
	(c)	360	(d)	2^6
	Ans	(c) 360		
iv.	For the sequence $1, 7, 25, 79, 241, 727, \cdots$, simple formula for (a_n) is			
	(a)	$3^n + 2$	(b)	$3^n - 2$
	(c)	$-3^n + 4$	(d)	$n^2 - 2$
	Ans	(b) $3^n - 2$		
v.	If X, Y are finite sets and there is an surjective function $f: X \to Y$ then			
	(a)	$\lvert X\rvert = \lvert Y\rvert$	(b)	$\lvert X\rvert \le \lvert Y\rvert$
	(c)	$\lvert X\rvert \ge \lvert Y\rvert$	(d)	None of these
	Ans	(c) $\lvert X\rvert \ge \lvert Y\rvert$		
vi.	How many ways are there to pick a man and a woman who are not husband and wife from a group of n married couples?			
	(a)	$n!$	(b)	$n(n-1)$
	(c)	$n+(n-1)$	(d)	None of these
	Ans	(b) $n(n-1)$		
vii.	Let $S(n, k)$ denote the Stirling number of second kind on n-set into k-disjoint nonempty unordered subsets, then $S(n, 1)$ is			
	(a)	1	(b)	0
	(c)	n	(d)	None of these
	Ans	(a) 1		
viii.	How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?			
	(a)	101	(b)	102

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	(c)	100	(d)	None of these
	Ans	(b) 102		
ix.	17 students are present in a class. In how many ways, can they be seated on 2 circular tables having of 8 and 9 chairs?			
	(a)	$\binom{17}{8}$ 9! 8!	(b)	$\binom{17}{8}$ 8! 7!
	(c)	9! 8!	(d)	None of these
	Ans	(b) $\binom{17}{8}$ 8! 7!		
x.	If p is a prime and $n > 0$, then			
	(a)	$\phi(p^n) = p^n\left(1 + \frac{1}{p}\right)$	(b)	$\phi(p^n) = p\left(1 - \frac{1}{p}\right)$
	(c)	$\phi(p^n) = p^n\left(1 - \frac{1}{p}\right)$	(d)	None of these
	Ans	(b) $\phi(p^n) = p^n\left(1 - \frac{1}{p}\right)$		
Q2.	Attempt any **ONE** question from the following: (08)			
a)	i.	Prove that for $n > 1$, the number of even permutations is equal to the number of odd permutations , each set having $\frac{n!}{2}$ elements.		

	Ans	Let S_n denotes set of all permutations of set $S = \{1,2,..,n\}$. $\therefore |S_n| = n!$ Since $I_n(identity\ permutation) \in S_n$ and it is even. Also for > 1 , $(1,2) \in S_n$ and it is odd. $\therefore$ S_n contains even as well as odd permutations. Let k_1 = odd permutations in S_n and k_2 = even permutations inS_n. Then $k_1 + k_2 = n!$ ..(1) As $(1,2) \in S_n$, for every even permutation α , we have corresponding odd permutation $\alpha(1,2)$. Thus there are at least as many odd permutation as even permutations. $\therefore k_1 \geq k_2$ ..(2) On other hand, for every odd permutation , we have corresponding even permutation $\beta(1,2)$. Thus there are at least as many even permutation as odd permutations. $\therefore k_1 \leq k_2$ ..(3) $\therefore k_1 = k_2$ ..(4) Thus from(1) $\therefore k_1 = k_2 = \frac{n!}{2}$	1 1 1 1 1 1

