① Q.P. Code: 54960

(3 Hours) [Total Marks: 100]

Note: (*i*) All questions are compulsory. (*ii*) Figures to the right indicate marks for respective parts.

Q.1	Choose correct alternative in each of the following (20)			
i.	$T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation if $\forall u,v \in V, \alpha,\beta \in \mathbb{R}$, then			
	(a)	$T(\alpha u+\beta v)=\alpha T(u)+\beta T(v)$	(b)	$T(\alpha uv)=\alpha T(u)\cdot T(v)$
	(c)	$T(\alpha u+\beta v)=\alpha T(u)\cdot\beta T(v)$	(d)	None of the above
	Ans	a		
ii.	If $T:U\to V$ is a linear transformation then			
	(a)	$T(0)=0$	(b)	$T(-u)=-T(u), \forall u\in U$
	(c)	$T(u_1-u_2)=T(u_1)-T(u_2)$ $,\forall u_1,u_2\in U$	(d)	All of these.
	Ans	d		
iii.	Which of the following is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$?			
	(a)	$T(x,y)=(xy,y)$	(b)	$T(x,y)=(x+1,y+1)$
	(c)	$T(x,y)=(x+y, x-y)$	(d)	All the above
	Ans	c		
iv.	Let A be a $m\times n$ matrix then			
	(a)	Rank A = number of non-zero columns of A	(b)	Rank A= number of linearly independent rows of A
	(c)	Rank A = number of non-zero rows of A	(d)	None of these
	Ans	b		
v.	Let $E=\begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix}$ then E^{-1} is			
	(a)	$\begin{pmatrix}1&0&0\\0&-3&0\\0&0&1\end{pmatrix}$	(b)	$\begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix}$
	(c)	$\begin{pmatrix}1&0&0\\0&1/3&0\\0&0&1\end{pmatrix}$	(d)	$\begin{pmatrix}1&0&0\\0&-1/3&0\\0&0&1\end{pmatrix}$
	Ans	(c)		
vi.	Let A be the matrix with If $\tilde{\tilde{A}}$ is Adjoint(A) then inverse of A is given by			
	(a)	$\det(A).\tilde{\tilde{A}}$	(b)	$\det(\tilde{\tilde{A}}).\tilde{\tilde{A}}$
	(c)	$\tilde{\tilde{A}}/\det(A)$	(d)	$A.\tilde{\tilde{A}}$
	Ans	(c)		
vii.	A $m\times n$ non-homogeneous system of linear equations $AX=b$ has a solution if and only if			
	(a)	$Rank\ A = Rank[A\|b]$	(b)	$Rank\ A > Rank[A\|b]$
	(c)	$Rank\ A < Rank[A\|b]$	(d)	none of these
	Ans	(a)		

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viii.	Which of the following groups is non-abelian			
	(a)	V_4	(b)	S_3
	(c)	C_{13}	(d)	None of the above
	Ans	(b)		
ix.	The inverse of i in the multiplicative group $\{-1,1,i,-i\}$ is			
	(a)	1	(b)	i
	(c)	$-i$	(d)	-1
	Ans	(c)		
x.	Let O denote the set of odd integers. Then			
	(a)	O forms a group under the operation of addition	(b)	O forms a group under the operation of multiplication
	(c)	O does not forms a group under the operation of addition	(d)	None of the above
	Ans	(c)		
Q2.	Attempt any **ONE** question from the following: (08)			
a)	i.	State and prove Rank-Nullity Theorem.		

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Ans

Proof : We have $T : V \to W$, be a linear transformation, $\ker T \subsetneq V$ is a subspace of V.

Let $\dim V = n$, $\dim \ker T = r$, $\dim W = m$

Let $B = \{u_1, u_2, \cdots\cdots, u_r\}$ be basis of ker T As ker T is subspace of V, B is a linearly independent subset of V and hence can be extended to a basis of V.

