M-3 SOLUTION SET-1 code.No. 00054561

(3 Hours) [Total Marks: 100]

Note: (i) All questions are compulsory.
(ii)Figures to the right indicate marks for respective parts.

Q.1	Choose correct alternative in each of the following			(20)
i.	The number of elements in S_6 is			
	(a)	6	(b)	120
	(c)	720	(d)	36
	Ans	(c) 720		
ii.	If $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}, \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$ then $(\sigma \circ \pi)(4) =$ ______			
	(a)	3	(b)	1
	(b)	2	(d)	5
	Ans	(b) 1		
iii.	Signature of identity permutation in S_n is			
	(a)	-1	(b)	0
	(c)	1	(d)	Depends on n
	Ans	(c) 1		
iv.	For the sequence $a_n = 6(1/3)^n$, a_4 is			
	(a)	$2/25$	(b)	$2/27$
	(c)	$2/19$	(d)	$2/13$
	Ans	(b) $2/27$		
v.	If X, Y are finite sets and there is an injective function $f: X \to Y$ then			
	(a)	$\|X\| = \|Y\|$	(b)	$\|X\| \leq \|Y\|$
	(c)	$\|X\| \geq \|Y\|$	(d)	None of these
	Ans	(b) $\|X\| \leq \|Y\|$		
vi.	Let $S(n,k)$ denote the Stirling number of second kind on n-set into k-disjoint non-empty unordered subsets, then $S(n, n)$ is			
	(a)	1	(b)	0
	(c)	n	(d)	None of these
	Ans	(a) 1		
vii.	What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?			
	(a)	25	(b)	6
	(c)	26	(d)	None of these
	Ans	(c) 26		
viii.	In how many ways can the letters of the word HAMMER be arranged in a row?			

	(a)	72	(b)	144
	(c)	360	(d)	None of the above
	Ans	(c) 360		
$ix.$	For $n > 2,\ \phi(n)$ is			
	(a)	Prime number	(b)	Even number
	(c)	Odd number	(d)	None of the above
	Ans	(b) Even number		
$x.$	Which of the following is derangement of the permutation 12345?			
	(a)	21543	(b)	21453
	(c)	12345	(d)	None of the above
	Ans	(b) 21453		

Q2.	Attempt any **ONE** question from the following:		(08)
$a)$	i.	Prove that any permutation in S_n can be expressed as a product of disjoint cycles.	
	Ans	Let S_n denotes set of all permutation on $S = \{1,2, \ldots, n\}$. Let $\sigma \in S_n$. If σ is identity then $\sigma = (1)(2) \ldots (n)$ [required disjoint cycles]. If σ is not identity then , first put down cycles of length one with the help of element which are remain unchanged under σ. Now choose any element from the set say x (other than cycles of length one). Permute σ on x until we reach to x again. If the permutation takes all the element then we get a cycle say $C_x = (x, \sigma(x), \sigma^2(x), \ldots \sigma^r(x))$. Thus above cycle along with cycles of length one which are required disjoint cycles. If above permutation does not take all elements , then choose the another element say y which is not already used and repeat the same above process. After a finite number of steps , the process will terminate as there are only finite element in S. Thus we get product of disjoint cycles.	
	ii.	Define Linear Homogeneous recurrence relation of degree n. Show that if the characteristic equation $x^2 - a_1x - a_2 = 0$ of the recurrence relation $h_n = a_1h_{n-1} + a_2h_{n-2}$ has a single non-zero roots q_1 then $h_n = c_1q_1^n + c_2nq_1^n$ is the generalsolution of the recurrence relation$h_n = a_1h_{n-1} + a_2h_{n-2}$.	