③

	ii.	Define Linear Homogeneous recurrence relation of degree n. Show that if the characteristic equation $x^2 - a_1x - a_2 = 0$ of the recurrence relation $h_n = a_1h_{n-1} + a_2h_{n-2}$ has a single non-zero roots q_1 then $h_n = c_1q_1^n + c_2nq_1^n$ is the generalsolution of the recurrence relation$h_n = a_1h_{n-1} + a_2h_{n-2}$.	
	Ans	Definition: (2 marks) Given recurrence relation $h_n = a_1h_{n-1} + a_2h_{n-2}$ ... (1) Its characteristic equation $x^2 - a_1x - a_2 = 0$ ... (2) As q_1 is root of (2) then ${q_1}^2 - a_1q_1 - a_2 = 0$ $\Rightarrow {q_1}^2 = a_1q_1 + a_2$ $h_n = c_1q_1^n + c_2nq_1^n$ $= c_1q_1^{n-2}.{q_1}^2 + c_2nq_1^{n-2}.{q_1}^2$ $= c_1q_1^{n-2}.(a_1q_1 + a_2) + c_2nq_1^{n-2}.(a_1q_1 + a_2)$ ... ... $= c_1h_{n-1} + c_2h_{n-2}$ ...(3) Equation (3) satisfies the equation (1).	2 2 4
Q.2	Attempt any **TWO** questions from the following:		(12)
$b)$	i.	For the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 8 & 1 & 9 & 3 & 7 & 6 & 2 \end{pmatrix}$ (I) Express σ in one row notation. (II) Find the inverse of σ. (III) Express σ as a product of transposition and find the sign of σ.	
	Ans	(2M+2M+2M) (I) $\sigma = (1\,5\,9\,2\,4)(3\,8\,6)$ (II) $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 6 & 2 & 1 & 8 & 7 & 3 & 5 \end{pmatrix}$ (III) $\sigma = (1\,4)(1\,2)(1\,9)(1\,5)(3\,6)(3\,8)$ Or $\sigma = (1\,5)(5\,9)(9\,2)(2\,4)(3\,8)(8\,6)$ $Sign(\sigma) = 1$	2 2 2
	ii.	Solve the following recurrence relations using iteration method: (a) $c_n = (-1.1)c_{n-1}, c_1 = 5$ (b) $d_n = d_{n-1} - 2,\ d_1 = 0$	
	Ans	(a) $c_n = 5(-1.1)^{n-1}$ (b) $d_n = -2(n-1)$	3 3
	iii.	Define Signature of permutation. Prove that , if $\alpha, \beta \in S_n$ then $sgn(\alpha\beta) = sgn(\alpha) \times sgn(\beta)$.	

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	Ans	Definition: $sgn(\alpha) = \prod_{1\le i<j\le n} \frac{\alpha(i)-\alpha(j)}{i-j}$	2
		Let $\alpha, \beta, \in S_n$, By def $sgn(\alpha\beta) = \prod_{1\le i<j\le n} \frac{(\alpha\beta)(i)-(\alpha\beta)(j)}{i-j}$ $= \prod_{1\le i<j\le n} \frac{(\alpha\beta)(i)-(\alpha\beta)(j)}{(\beta)(i)-(\beta)(j)} \times \frac{(\beta)(i)-(\beta)(j)}{i-j}$ $= \prod_{1\le i<j\le n} \frac{\alpha(\beta(i))-\alpha(\beta(j))}{\beta(i)-\beta(j)} \times \prod_{1\le i<j\le n} \frac{(\beta)(i)-(\beta)(j)}{i-j}$ Let $\beta(i) = a,\ \beta(j) = b$ $= \prod_{1\le i<j\le n} \frac{\alpha(a)-\alpha(b)}{a-b} \times sgn(\beta)$ $= sgn(\alpha) \times shn(\beta)$	2 2
	iv.	Solve the linear homogeneous recurrence relation $h_n = h_{n-1} + h_{n-2}$, $n \ge 3$ $h_1 = 1, h_2 = 1$ by using characteristic equation.	
	Ans	$h_n = \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(-\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$	6
Q3.	Attempt any **ONE** question from the following:		(08)
a)	i.	Define finite set. Show that in any set X of people there are two members of X who have the same number of friends in X. (It is assumed that $\|X\|$ is at least 2, and if x is a friend of y then y is a friend of x.)	
	Ans	Def: A set A is said to be finite if it is empty or if $\|A\| = n$ for some $n \in \mathbb{N}$. i.e. if there is a bijection $f: A \to \mathbb{N}_n$ for some $n \in \mathbb{N}$.	2
		Let $\|X\| = m$. Define a function f on X, such that $f(x)$ = number of friends of x.	2
		Then $f(x)$ values are 0, 1, 2, …, m-1. If $f(x) = 0$ for some x, then $f(x)$ cannot be m - 1 for any $x \in X$. Thus, f is not injective. Therefore, there must be a and b, such that $f(a) = f(b)$.	2
		Similarly, If $f(x) = m$ - 1 for some x, then $f(x)$ cannot be 0 for any $x \in X$. And again we get $f(a) = f(b)$ for some a and b in X.	2
	ii.	State the Pigeonhole Principle. Show that in any set of 6 people there are three who are mutual strangers or mutual friends.	