Let $B_1 = \{u_1, u_2, \cdots\cdots, u_r, u_{r+1}, \cdots, u_n\}$ be a basis of V, obtained by extension of B.

Let $w_i = T(V_{r+i})$, $\forall\ i = 1, \cdots, n-r$. 1

Claim : $B_2 = \{w_1, w_2, \cdots\cdots, w_{n-r}\}$ forms a basis of $I_m T$

Let us prove first. B_2 is linearly independent 1

Let $a_1, a_2, \cdots\cdots a_{n-r}$ be scalars such that

$a_1 w_1 + a_2 w_2 + \cdots\cdot + a_{n-r} w_{n-r} = 0$

But $T(u_{r+1}) = w_1, T(u_{r+2}) = w_2, \cdots\cdots T(u_n) = w_{n-r}$

$\therefore\ a_1 T(u_{r+1}) + a_2 T(u_{r+2}) + \cdots + a_{n-r} T(u_n) = 0$ 1

$\therefore\ T(a_1 u_{r+1} + a_2 u_{r+2} + \cdots\cdots + a_{n-r} u_n) = 0$

$$\Rightarrow T\left(\sum_{i=1}^{n-r} a_i u_{r+i}\right) = 0$$ 1

$$\Rightarrow \sum_{i=1}^{n-r} a_i u_{r+i} \in \ker T$$

$\Rightarrow \exists\ b_1, b_2, \cdots, b_r$ scalar s.t.

$$\sum_{i=1}^{n-r} a_i u_{r+i} = \sum_{j=1}^{r} b_j u \cdots\cdots \text{as B is basis of ker T}$$ 1

$\Rightarrow b_1 u_1 + b_2 u_2 + \cdots\cdots + b_r u_r - (a_1 u_{r+1} + \cdots\cdot + a_{n-r} u_n) = 0$

As $B_1 = \{u_1, u_2, \cdots, u_r, u_{r+1}, \cdots\ u_n\}$ is lin independent

$\Rightarrow b_1 = b_2 = b_r = 0$ and 1

$a_1 = a_2 = a_{n-r} = 0$

$\Rightarrow \{w_1, w_2, \cdots, w_{n-r}\}$ is linearly independent $\cdots$ (I) 1

Claim : $\{w_1, w_2, \cdots, w_{n-r}\}$ spans $I_m(T)$

Let $w \in I_mT$

$\Rightarrow \exists\, v \in V$ such that $T(v) = w$.

As $B_1 = \{u_1, u_2, \cdots, u_r, u_{r+1}, u_{r+2}, \cdots u_n\}$ is a basis of V. 1

$\Rightarrow \exists\, b_1, b_2, \cdots, b_n \in \mathbb{R}$ such that

$v = b_1u_1 + b_2u_2 + \cdots + b_ru_r + b_{r+1}u_{r+1} + \cdots + b_nu_n$

As T is linear,

$\therefore\ T(v) = T(b_1u_1 + \cdots + b_ru_r) + T(b_{r+1}, u_{r+1} + \cdots + b_nu_n)$

$= b_1T(u_1) + b_2T(u_2) + \cdots + b_rT(u_r) + b_r + 1T(u_{r+1}) + \cdots b_nT(u_n) \cdots$

as T is linear

$$\therefore\ T(v) = b_{r+1}T(u_{r+1}) + b_{r+2}T(u_{r+2}) + \cdots + b_nT(u_n)$$

as $u_1, u_2, \cdots, u_r \in \ker T$

$$\Rightarrow T(v) = b_{r+1}w_1 + b_{r+2}w_2 + \cdots + b_nw_{n-r}$$

$$\Rightarrow w = b_{r+1}w_1 + b_{r+2}w_2 \cdots + b_nw_n.$$

$\Rightarrow w \in \text{span}\{w_1, \cdots, w_{n-r}\}$

$\Rightarrow$ If $w \in I_mT \Rightarrow w \in \text{span}\{w_1, w_2, \cdots, w_{n-r}\}$