	Ans	Definition: (2 marks) Given recurrence relation $h_n = a_1 h_{n-1} + a_2 h_{n-2}$ ... (1) Its characteristic equation $x^2 - a_1 x - a_2 = 0$ ... (2) As q_1 is root of (2) then $q_1{}^2 - a_1 q_1 - a_2 = 0$ $\Rightarrow q_1{}^2 = a_1 q_1 + a_2$ $h_n = c_1 q_1^n + c_2 n q_1^n$ $= c_1 q_1^{n-2} . q_1{}^2 + c_2 n q_1^{n-2} . q_1{}^2$ $= c_1 q_1^{n-2} . (a_1 q_1 + a_2) + c_2 n q_1^{n-2} . (a_1 q_1 + a_2)$ ... ... $= c_1 h_{n-1} + c_2 h_{n-2}$ ...(3) Equation (3) satisfies the equation (1).
Q.2	Attempt any **TWO** questions from the following: (12)	
b)	i.	For the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 1 & 8 & 3 & 7 & 9 & 6 & 2 \end{pmatrix}$ (I) Express σ in one row notation. (II) Find the inverse of σ. (III) Express σ as a product of transposition and find the sign of σ.
	Ans	(2M+2M+2M) (I) $\sigma = (1\ 4\ 8\ 6\ 7\ 9\ 2\ 5\ 3)$ (II) $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 9 & 5 & 1 & 2 & 8 & 6 & 4 & 7 \end{pmatrix}$ (III) $\sigma = (1\ 3)(1\ 5)(1\ 2)(1\ 9)(1\ 7)(1\ 6)(1\ 8)(1\ 4)$ Or $\sigma = (1\ 4)(4\ 8)(8\ 6)(6\ 7)(7\ 9)(9\ 2)(2\ 5)(5\ 3)$ $Sign(\sigma) = 1$
	ii.	Define even permutation and odd permutation. Show that : (I) A product of two even permutations is an even permutation. (II) A product of even permutation and odd permutation is an odd permutation.
	Ans	Def. : Even permutation (1 mark) Def. : Odd permutation (1 mark) (I) (2 marks) α, β – even permutations , then $\alpha = 2p$, $\beta = 2k$ Where p , k are no. of transpositions. $\therefore \alpha\beta = 2p + 2k = 2(p + k)$

(9)

		$\therefore\ \alpha\beta$ is even. (II) (2 ,arks) ρ – even permutations and σ – odd permutation , then $= 2n\ ,\ \sigma = 2m + 1$ Where p , k are no. of transpositions. $\therefore \rho\sigma = 2n + 2m + 1 = 2(n+m) + 1$ $\therefore\ \rho\sigma$ is odd.	
	iii.	Find recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive $0's$. How many such bit strings are there of length five?	
	Ans	$a_n = a_{n-1} + a_{n-2}$, $a_1 = 2$, $a_2 = 3$	
	iv.	Solve the linear homogeneous recurrence relation $h_n = 5h_{n-1} - 6h_{n-2}, h_0 = 1, h_1 = 0$ by using characteristic equation.	
	Ans	$a_n = 3 \times 2^n + (-2) \times 3^n$	
Q3.	Attempt any **ONE** question from the following:		(08)
a)	i.	Define countable set and give an example of the same.. Also show that the set of all integers is countable.	
	Ans	Def: A set which is empty, finite or denumerable is called countable set. The set of integers is a countable set. The set $\mathbb{Z}$is infinite set. If we show that there is a one-to-one correspondence between $\mathbb{N}$ to $\mathbb{Z}$ then $\mathbb{Z}$ will be countably infinite. Define f: $\mathbb{N} \to \mathbb{Z}$as $f(n) = \frac{n}{2}$ if n is even $= \frac{n-1}{2}$ if n is odd We can show that function f: $\mathbb{N} \to \mathbb{Z}$ is one-one and onto Therefore, Z is countable.	2 1 1 4
	ii.	State the recurrence relation for $S(n, k)$ and find the value of $S(7, 4)$.	