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	Ans	Statement: If there are n pigeonholes and $n+1$ pigeons, then at least one of the pigeonholes must contain at least two pigeons. Fix any person from the group of 6 people. Call him X. Form two pigeonholes, F and S. F will contain friends of X and S will contain those, who are strangers to X. By strong form of Pigeonhole Principle, one of these pigeonholes must contain at least three people. Suppose F contains (at least) three people, B, C, D. If any two of these are friends of each other, then we get three who are mutual friends. If no two of B, C and D are friends of each other, then we get three who are mutual strangers. Similarly, it can be shown if there are (at least) three people in S.	2 2 3 1
Q3.	Attempt any **TWO** questions from the following:		(12)
b)	i.	Show that interval [0,1] is uncountable.	
	Ans	Consider a set A = $\{1/n , n \in \mathbb{N}\}$ $\mathbb{N}$ is infinite so A is also infinite Therefore, A is infinite, then [0, 1] is infinite. Using contradiction prove that [0,1] is uncountable.	2 2 2
	ii.	Find Stirling number of second kind $S(n, k)$ for $n = 5$ and $k = 1, 2$ by actually partitioning.	
	Ans	Let $k = 1$. Then $\{\{a, b, c, d, e\}\}$ is the only partition possible. Hence, $S(5, 1) = 1$. Let $k = 2$. Then, $\{\{a\}, \{b, c, d, e\}\}$, $\{\{b\}, \{a, c, d, e\}\}$, $\{\{c\}, \{a, b, d, e\}\}$, $\{\{d\}, \{a, b, c, e\}\}$, $\{\{e\}, \{a, b, c, d\}\}$, $\{\{a, b\}, \{c, d, e\}\}$, $\{\{a, c\}, \{b, d, e\}\}$, $\{\{a, d\}, \{b, c, e\}\}$, $\{\{a, e\}, \{b, c, d\}\}$, $\{\{b, c\}, \{a, d, e\}\}$, $\{\{b, d\}, \{a, c, e\}\}$, $\{\{b, e\}, \{a, c, d\}\}$, $\{\{c, d\}, \{a, b, e\}\}$, $\{\{c, e\}, \{a, b, d\}\}$, $\{\{d, e\}, \{a, b, c\}\}$ are the only partitions.	2 3

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		Hence, $S(5, 2) = 15$.	
	iii.	Prove by mathematical induction $S(n, n-1) = {}^nC_2$.	
	Ans	P(1) is true, P(2) is true Assume the result is true for p(m) Hence prove p(m+1)	2 2 2
	iv.	In how many ways can we draw (a) a heart and a spade (b) a king and an ace from an ordinary deck of 52 playing cards?	
	Ans	a) $C(13, 1) \times C(13, 1) = 13 \times 13 = 169$ b) $C(4, 1) \times C(4, 1) = 4 \times 4 = 16$	3 3
Q4.	Attempt any **ONE** question from the following:		(08)
$a)$	i.	State and prove The Multinomial Theorem.	
	Ans	The Multinomial Theorem: Let n be a non-negative integer. Then : $(x_1 + x_2 + \cdots + x_r)^n =$ $\sum_{\substack{n_1+n_2+\cdots+n_r=n \\ n_1,n_2,\ldots,n_k(\geq 0)}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$. Proof: We write $(x_1 + x_2 + \cdots + x_r)^n$ as a product of n factors, each equal to $(x_1 + x_2 + \cdots + x_r)$. We expand this product using the distributive law and collect like terms. For each of the n factors we choose one of the r numbers $x_1, x_2, \ldots, x_r$ and form their product. There are r^n terms that result in this way, and each can be arranged in the form $x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$, where $n_1, n_2, \ldots, n_k$ are non-negative integers whose sum is n. We obtain the term $x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$, by choosing x_1 from the n_1 of the n factors, x_2 from the n_2 of the remaining $n - n_1$ factors,…, x_r from n_r factors $n - n_1 - n_2 - \cdots - n_r$ i.e. n_r factors. Thus, by the multiplication principle, the number of times the term $x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$ occurs is given by	2