$\Rightarrow \{w_1, w_2, \cdots, w_{n-r}\}$ spans $I_mT \ldots$ (II)

$\Rightarrow \{w_1, w_2, \cdots w_{n-r}\}$ forms a basis of ImT $\cdots$ From (I) and (II)

$\therefore\ \dim(I_mT) = n - r$

$\dim(\ker T) = r$

$\dim v = n = \dim(I_mT) + \dim(\ker T) = r + n - r.$

$\therefore$ $\boxed{\dim(V) = \dim(\ker T) + \dim(I_mT)}$

as $\boxed{n = r + n - r}$

$\therefore$ Rank nullity theorem is verified.

ii. Show that the vector space of all polynomials in x of degree less than or equal to n is isomorphic to $\mathbb{R}^{n+1}$

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	Ans	i) $P_n[x] = \{a_0 + a_1x + a_2x^2 + \dots + a_nx^n : a_0, a_1, \dots a_n \in \mathbb{R}\}$ $\mathbb{R}^{n+1} = \{(a_0, a_1, a_2 \dots a_n) : a_0, a_1, \dots a_n \in \mathbb{R}\}$ Define $T : \mathbb{R}^{n+1} \to P_n[x]$ as $T(a_0, a_1, \dots a_n) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ --- (1) $\ker T = \{(a_0, a_1, \dots a_n) : a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0\}$ $= \{(a_0, a_1, \dots a_n) : a_0 = 0, a_1 = 0, \dots a_n = 0\}$ $= \{(0, 0, \dots 0)\}$ $\therefore$ KerT is one-one ---- (1) Also if $b_0 + b_1x + \dots + b_nx^n \in P_n[x]$ is any then $\exists (b_0, b_1, \dots b_n) \in \mathbb{R}^{n+1}$ such that $T(b_0, b_1, \dots b_n) = b_0 + b_1x + \dots + b_nx^n$ $\therefore$ T is onto --- (1) $\therefore$ T is bijective map Also $T(\alpha\bar{a} + \beta\bar{b})$ where $\bar{a} = (a_0, \dots a_n)$, $\bar{b} = (b_0, \dots b_n)$ $= T((\alpha a_0 + \beta b_0, \alpha a_1 + \beta b_1, \dots \alpha a_n + \beta b_n))$ $= (\alpha a_0 + \beta b_0)x^0 + (\alpha a_1 + \beta b_1)x^1 + \dots + (\alpha a_n + \beta b_n)x^n$ $= \alpha a_0x^0 + \alpha a_1x^1 + \dots + \alpha a_nx^n + \beta b_0x^0 + \beta b_1x^1 + \dots + \beta b_nx^n$ $= \alpha(a_0 + a_1x + \dots + a_nx^n) + \beta(b_0 + b_1x + \dots + b_nx^n)$ $= \alpha T(\bar{a}) + \beta T(\bar{b})$ $\therefore$ T is also a linear map --- (2) $\therefore$ T is a bijective linear map hence an isomorphism $\therefore P_n[x] \cong \mathbb{R}^{n+1}$	1 1 1 1 1 1 1 1
Q.2	Attempt any **TWO** questions from the following: (12)		
b)	i.	Show that F is non-singular where $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $F(x,y,z) = (x + y - 2z, x + 2y + z, 2x + 2y - 3z)$.	
	Ans	Let $(x,y,z) \in \ker T$ so that $x + y - 2z = 0, x + 2y + z = 0, 2x + 2y - 3z = 0$	1