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	Ans	$S(n,k) = S(n-1,k-1) + kS(n-1,k)$, where $2 \le k \le n-1$ $S(7,4) = S(6,3) + 4S(6,4)$ $S(6,3) = S(5,2) + 3S(5,3)$ Now $S(5,2) = 2^{5-1} - 1 = 15$ $S(5,3) = S(4,2) + 3S(4,3)$ Now $S(4,2) = 2^{4-1} - 1 = 7$ Also, $S(4,3) = C(4,2) = 6$ Thus, $S(5,3) = 7 + 3 \times 6 = 25$ Hence, $S(6,3) = 15 + 3 \times 25 = 90$ Now, $S(6,4) = S(5,3) + 4S(5,4)$ $S(5,3) = 25$, calculated already. And $S(5,4) = C(5,2) = 10$. So, $S(6,4) = 25 + 4 \times 10 = 65$. Hence, $S(7,4) = 90 + 4 \times 65 = 90 + 260 = 350$	1 1 3 3
Q3.	Attempt any **TWO** questions from the following:		(12)
b)	i.	Show that interval [0,1] is uncountable.	
	Ans	Consider a set A = { $1/n$, $n \in \mathbb{N}$ } $\mathbb{N}$ is infinite so A is also infinite Therefore, A is infinite, then [0, 1] is infinite. Using contradiction prove that [0,1] is uncountable	1 1 4
	ii.	Let $A = \{a, b, c, d\}$. Find Stirling number of second kind for $k = 1, 2, 3, 4$ by actually partitioning of A into k parts.	
	Ans	Let $k = 1$. Then $\{\{a, b, c, d\}\}$ is the only partition possible. Hence, $S(4, 1) = 1$. Let $k = 2$. Then, $\{\{a\}, \{b, c, d\}\}$, $\{\{b\}, \{a, c, d\}\}$, $\{\{c\}, \{a, b, d\}\}$, $\{\{d\}, \{a, b, c\}\}$, $\{\{a, b\}, \{c, d\}\}$, $\{\{a, c\}\}, \{b, d\}\}$, $\{\{a, d\}, \{b, c\}\}$ are the only partitions. Hence, $S(4, 2) = 7$. Let $k = 3$. Then, $\{\{a, b\}, \{c\}, \{d\}\}$, $\{\{a, c\}, \{b\}, \{d\}\}$, $\{\{a, d\}, \{b\}, \{c\}\}$, $\{\{b, c\}, \{d\}, \{a\}\}$, $\{\{b, d\}, \{a\}, \{c\}\}$, $\{\{c, d\}, \{a\}, \{b\}\}$ are the only partitions. Hence, $S(4, 3) = 6$.	1 2 2

(6)

		Finally, let $k = 4$. Then $\{\{a\}, \{b\}, \{c\}, \{d\}\}$ is the only partition possible. Therefore, $S(4, 4) = 1$.	1
	iii.	Prove by mathematical induction $S(n, n-1) = {}^{n}C_{2}$.	
	Ans	P(1) is true, P(2) is true Assume the result is true for p(m) Hence prove p(m+1)	1 1 4
	iv.	How many different 4-letter radio station call letters (upper case) can be made a) if the first letter must be a K or W and no letter may be repeated? b) if repeats are allowed (but the first letter is a K or W). c) starting with K or W) with no repeats and ending in R?	
	Ans	a) $2 \times 25 \times 24 \times 23 = 27{,}600$ b) $2 \times 26 \times 26 \times 26 = 35{,}152$ c) $2 \times 24 \times 23 = 1{,}104$	2 2 2
Q4.	Attempt any **ONE** question from the following:		(08)
a)	i.	Prove by giving Combinatorial argument: $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$.	
	Ans	Let T be a $(m+n)$-set with m objects of type I and the remaining n objects of typeII. We will find the number of r-subsets of T in two different ways. We know that the number of r-subsets of a $(m+n)$-set is$\binom{m+n}{r}$ … (*) Any r-subset of T will contain k number of objects of type I and the remaining $r-k$ objectsof type II, where $0 \le k \le r$. The number of ways by which we select i objects of type I is$\binom{m}{k}$and the number of waysby which we select $r-k$ objects of type II is$\binom{n}{r-k}$. So, by the Multiplication Principle, the number of ways by which we select k objects of typeI and $r-k$ objects of type II is$\binom{m}{k}\binom{n}{r-k}$. Now, since k can be 0 *or* 1 *or* … *or* r. ∴by the Addition Principle,	