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		$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$ which is equal to $\binom{n-n_1}{n_1,n_2,\ldots,n_k} = \frac{n!}{n_1!\,n_2!\ldots n_r!}$. Hence, the sum of the r^n terms is $\sum_{\substack{n_1+n_2+\cdots+n_r=n\\ n_1,n_2,\ldots,n_k(\geq 0)}} \binom{n}{n_1,n_2,\ldots,n_r} x_1^{n_1}x_2^{n_2}\ldots x_k^{n_k}$. Thus, $(x_1+x_2+\cdots+x_r)^n =$ $\sum_{\substack{n_1+n_2+\cdots+n_r=n\\ n_1,n_2,\ldots,n_k(\geq 0)}} \binom{n}{n_1,n_2,\ldots,n_r} x_1^{n_1}x_2^{n_2}\ldots x_k^{n_k}$.	6
	ii.	For Euler's ϕ function prove that: If $n = p_1^{\alpha_1}\, p_2^{\alpha_2} \ldots p_t^{\alpha_t}$ is the prime factorization of n, then, $\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_t}\right)$.	
	Ans	Let A_i denote the subset of $\mathbb{N}_n$ which consists of the multiples of $p_j (1 \leq j \leq t)$. Then $\phi(n) = n - \|A_1 \cup A_2 \cup \ldots \cup A_t\|$ $= n - [\sum_{i=1}^{i=t}\|A_i\| - \sum_{1\leq i<j\leq t}\|A_i \cap A_j\| + \sum_{1\leq i<j<k\leq t}\|A_i \cap A_j \cap A_k\| - \ldots$ $+(-1)^{t+1}\|\bigcap_{i=1}^{t} A_i\|]$ $= n - \sum_{i=1}^{i=t}\|A_i\| + \sum_{1\leq i<j\leq t}\|A_i \cap A_j\| - \sum_{1\leq i<j<k\leq t}\|A_i \cap A_j \cap A_k\| - \ldots$ $+(-1)^{t}\|\bigcap_{i=1}^{t} A_i\|$ $= n - \alpha_i + \alpha_2 - \ldots\ldots + (-1)^t\alpha_t$ Where α_i is the sum of the cardinalities of the intersections of $A_1, A_2, \ldots, A_n$ taken i at a time. The intersection $A_{j1} \cap A_{j2} \cap \ldots \cap A_{ji}$ contains the multiples of $p = p_{ji} \times p_{j2} \times \ldots \times p_{ji}$ in $\mathbb{N}_n$ and these are just the integers, $p, 2p, 3p, \ldots, \left(\frac{n}{p}\right)p$. $\therefore$ the cardinality of this intersection is $\frac{n}{p}$, and α_i is the sum of all terms of type, $\frac{n}{p} = n\left(\frac{1}{p_{ji}}\right)\left(\frac{1}{p_{j2}}\right)\ldots\left(\frac{1}{p_{ji}}\right)$. $\therefore\ \phi(n) = n - n\left(\frac{1}{p_1}+\frac{1}{p_2}+\cdots\ \ldots+\frac{1}{p_t}\right) + n\left(\frac{1}{p_1p_2}+\frac{1}{p_1p_3}+\ldots\right)$ $+\cdots\ldots\ldots+(-1)^t\, n\left(\frac{1}{p_1p_2p_3\ldots p_t}\right)$ $= n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_t}\right)$ $\therefore \phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_t}\right)$.	2 2 2 2