		The matrix corresponding to this system $\begin{pmatrix}1&1&-2\\1&2&1\\2&2&-3\end{pmatrix}$ whose row reduced form is $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$. Thus $x=y=z=0 \Rightarrow ker\,F=\{0\}$ Therefore F is non-singular.	2 2 1
	ii.	If $T:\mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x,y,z)=(x+2y-z,y+z,x+y-2z)$. Find basis and the dimension of Ker T.	
	Ans	Let $(x,y,z)\in ker\,T$ so that $x+2y-z=0, y+z=0, x+y-2z=0$ The matrix corresponding to this system $\begin{pmatrix}1&2&-1\\0&1&1\\1&1&-2\end{pmatrix}$ whose row reduced form is $\begin{pmatrix}1&0&-3\\0&1&1\\0&0&0\end{pmatrix}$. Thus putting $z=t$, we get $y=-t$, and $x=3t$ $\Rightarrow ker\,F=\{t(3,-1,1)/t\in\mathbb{R}\}$. Therefore the basis of $ker\,F$ is $(3,-1,1)$ and is of dimension 1.	1 2 1 1 1
	iii.	Show that any n-dimensional real vector space is isomorphic to $\mathbb{R}^n$.	
	Ans	Let $x_1, x_2 \ldots\ldots x_n$ be a basis of V of dim n. Therefore for $x\in V, x=\sum c_i x_i \quad \forall c_i\in\mathbb{R}$ Thus a LT is defined by $T(x)=(c_1,\ldots\ldots\ldots c_n)$ is one-one. Since Dim V = n = dim $\mathbb{R}^n$, T is onto. Therefore T is invertible. Therefore $V\cong\mathbb{R}^n$.	1 1 2 1 1
	iv.	$P_3[\mathbb{R}]$ denote the vector space of all polynomials over $\mathbb{R}$ of degree 3 or less an $D(f(x))=\frac{df(x)}{dx}, \forall f(x)\in P_3[\mathbb{R}]$ denote the differentiation mapping. Let $B=\{1,x,x^2,x^3\}$ be the basis. Find $[m(D)]_B^B$.	
	Ans	$Dv_1=0=0v_1+0v_2+0v_3+0v_4$ $Dv_2=1=1v_1+0v_2+0v_3+0v_4$ $Dv_3=2x=0v_1+2v_2+0v_3+0v_4$ $Dv_4=3x^2=0v_1+0v_2+3v_3+0v_4$	4 2

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		$\Rightarrow [m(D)]_B^B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	
Q3.	Attempt any **ONE** question from the following: (08)		
$a)$	i.	Show that elementary row operations do not change the row rank of $A \in M_n(\mathbb{R})$.	
	Ans	Let $A \in M_n(\mathbb{R})$. Let $A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}$ where A_i is ith row of A Let$< A_1, A_2, \dots, A_i, \dots A_n >$ denote the row space of A B be the marix obtained from A by applying one of the following row operation. Elementary operation of 1st type: Let B be the matrix obtained from A by applying row operation $R_i \leftrightarrow R_j$ Row rank of $B = \dim < A_1, A_2, \dots, A_j, \dots, A_i, \dots A_n >$ $= \dim < A_1, A_2, \dots, A_i, \dots, A_j, \dots A_n >$ $=$ Row rank of A	2
		Elementary operation of 2nd type: Let B be the matrix obtained from A by applying row operation $R_i \leftrightarrow cR_i$ Row rank of $B = \dim < A_1, A_2, \dots, cA_i, \dots A_n >$ $= \dim < A_1, A_2, \dots, A_i, \dots A_n >$ $(\because < A_1, A_2, \dots, cA_i, \dots A_n > = < A_1, A_2, \dots, A_i, \dots A_n >)$ $=$ Row rank of A	2
		Elementary operation of 3rd type: Let B be the matrix obtained from A by applying row operation $R_j \leftrightarrow R_j + cR_i$ Row rank of $B = \dim < A_1, A_2, \dots, A_i, \dots, A_j + cA_i, \dots A_n >$ $= \dim < A_1, A_2, \dots, A_i, \dots, A_j, \dots A_n >$ $(\because < A_1, A_2, \dots, A_i, \dots, A_j + cA_i, \dots A_n > = < A_1, A_2, \dots, A_i, \dots, A_j, \dots A_n >)$ $=$ Row rank of A	2
		Elementary row operations do not change the row rank of matrix A.	2
	ii.	Let $A^1, A^2 \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Show that I) $\det(A^1, A^2) = 0$ iff $\{A^1, A^2\}$ is linearly dependent. II) $\det(A^1 + cA^2, A^2) = \det(A^1, A^2)$.	