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		the number of r-subsets of T is $\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$ … (**) From (*) and (**) we have $\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}=\binom{m+n}{r}$.
	ii.	Show that number of non-negative integer solutions to the equation $x_1+x_2+\ldots+x_k=n$ is $\binom{n+k-1}{k-1}$
	Ans	Above problem is equivalent to, distributing n undistinguishable balls among k persons, each one getting $x_1, x_2, \ldots, x_k$ balls respectively. First let us consider the case when each person gets at least one ball, i.e. $x_i \geq 1$, for $i = 1,2 \ldots, k$. Keeping all undistinguishable balls in a line. If we consider a gap between two consecutive balls, then there are, a total of $n-1$ gaps. ⊗⊗⊗⊗⊗⊗.......⊗ ⊗⊗⊗ (n balls, $n-1$ gaps) Now out of $n-1$ gaps if we select any $k-1$ gaps, there will be k partitions. This can be done by $\binom{n-1}{k-1}$ numbers of ways. ∴ number of ways n undistinguishable balls are distributed among k persons are $\binom{n-1}{k-1}$ (1) Now we shall discuss the case when each $x_i \geq 0$ (non- negative). Adding k to both sides of the equation, $x_1+x_2+\ldots+x_k=n$ we get, $x_1+x_2+\ldots+x_k+k=n+k$ $\therefore (x_1+1)+(x_2+1)+\ldots(x_k+1)=n+k$ Put $x_1+1=y_1,\ x_2+1=y_2, \ldots, x_k+1=y_k$ we get, $y_1+y_2+\ldots+y_k=n+k$, (2) here each $y_i \geq 1, i = 1,2, \ldots k,$ equation (2)has $\binom{n+k-1}{k-1}$ positive integer solutions, [by (1)] As each $y_i = x_i+1, i = 1,2,\ldots k.\ i.e.\ y_i \geq 1$ $\Rightarrow x_i+1 \geq 1$, $i = 1,2,..k$ $\Rightarrow x_i \geq 0 \forall\ i = 1,2 \ldots n$ ∴ number of non-negative integer solutions to the equation $x_1+x_2+\ldots+x_k=n$ is $\binom{n+k-1}{k-1}$
Q4.	Attempt any **TWO** questions from the following:	(12)

b) i. State Pascal's identity and prepare Pascal's triangle up to $n = 7$.

Ans Pascal Identity: Let n and k be positive integers, then
$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

$n:k$	0	1	2	3	4	5	6	7	8
1	1								
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5				
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	

ii. 10 people including 2 who do not wish to sit next to each other are to be seated on a round table. How many circular seating arrangements are there for them to sit?

Ans Let the ten people be$P_1, P_2, \dots, P_{10}$, where P_1 and P_2 are the two people who do not wishto sit together.
The total number of circular permutations is $(10-1)! = 9!$.
Now first we will find the number of circular permutations in which P_1 and P_2 are together.
If we want the number of circular permutations in which P_1 and $P_2$are together, we have to considerP_1 and P_2 together as one person.
So, now there are 9 people and there are 8! ways by which we can be arrange them in circulararrangement.
Now these 2people can be arranged among themselves in 2 ways.
So, the total number of ways by which we can arrange these 10 people in a circular manner so thatP_1 and P_2 are always together is $2 \times 8!$.
Hence, the total number of circular arrangements in which P_1 and $P_2$are not together is$9! - 2 \times 8! = 8!(9-2) = 8! \times 7$.

iii. A professor in a Discrete Mathematics class passes out a form asking students to check all the Mathematics and Computer Science courses they have recently taken. The finding is that out of a total of 50 students in the class, 30 took Precalculus; 16 took both Precalculus and Java; 18 took Calculus: 8 took both Calculus and Java; 26 took Java, 9 took both Precalculus and Calculus and 6 took all three courses. How many students did not take any of the three courses?