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Q4.	Attempt any **TWO** questions from the following:			(12)
b)	i.	Prove by giving a Combinatorial argument: $\sum_{k=0}^{n}\binom{n}{k} = 2^n$.		
	Ans	First we will show that the number of subsets of a n set is 2^n. Let $S = \{x_1, x_2, \dots, x_n\}$ be any n-set and A be any subset of S. Now, The element x_1 has two choices, $x_1 \in A$ or $x_1 \notin A$. Similarly the element x_2 has two choices, $x_2 \in A$ or $x_2 \notin A$. Continuing this way, we can say that every element of S has two choices. Depending upon these choices, different subsets are formed. Hence by the Multiplication Principle, the number of subsets of S are $2 \times 2 \times \dots n$ times $= 2^n$.....(*) Now, consider the L.H.S. of $\sum_{k=0}^{n}\binom{n}{k} = 2^n$. A subset of S can be of size 0 or 1 or 2 ... or n. The number of subsets of S of size 0 is $1 = \binom{n}{0}$. The number of subsets of S of size 1 is $\binom{n}{1}$. The number of subsets of S of size 2 is $\binom{n}{2}$. . . . The number of subsets of S of size n is $\binom{n}{n}$. Hence, by the Addition Principle, the total number of subsets of S is $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$.....(**) By from (*) and (**) we get, $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = \sum_{k=0}^{n}\binom{n}{k}$.	3 2 1	
	ii.	How many 8 letter words can by constructed by using the 26 letters of the English alphabets if each word can contain 3, 4 or 5 vowels? It is understood that there is no restriction on the number of times a letter can be used in constructing a word.		
	Ans	We count the number of words according to the number of vowels they contain and then use the Addition Principle. First, consider words with 3 vowels. Consider any 8 letter word. The 3 positions occupied by the vowels can be chosen in $\binom{8}{3}$ ways and then the other 5 positions are taken by the consonants. There are total 5 vowels and 21 consonants in the alphabets. Since the	2	

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		repetition of letters is allowed, the 3 places of vowels can be filled in 5^3 ways and 5 places of consonants are filled in 21^5. So, the total number of words with 3 vowels is $\binom{8}{3}5^3 21^5$. In a similar way, we can see that the number of words with 4 vowels is $\binom{8}{4}5^4 21^4$ and the number of words with 5 vowels is $\binom{8}{5}5^5 21^3$. The total number of words of the required type is $\binom{8}{3}5^3 21^5 + \binom{8}{4}5^4 21^4 + \binom{8}{5}5^5 21^3$.	2 2
	iii.	Find the number of positive integers from 1 to 600 (both inclusive) which are not divisible by 2 , 3 and 5.	
	Ans	Let A = set of all numbers divisible by 2. B = set of all numbers divisible by 3. C = set of all numbers divisible by 5. $\therefore |A| = \left\lfloor \frac{600}{2} \right\rfloor = 300$ $|B| = \left\lfloor \frac{600}{3} \right\rfloor = 200$ $|C| = \left\lfloor \frac{600}{5} \right\rfloor = 120$ $|A \cap B| = \left\lfloor \frac{600}{\text{LCM }(2,3)} \right\rfloor = \left\lfloor \frac{600}{6} \right\rfloor = 100$ $|B \cap C| = \left\lfloor \frac{600}{\text{LCM }(3,5)} \right\rfloor = \left\lfloor \frac{600}{15} \right\rfloor = 40$ $|A \cap C| = \left\lfloor \frac{600}{\text{LCM }(2,5)} \right\rfloor = \left\lfloor \frac{600}{10} \right\rfloor = 60$ $|A \cap B \cap C| = \left\lfloor \frac{600}{\text{LCM }(2,3,5)} \right\rfloor = \left\lfloor \frac{600}{30} \right\rfloor = 20$ $\therefore$ By Inclusion Exclusion principle, the number of positive integers from 1 to 600 (both inclusive) which are divisible by 2 , 3 and 5. $|A \cup B \cup C|$ $= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$ $= 300 + 200 + 120 - 100 - 40 - 60 + 20$ $=440$ $\therefore$ numbers not divisible by 2, 3 and 5 are $= 600 - |A \cup B \cup C|$ $= 600 - 440$ $= 160$ $\therefore$ 160 Numbers are not divisible by 2, 3 and 5.	2 2 2