Ans

I)

($\Rightarrow$)

Given: $\det(A^1, A^2) = 0$

T.P.T : $\{A^1, A^2\}$ is linearly dependent.

Suppose $\{A^1, A^2\}$ is linearly independent

$\therefore \{A^1, A^2\}$ is the basis of $\mathbb{R}^2$

Let $E^1 = \alpha_1 A^1 + \alpha_2 A^2$ and $E^2 = \beta_1 A^1 + \beta_2 A^2$

$$\det(E^1, E^2) = \det(\alpha_1 A^1 + \alpha_2 A^2, \beta_1 A^1 + \beta_2 A^2)$$
$$= (\alpha_1\beta_2 - \alpha_2\beta_1)\det\,(A^1, A^2)$$
$$= 0.\ \text{which is a contradiction.}$$

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($\Leftarrow$)

Given: $\{A^1, A^2\}$ is linearly dependent.

T.P.T : $\det(A^1, A^2) = 0$

As $\{A^1, A^2\}$ is linearly dependent

$\therefore$ Let $A^2 = cA^1, c \neq 0, c \in \mathbb{R}$

$$\det(A^1, A^2) = \det(A^1, cA^1)$$
$$= c\det(A^1, A^1)$$
$$= 0$$

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II)

$$\det(A^1 + cA^2, A^2) = \det(A^1, A^2) + \det(cA^2, A^2)$$
$$= \det(A^1, A^2) + c\det(A^2, A^2)$$
$$= \det(A^1, A^2)$$

2

Q3. Attempt any **TWO** questions from the following: (12)

b) i. Define adjoint of a matrix. Find A^{-1} for $A = \begin{pmatrix} 1 & -1 & 2 \\ 4 & 0 & 6 \\ 0 & 1 & -1 \end{pmatrix}$ using adjoint.

Ans

For $A, \in M_n(\mathbb{R})$,

Let A_{ij} be the matrix obtained from A by deleting its ith row and jth column

Let $c_{ij} = (-1)^{i+j} \det A_{ij}$

$C = (c_{ij})$ is called matrix of cofactors

$adj(A) := C^t$

Given matrix is $A = \begin{pmatrix} 1 & -1 & 2 \\ 4 & 0 & 6 \\ 0 & 1 & -1 \end{pmatrix}$ 2

Matrix of cofactors is $C = \begin{pmatrix} -6 & 4 & 4 \\ 1 & -1 & -1 \\ -6 & 2 & 4 \end{pmatrix}$

2

$\text{Adj}(A) = C^t = \begin{pmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{pmatrix}$

$A^{-1} = \frac{1}{\det A} Adj(A) = \frac{1}{-2}\begin{pmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{pmatrix}$