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	Ans	Let A = Set of students who have taken Precalculus. B = set of students who have taken Calculus. C = set of students taken Java course. $\therefore |A| = 30, |B| = 18, \ |C| = 26, |A \cap B \cap C| = 6$ $|A \cap B| = 9, \quad |A \cap C| = 16, \quad |B \cap C| = 8$ $\therefore$ By inclusion exclusion principle, number of students who have taken atleast one of the courses are $|A \cup B \cup C|$ $= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$ $= 30 + 18 + 26 - 9 - 16 - 8 + 6$ $= 47$ Total number of students are 50. Number of students who did not take any of the three courses $= 50 - |A \cup B \cup C|$ $= 50 - 47$ $= 3$	
	iv.	Determine the total number of integral solutions of $x_1 + x_2 + x_3 = 12$ with $x_1 \geq -2, \ x_2 \geq 4$ and $x_3 \geq 2$.	
	Ans	Given that $x_1 \geq -2, \ x_2 \geq 4$ and $x_3 \geq 2$ Let $x_1 = y_1 - 2, \ x_2 = y_2 + 4,$ and $x_3 = y_3 + 2,$ $\therefore y_1, y_2, y_3 \geq 0$ $\therefore x_1 + x_2 + x_3 = 12$ becomes, $y_1 + y_2 + y_3 = 8$, we have, $k = 3, \quad n = 8$ $\therefore$ number of non-negative integral solutions for $y_1 + y_2 + y_3 = 8$ are $\binom{n+k-1}{k-1} = \binom{8+3-1}{3-1}$ $= \binom{10}{2}$ $= 45$	
			(20)
Q5.	Attempt any **FOUR** questions from the following:		
$a)$	If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ find whether $\alpha \circ \beta = \beta \circ \alpha$.		
Ans	$\alpha\beta \neq \beta\alpha$		
$b)$	Suppose that a person deposits Rs 10,000 in a saving account at a bank yielding 11 percent per year with interest compounded annually. How much will be in the account after 30 years? Solve using an appropriate recurrence relation.		

Ans	$a_n = (1.11)^n\, a_{n-1}$ $a_{30} = 10000 \times (1.11)^{30} \approx 228923$	
c)	In Algebra class, 32 of the students are boys. Each boy knows five of the girls in the class and each girl knows eight of the boys. How many girls are in the class?	
Ans	If we put each boy in rows and each girl in column. Then total of each row is 5 and there are 32 rows. total number $= 32 \times 5 = 160$ Since each girl knows eight boys,. Let there are n girls total number = 8n 8n = 160 Number of girls = 20 .	2 2 1
d)	State strong form of Pigeonhole Principle (Extended Pigeonhole Principle). Show that among any five points inside an equilateral triangle of side length 1, there exist at least two points whose distance is at most $\frac{1}{2}$.	
Ans	Let $q_1, q_2, \ldots\ldots\ldots q_n$ be positive integers. If $q_1 + q_2 + \ldots\ldots\ldots + q_n - n + 1 > 0$ objects are put into n boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 objects, or the n^{th} box contains at least q_n objects. Split the equilateral triangle ABC into 4 smaller triangles by connecting the midpoints. Each of these small triangles is a pigeonhole and the five points are pigeons. Therefore, at least one of these small triangles must contain at least two points. Obviously the distance between such two points is at the most $\frac{1}{2}$.	1 1 2 1
e)	Find the term that does not contain x in the complete expansion of $\left(x^2 + x + \frac{1}{x}\right)^4$.	
Ans	By Multinomial theorem $\left(x^2 + x + \frac{1}{x}\right)^4 =$ $\sum_{\substack{n_1+n_2+n_3=4 \\ n_1,n_2,n_3(\geq 0)}} \binom{4}{n_1, n_2, n_3} (x^2)^{n_1} x^{n_2} \left(\frac{1}{x}\right)^{n_3} =$ $\sum_{\substack{n_1+n_2+n_3=4 \\ n_1,n_2,n_3(\geq 0)}} \binom{4}{n_1, n_2, n_3} x^{2n_1+n_2-n_3}$.	

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	To find the term that does not contain x we must have $2n_1 + n_2 - n_3 = 0$ in addition to $n_1 + n_2 + n_3 = 4$ and $n_1, n_2, n_3 (\geq 0)$. The only values of n_1, n_2, and n_3 satisfying these constraints is $n_1 = 0, n_2 = 2, n_3 = 2$. ∴the only term that does not contain x in the complete expansion of $\left(x^2 + x + \frac{1}{x}\right)^{10}$ is $\binom{4}{0,2,2} = \frac{4!}{0!2!2!} = 6$.	
f)	Find ϕ (2400).	
Ans	$$\begin{aligned} &\phi(2400) \\ &= \phi(2^5.3^1.5^2) \\ &= \phi(2^5)\phi(3^1)\phi(5^2) \\ &= 2400\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) \\ &= 2400\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right) \\ &= 2400 \times \frac{4}{15} \\ &= 640 \end{aligned}$$	

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