	iv.	(a) Show that $D_n = nD_{n-1} + (-1)^n, n \geq 2$. (b) In how many ways can the integers 0, 1, 2, …, 9 be permutated so that exactly 4 of the integers are in natural position?	
	Ans	(a) We have, LHS= D_{k+1} $= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}\right)$ $= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots + \frac{(-1)^{n-1}}{(n-1)!}\right) + n!\,\frac{(-1)^n}{n!}$ $= n(n-1)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots + \frac{(-1)^{n-1}}{(n-1)!}\right) + (-1)^n$ $= nD_{n-1} + (-1)^n$ $=$ RHS $\therefore\ D_n = nD_{n-1} + (-1)^n, n \geq 2$	3
		(b) Out of the 9 integers any 4 can be selected by $C(9,4)$ ways. These 4 integers can be arranged in natural position in only 1 way. The remaining 5 integers should be deranged. This can be done in D_5 ways. $\therefore$ by multiplication principle, the required number of permutations is $C(9,4)D_5 = \frac{9!}{4!5!} \times 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) =$ $126 \times 44 = 5544$ ways.	3
Q5.	Attempt any **FOUR** questions from the following:		(20)
a)	Show that the number of elements in S_n is $(n!)$. List all the elements in S_3.		
Ans	Let $S = \{1,2,..,n\}$ and each permutation is bijection from S to S. $\therefore$ image of 1 to n elements can be selected in following ways 1^{st} element image ...n ways 2^{nd} element image ...n-1 ways n^{th} element image ...1 way Total ways=$1 \times 2 \times .. \times n = n!$		2
	Elements of S_3 $P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$		3
b)	Prove that $b_n < \left(\frac{5}{2}\right)^n$ for the recurrence relation $b_n = b_{n-1} + 2b_{n-2}$, $b_1 = 1, b_2 = 3, n \geq 3$.		

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Ans	$P(n): b_n < \left(\frac{5}{2}\right)^n$ $P(1)$ is true Assume $P(k)$ Prove $P(k+1)$	 1 1 3
$c)$	In Algebra class, 32 of the students are boys. Each boy knows five of the girls in the class and each girl knows eight of the boys. How many girls are in the class?	
Ans	If we put each boy in rows and each girl in column. Then total of each row is 5 and there are 32 rows. total number $= 32 \times 5 = 160$ Since each girl knows eight boys,. Let there are n girls total number = 8n 8n = 160 Number of girls = 20 .	2 2 1
$d)$	State strong form of Pigeonhole Principle (Extended Pigeonhole Principle). Given 5 points in the plane with integer coordinates, show that there exists a pair of points whose midpoint also has integer coordinates.	
Ans	Let $q_1, q_2, \ldots\ldots q_n$ be positive integers. If $q_1 + q_2 + \ldots\ldots + q_n - n + 1 > 0$ objects are put into n boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 objects, or the n^{th} box contains at least q_n objects. Let there be four pigeonholes: EE, OO, EO, OE. The five points be the five pigeons. Therefore, by Pigeonhole Principle, there must exist at least two points in one of these pigeonholes. Suppose, there are (at least) two points *a* and *b* in EE, it means both the coordinates of *a* and *b* are even numbers. Clearly their midpoint also has integer coordinates. Similarly, it can be shown if either OO, EO or OE contains (at least) two points.	1 2 1 1

e) Prepare Pascal's triangle up to $n = 8$.

Ans

$n:k$	0	1	2	3	4	5	6	7	8
1	1								
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5				
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

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f) Define Euler ϕ function. Hence find $\phi(480)$.

Ans

Euler function $\phi: \mathbb{N} \to \mathbb{N}$ is defined by: $\phi(n)$ = the number of positive integers less than or equal to n which are relatively prime to n. 1

$\phi(480) = \phi(2^5.3^1.5^1)$

$= \phi(2^5)\phi(3^1)\phi(5^1)$ by definition of ϕ function.

$= 2^5\left(1-\frac{1}{2}\right)3\left(1-\frac{1}{3}\right)5\left(1-\frac{1}{5}\right)$

$= 2^5\left(\frac{1}{2}\right)3\left(\frac{2}{3}\right)5\left(1-\frac{1}{5}\right)$ 2

$= 32\left(\frac{1}{2}\right)3\left(\frac{2}{3}\right)5\left(\frac{4}{5}\right)$

$= 128$ 2

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