			2
	ii.	Let A= $\begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$, find elementary matrices E_1, E_2 such that $E_2E_1A= I_2$.	
	Ans	$\begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 5R_1} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore E_1 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$	3 3
	iii.	Let $A \in M_{m \times n}(\mathbb{R})$. Show that dimension of the solution space of the system of linear equations $AX = 0$ equals $n - \text{rank}\, A$	
	Ans	Define a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ such that $T(X) = AX$ $\therefore$ By rank nullity theorem n= Rank T +nullity T $\therefore$nullity $T = n -$Rank T......(1) Nullity T =Dim Ker T $= Dim\{X \in \mathbb{R}^n \mid T(X) = 0\}$ $= Dim\{\{X \in \mathbb{R}^n \mid AX = 0\}$ =Dim of solution space of the system $(AX = 0)$.....(2) From (1) and (2) Dim of solution space of the system $(AX = 0) = n -$Rank T	3 3
	iv.	Solve the following system of linear equations using Cramer's rule: $x + y + z = 6$, $2x - y + z = 3$, $4x - y - z = -1$	
	Ans	$x + y + z = 6$, $2x - y + z = 3$, $4x - y - z = -1$The corresponding non-homogeneous system is $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix}$ $x = \dfrac{det\begin{pmatrix} 6 & 1 & 1 \\ 3 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix}}{\det\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & -1 & -1 \end{pmatrix}} = \dfrac{10}{10} = 1$	2 2

$$y = \frac{\det\begin{pmatrix}1 & 6 & 1\\ 2 & 3 & 1\\ 4 & -1 & -1\end{pmatrix}}{\det\begin{pmatrix}1 & 1 & 1\\ 2 & -1 & 1\\ 4 & -1 & -1\end{pmatrix}} = \frac{20}{10} = 2$$

2

$$z = \frac{\det\begin{pmatrix}1 & 1 & 6\\ 2 & -1 & 3\\ 4 & -1 & -1\end{pmatrix}}{\det\begin{pmatrix}1 & 1 & 1\\ 2 & -1 & 1\\ 4 & -1 & -1\end{pmatrix}} = \frac{30}{10} = 3$$

Q4.	Attempt any **ONE** question from the following: (08)		
a)	i.	Define Group. For any positive integer n, Prove that $\mathbb{Z}_n$ the set of residue classes modulo $n$is a group under addition modulo n.	
	Ans	Definition of Group	2
		For any positive integer n, we have $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{n-1}\}$. Closure property: Let $\bar{a}, \bar{b} \in \mathbb{Z}_n$ By division algorithm, there exist unique integers q and r such that $a + b = nq + r$ where $0 \le r < n$. $\therefore \overline{a+b} = \bar{r} \in \mathbb{Z}_n$	2
		Associative law: Since addition is associative for real numbers, Associative law holds for $\mathbb{Z}_n$	1
		Prove Identity: $e = \bar{0}$	1
		Prove Inverse: $\overline{n-a}$ is the inverse of $\bar{a}$	2
	ii.	Define a subgroup of a group? Show that a nonempty subset H of G is a subgroup of a group G if and only if $ab^{-1} \in H, \forall a, b \in H$.	
	Ans	A subset $H$of a group $G$is a subgroup of G if and only if $H$is itself a group under the group operation of G.	2
		($\Rightarrow$) Let H be a subgroup of G and H be a non-empty subset of G. Let $a, b \in H$ then$b^{-1} \in H$ ($\because H$ is a subgroup of $G \therefore H$ is a group. $\therefore$ every element of H has an inverse in H) Now $\because a, b^{-1} \in H \therefore ab^{-1} \in H$ ($\because H$ is a subgroup of $G \therefore H$ is a group. $\therefore H$ is closed	3

under multiplication)

($\Leftarrow$) Let $ab^{-1} \in H, \forall a, b \in H$.

Now $\because H \neq \phi . \therefore$ let $a \in H$. $\therefore e = aa^{-1} \in H$. $\therefore H$ has multiplicative identity.

Next, $a \in H \Rightarrow (\because e \in H)\ a^{-1} = ea^{-1} \in H$. $\therefore$ every element of H has an inverse in H

Also now, $a, b \in H \Rightarrow a, b^{-1} \in H \Rightarrow ab = a(b^{-1})^{-1} \in H$. $\therefore H$ is closed under multiplication. **3**

$\therefore H$ is a subgroup of G.

Q4. Attempt any **TWO** questions from the following: (12)

b) i. In any group show that:
(p) the identity element is unique and
(q) the inverse of every element is unique.

Ans (p) Let G be a group. Suppose that e and e_1 are both identity elements of G then to prove that $e = e_1$.
Now, $ee_1 = e_1$ ($\because e$ is the identity element of G)
Also, $ee_1 = e$ ($\because e_1$ is the identity element of G)
Thus, $e = e_1$, as required. **3**
(q) Let $x \in G$, then $\because G$ is a group $\therefore x$ has an inverse.
Now t.p.t. the inverse of x is unique.
i.e. t.p.t. if y and y_1 are inverses of x then $y = y_1$.
$\because$ multiplication is associative in G, $\therefore y(xy_1) = (yx)y_1$
Now, $\because y_1$ is the inverse of x. $\therefore LHS = y(xy_1) = ye = y$.
//ly, $\because y$ is the inverse of x. $\therefore RHS = (yx)y_1 = ey_1 = y_1$.
$\therefore y = y_1$ as required. **3**

ii. Construct composition table of $G = \{\overline{5}, \overline{10}, \overline{15}, \overline{20}\}$ under multiplication modulo 40 and find order of all its elements.

Ans $G = \{\overline{5}, \overline{15}, \overline{25}, \overline{35}\}$ **2**

$\times_{40}$	$\overline{5}$	$\overline{15}$	$\overline{25}$	$\overline{35}$
$\overline{5}$	$\overline{25}$	$\overline{35}$	$\overline{5}$	$\overline{15}$
$\overline{15}$	$\overline{35}$	$\overline{25}$	$\overline{15}$	$\overline{5}$
$\overline{25}$	$\overline{5}$	$\overline{15}$	$\overline{25}$	$\overline{35}$
$\overline{35}$	$\overline{15}$	$\overline{5}$	$\overline{35}$	$\overline{25}$

$\overline{25}$ is the identity. $\therefore o(\overline{25}) = 1$ **1**
$\overline{5}^2 = \overline{25}$ $\therefore o(\overline{5}) = 2$ **1**
$\overline{15}^2 = \overline{25}$ $\therefore o(\overline{15}) = 2$ **1**
$\overline{35}^2 = \overline{25}$ $\therefore o(\overline{35}) = 2$ **1**

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	iii.	In a group G prove the following: (p) If $(ab)^2 = a^2b^2, \forall a, b \in G$ then G is abelian. (q) If G is abelian then $(ab)^n = a^n b^n, \forall n \in \mathbb{N}$.	
	Ans	(p) $\forall a, b \in G, (ab)^2 = a^2b^2 \Rightarrow (ab)(ab) = (aa)(bb) \Rightarrow a(ba)b = a(ab)b \Rightarrow (ba)b = (ab)b$, by LCL$\Rightarrow ba = ab$, by RCL. $\therefore G$ is abelian. (q) We prove the result by induction on n. $n = 1$: $(ab)^1 = ab = a^1b^1$ is true. Assume by i.h. the result for $n = k$, then to prve the same for $n = k + 1$. We have,$(ab)^{k+1} = (ab)^k(ab) \underset{\text{by i.h.}}{=} a^kb^kab \underset{\because G \text{ is abelian.}}{=} a^kab^kb = a^{k+1}b^{k+1}$. This completes the induction step and the proof.	2 4
	iv.	Let G be a group and $a \in G$. Prove that $N(a) = \{x \in G: ax = xa\}$ is a subgroup of G.	
	Ans	Let e be the identity of group G and $a \in G$. Since $ae = ea, e \in N(a)$. Hence, $N(a)$ is a non-empty subset of G. Consider any $x, y \in N(a) \therefore ax = xa$ and $ay = ya$ Note $ay^{-1} = (y^{-1}y)ay^{-1} = y^{-1}(ya)y^{-1}$ $= y^{-1}(ay)y^{-1} = y^{-1}a(yy^{-1}) = y^{-1}a$ $\therefore a(xy^{-1}) = (ax)y^{-1} = (xa)y^{-1}$ $= x(ay^{-1}) = x(y^{-1}a) = (xy^{-1})a \quad \therefore xy^{-1} \in N(a)$ Hence$N(a) = \{x \in G: ax = xa\}$ is a subgroup of G	1 1 2 2
Q5.	Attempt any **FOUR** questions from the following: (20)		
a)	Prove that if $T: V \to V'$ is a linear transformation then T is injective if and only if $\ker T = \{0\}$.		
Ans	**Proof:** Suppose T is injective and $x \varepsilon \ker T$. Then $Tx = o$. However $To = o$. $\therefore$ $Tx = To$ forcing x to be o. Thus $\ker T = \{o\}$. Conversely, suppose $\ker T = \{o\}$. If $Tx = Ty$, then since T is l.t. we have $T(x - y) = Tx - Ty = o$, implying that $x - y \varepsilon \ker T$. $\therefore$ $x - y = o$ or $x = y$. This proves that T is injective.		1 1 2 1
b)	Check whether the following linear transformation is an isomorphism $T: \mathbb{R}^3 \to \mathbb{R}^3, T(x, y, z) = (x + y, x - z, y + 2z)$		
Ans	Let $(x, y, z) \in \ker T$ so that $x + y = 0, x - z = 0, y + 2z = 0$		1 2

(13)

	The matrix corresponding to this system $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ whose row reduced form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Thus we get, $z=0, y=0$, and $x=0 \Rightarrow ker\, F=\{0\}$. And dim V = dim W. Therefore T is an isomorphism.	1 1
c)	Define Rank of the Matrix. Find Rank of the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 1 \\ 1 & -3 & 1 \end{pmatrix}$	
Ans	Rank A= Dimension of row space of A = number of linearly independent rows of A = Dimension of column space of A = number of linearly independent columns of A Observe that $R_3 = R_2 - 2R_1$ $\therefore$number of linearly independent rows of $A = 2$ Rank A=2.	2 3
d)	For $A, B \in M_n(\mathbb{R})$, if A is invertible show that I) $\det(A^{-1}) = (\det A)^{-1}$ II) $\det(ABA^{-1}) = \det B$ III) $\det(A^tB^t) = \det A \,.\det B$	
Ans	We know that $AA^{-1} = I$ $\therefore det\ AA^{-1} = det\ I$ $\therefore detA.detA^{-1} = 1$ $\therefore detA^{-1} = (detA)^{-1}$ $detABA^{-1} = detA.detB.detA^{-1}$ $= detA.detB.(detA)^{-1}$ $= detB$ $detA^tB^t = detA^t detB^t$ $= detA.detB$	2 2 1
e)	List all elements of $U(15)$ and find their orders.	
Ans	$U(15) = \{\bar{1},\bar{2},\bar{4},\bar{7},\bar{8},\overline{11},\overline{13},\overline{14}\}$ and $o(\bar{1}) = 1;\ o(\bar{4}) = o(\overline{11}) = o(\overline{14}) = 2;\ o(\bar{2}) = o(\bar{8}) = o(\bar{7}) = o(\overline{13}) = 4.$	1 4
f)	Let G be an abelian group then show that $H = \{g^2/g \in G\}$ is a subgroup of G.	

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Ans	Since $e = e^2 \in H, H$ is a non-empty subset of G. Consider any $x, y \in H. \therefore x = g^2$ and $y = h^2$ for some $g, h \in G$. Then $xy^{-1} = g^2(h^2)^{-1} = g^2(h^{-1})^2 = ggh^{-1}h^{-1}$ $= gh^{-1}gh^{-1}$ ... since G is abelian $= (gh^{-1})^2$ where $gh^{-1} \in G$ since G is a group. $\therefore xy^{-1} \in H$ Hence, H is a subgroup of G.	1 1 2 1

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